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A Study on the Errors in the Free-Gyro Positioning System (I)

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Abstract : *This paper is to develop the position error equation of in the free-gyro positioning system by using two free gyros. First, the determination of a position is analyzed on the ellipsoid of the Earth and the type of the errors is defined. Finally the position error equation is introduced and developed, based on the definition of the type of errors which may be involved in the FPS.*

Key words : *Free gyro, Zenith angle, Position error equation, FPS, Sensor error, Danger zone, Error of position*

1. Introduction

A free gyro positioning system (FPS) is to determine the position of a vehicle by using two free gyros(Park & Jeong, 2004). It is an active positioning system like an inertial navigation system (INS) in view of obtaining a position without external source. However, a FPS is to determine its own position by using the angle between the vertical axis of local geodetic frame and the axis of free gyro, while an INS is to do so by measuring its acceleration(Lawrence, 1998).

In general the INS comprises a set of inertial measurement units (IMU's). It contains both accelerometers and gyros, the platform on which they are mounted, including the stabilization mechanism if so provided, and the calculating computer for transforming sensed accelerations and in some mechanizations, angles or angular rates into useful information for navigation such as position, velocity and attitude. So the INS is composed of a very complicated structure (Lawrence, 1998).

On the other hand, the FPS consists of a set of units with a vertical gyro, two free gyros and the computer that calculates navigational information. It is comparatively simpler than the INS. Although the FPS has this merit, the position error has not been decided yet. The system is thought to have position errors caused by the variation in Earth rate, measurement of time, and sensor errors, etc. In this paper the position error equations are introduced and developed, based on the definition of the type of errors which may be involved in the FPS. Then the determination of a position by two free gyros was analyzed by the transformation of the coordinate frame.

2. Determination of Vehicle's Position and Type of Errors Involved in the FPS

2.1 Determination of Vehicle's position

The FPS is to obtain a vehicle's position by using the property of free gyro, which is known as gyroscopic inertia or rigidity in space and vertical gyro. Park & Jeong(2004)'s method here is derived as follows.

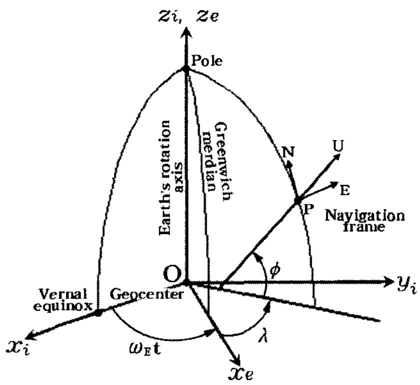

Fig. 1 Reference frame

When describing locations of points on or near the Earth's surface, we most naturally consider a system of coordinate frames. Each frame is an orthogonal, right-handed axis set. As shown in Fig. 1, the inertial frame has its origin at the center of the Earth and axes which are not rotating with respect to the vernal equinox, defined by the axes Ox_i, Oy_i, Oz_i, with Oz_i coincident with the Earth's polar axis. The earth frame has its origin at the center of the Earth, defined by the axes Ox_e, Oy_e, Oz_e, along the Earth's polar axis. The earth frame rotates with

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respect to the inertial frame at a rate of ω_E about the axis Oz_i. Finally, the navigation frame is a local geographic frame, which has its origin at the location of the navigation system, point P, and axes aligned with the directions of east(E), north(N) and the local vertical (U). The turn rate of the navigation frame, with respect to the Earth frame, ω_{en}, is determined by the motion of the point P with respect to the Earth.

First consider the transformation matrix C_i^n (Rogers 2000) from the inertial frame to the navigation frame which is simply given by eqn. (1).

$$\begin{aligned} C_i^n &= C_e^n C_i^e \\ &= \begin{bmatrix} -sin\lambda & cos\lambda & 0 \\ -sin\phi cos\lambda & -sin\phi sin\lambda & cos\phi \\ cos\phi cos\lambda & cos\phi sin\lambda & sin\phi \end{bmatrix} \begin{bmatrix} cos\omega_E t & sin\omega_E t & 0 \\ -sin\omega_E t & cos\omega_E t & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -sin(\lambda+\omega_E t) & cos(\lambda+\omega_E t) & 0 \\ -sin\phi cos(\lambda+\omega_E t) & -sin\phi sin(\lambda+\omega_E t) & cos\phi \\ cos\phi cos(\lambda+\omega_E t) & cos\phi sin(\lambda+\omega_E t) & sin\phi \end{bmatrix} \end{aligned} \quad (1)$$

Here, ω_E is the (presumably uniform) rate of Earth rotation, λ is the geodetic longitude, φ is the geodetic latitude and t denotes time. This transformation matrix C_i^n denotes the transformation from the unit vectors of axes in the inertial frame to those in the navigation frame. Consider an arbitrary gyro vector $g_v^i = [u_x, u_y, u_z]^T$ which is unit vector in the inertial frame. We obtain easily the gyro vector transformed in the navigation frame, $g_v^n = [E_u, N_u, U_u]^T$, as eqn. (2).

$$\begin{aligned} g_v^n &= C_i^n g_v^i \\ &= \begin{bmatrix} -sin(\lambda+\omega_E t) & cos(\lambda+\omega_E t) & 0 \\ -sin\phi cos(\lambda+\omega_E t) & -sin\phi sin(\lambda+\omega_E t) & cos\phi \\ cos\phi cos(\lambda+\omega_E t) & cos\phi sin(\lambda+\omega_E t) & sin\phi \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \\ &= \begin{bmatrix} -u_x sin(\lambda+\omega_E t) + u_y cos(\lambda+\omega_E t) \\ -u_x sin\phi cos(\lambda+\omega_E t) - u_y sin\phi sin(\lambda+\omega_E t) + u_z cos\phi \\ u_x cos\phi cos(\lambda+\omega_E t) + u_y cos\phi sin(\lambda+\omega_E t) + u_z sin\phi \end{bmatrix} \\ &= \begin{bmatrix} E_u \\ N_u \\ U_u \end{bmatrix} \end{aligned} \quad (2)$$

As shown in Fig. 2, the azimuth angle *a* and the zenith angle θ can be obtained respectively as eqn. (3) and eqn. (4), considering $\lfloor g_v \rfloor = 1$.

$$cos\theta = \frac{U_u}{|g_v|} = u_x cos\phi cos(\lambda+\omega_E t) + u_y cos\phi sin(\lambda+\omega_E t) + u_z sin\phi \quad (3)$$

$$tan\alpha = \frac{E_u}{N_u} = \frac{-u_x sin(\lambda+\omega_E t) + u_y cos(\lambda+\omega_E t)}{-u_x sin\phi cos(\lambda+\omega_E t) - u_y sin\phi sin(\lambda+\omega_E t) + u_z cos\phi} \quad (4)$$

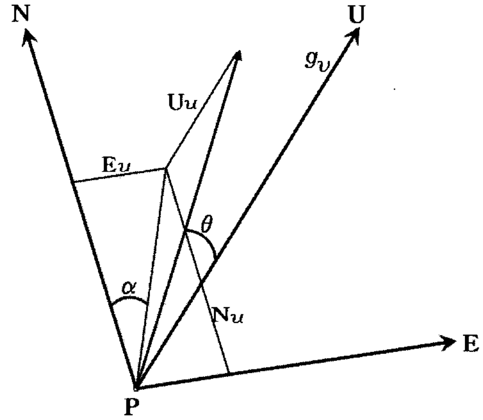

Fig. 2 Measurement quantities in the navigation frame

If we use two free gyros whose gyro vectors in eqn. (3) are $g_{va}^i = [u_{ax}, u_{ay}, u_{az}]^T$ and $g_{vb}^i = [u_{bx}, u_{by}, u_{bz}]^T$ respectively, we can determine the position (ϕ, λ) of a vehicle at the given zenith angles θ_a, θ_b. Once determining the position, we can also obtain its azimuth or heading by using eqn. (4). Park and Jeong(2004) already suggested the algorithm of how to determine a position.

2.2 Type of Errors involved in the FPS

Recalling eqn. (3), the FPS has the errors caused by the variation in Earth rate, ω_E, by the inaccuracy in time *t*, by the measurement sensors of zenith angle θ, and by the geometric arrangement of two free gyros.

1) Variation in Earth rate

It is known that the direction of the vernal equinox is changing. Magnitude, time scales, and causal mechanisms are still being sorted out, but these are some estimates (Grewal et al., 2001).

1. On the scale of millions to billions of years, the rotation of the earth is slowing down due to tidal effects. The rate of slowdown is currently estimated to be on the order of 0.2 parts per billion per year.
2. On the scale of millennia, redistribution of water during ice ages changes the moments of inertia of the Earth, with rotation rate varying inversely as the polar moment of inertia. A change of 20 arc seconds in the direction of the equinox is comparable to a change of about 100ppm in magnitude.
3. On the scale of years to millennia, there are changes in the internal flows in the hydrosphere and lithosphere. Apparent pole shifts on the earth observed in the last few decades are on the order of tens of meters.
4. On the scale of months to years, the global changes in

weather patterns known as El Niño are associated with a slowdown of the earth rotation rate on the several parts per billion.

5. On the scale of days to weeks, shifts in the north and south jet streams and associated weather patterns are suspected of altering the earth rotation rate on the order of parts per billion.

But, for navigation missions on time scales on the order of hours, these variations can be ignored. Here, the value of earth rate in the World Geodetic System 1984 (WGS 84) earth model is used. It is 7,292,115.0 x 10-11 rad/sec (NIMA, 2000)and represents a standard Earth rotating with a constant angular velocity. This is its sidereal rotation rate with respect to distant stars.

2) Measurement of time

In a vehicle time is assumed to be measured by a crystal clock. Even if a quartz wristwatch is rated by measuring time against an atomic clock's time broadcast, and worn on one's body to keep its temperature constant, the corrected time can easily be as accurate as 2 seconds per month. Nowadays the accuracy of crystal clock ranges from 0.01 to 0.001 second per day.

3) Sensor errors

The errors of the zenith angle are caused by the measurement sensors such as free gyros and a vertical gyro. The major sources of error which arise in mechanical gyros are fixed bias(g-independent), acceleration dependent bias (g-dependent), anisoelastic bias (g^2-dependent), anisoinertia errors, scale factor errors, cross-coupling errors, and angular acceleration sensitivity(Titterton & Weston, 1997).

The next chapter will investigate the error of position, assuming that the Earth rate is a constant.

3. Position Error Equation

For a development of the error equation, we differentially perturb eqn. (3), assuming that the Earth rate is a constant, and then interpret the perturbations (Jekeli, 2000) as errors resulting in a linear model. Here, considering tacitly the commutativity of the differential operator, δ, and the time differentiation operator, d/dt, then the differential perturbation of eqn. (3) is given by eqn. (5).

$$\begin{aligned} &-sin\theta\delta\theta - cos\phi[-u_x sin(\lambda+\omega_E t)+u_y cos(\lambda+\omega_E t)]\omega_E \delta t \\ &= [-sin\phi(u_x cos(\lambda+\omega_E t)+u_y sin(\lambda+\omega_E t))+u_z cos\phi]\delta\phi \\ &+ cos\phi[-u_x sin(\lambda+\omega_E t)+u_y cos(\lambda+\omega_E t)]\delta\lambda \end{aligned} \quad (5)$$

Now, two gyro vectors $g^i_{va}=[u_{ax}, u_{ay}, u_{az}]^T$, $g^i_{vb}=[u_{bx}, u_{by}, u_{bz}]^T$ and the corresponding zenith angles θ_a, θ_b, are considered. The perturbation matrix is arranged by eqn. (6).

$$\begin{bmatrix} Z_a \\ Z_b \end{bmatrix} = \begin{bmatrix} \Phi_a & M_a \\ \Phi_b & M_b \end{bmatrix} \begin{bmatrix} \delta\phi \\ \delta\lambda \end{bmatrix} \quad (6)$$

Here,

$$\begin{aligned} \Phi_a &= -sin\phi(u_{ax} cos(\lambda+\omega_E t)+u_{ay} sin(\lambda+\omega_E t))+u_{az} cos\phi \\ M_a &= cos\phi(-u_{ax} sin(\lambda+\omega_E t)+u_{ay} cos(\lambda+\omega_E t)) \\ Z_a &= -sin\theta_a \delta\theta_a - M_a \omega_E \delta t \end{aligned}$$

$$\begin{aligned} \Phi_b &= -sin\phi(u_{bx} cos(\lambda+\omega_E t)+u_{by} sin(\lambda+\omega_E t))+u_{bz} cos\phi \\ M_b &= cos\phi(-u_{bx} sin(\lambda+\omega_E t)+u_{by} cos(\lambda+\omega_E t)) \\ Z_b &= -sin\theta_b \delta\theta_b - M_b \omega_E \delta t \end{aligned}$$

This matrix can be written in the matrix-vector form

$$\Delta Z = \Psi \Delta x \quad \Leftrightarrow \quad \Delta x = \Psi^{-1} \Delta Z \quad (7)$$

where

$$\Delta x = \begin{bmatrix} \delta\phi \\ \delta\lambda \end{bmatrix} \qquad \Psi^{-1} = \begin{bmatrix} \Phi_a & M_a \\ \Phi_b & M_b \end{bmatrix}^{-1}$$

$$\Delta Z = \begin{bmatrix} Z_a \\ Z_b \end{bmatrix}$$

have been introduced. The covariance matrix for eqn. (7) is given by

$$cov(\Delta x) = \Psi^{-1} cov(\Delta Z)(\Psi^{-1})^T. \quad (8)$$

The assumption may be made that the sensor and time errors show a random behavior resulting in a normal distribution with expectation value zero and variance σ_s^2. Therefore, measured sensor and time values are linearly independent or uncorrelated. The $cov(\Delta Z)$ is represented by

$$cov(\Delta Z) = \sigma_s^2 I \quad (9)$$

where I is the unit matrix.
Substituting eqn. (9) into eqn. (8) yields

$$\begin{aligned} cov(\Delta x) &= \Psi^{-1}\sigma_s^2 I(\Psi^{-1})^T \\ &= \sigma_s^2\ \Psi^{-1}(\Psi^{-1})^T \\ &= \sigma_s^2\ (\Psi^T\Psi)^{-1} \\ &= \sigma_s^2 Q_x \end{aligned} \quad (10)$$

where $Q_x = (\Psi^T\Psi)^{-1}$. The cofactor matrix Q_x (Hofmann -Wellenhof B. et al, 2001) is a 2 x 2 matrix where two components are contributed by the gyro vectors g^i_{va}, g^i_{vb}. The elements of the cofactor matrix are denoted as

$$Q_x = \begin{bmatrix} q_{\phi\phi} & q_{\phi\lambda} \\ q_{\phi\lambda} & q_{\lambda\lambda} \end{bmatrix}. \quad (11)$$

In the cofactor matrix the diagonal elements are used for FDOP1) which is borrowed from GDOP in the GPS as we know and is defined as the geometry of two free gyros.

$$FDOP = \sqrt{q_{\phi\phi} + q_{\lambda\lambda}} \quad (12)$$

Therefore the error of a position in the FPS, fd_{rms}, is given by

$$fd_{rms} = FDOP\ \sigma_s \quad (13)$$

From eqn. (13) fd_{rms} can be computed easily. If in this case two-dimensional error distribution is close to being circular, the probability is about 0.63(Kaplan, 1996).

As an example Fig. 3 shows the variation of position error, fd_{rms}, with respect to the error in zenith angle at a given position. The following are assumed.

Position : Lat 35.456216°N, Long. 127.234261°E

Zenith angles:

θ_a = 84.14503077802169°

θ_b = 58.75116275701419°

Error in zenith angle :

0.1′ ~0.4′ ($1.6 \times 10^{-3} \sim 6.6 \times 10^{-3}$°)

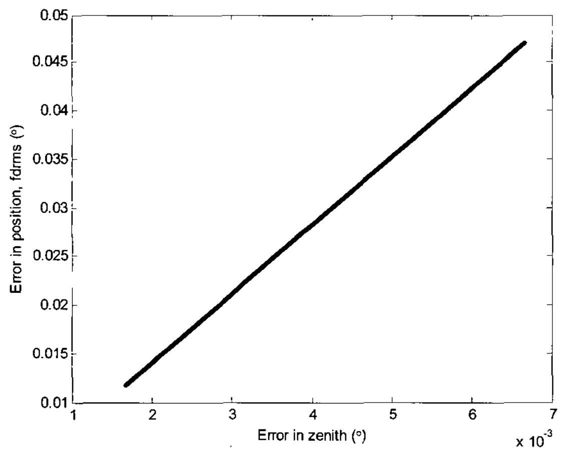

Fig. 3 Position error, fd_{rms} with respect to error in zenith

1) FDOP is the same as DOP(dilution of precision). The letter 'F' is short form of free gyro.

4. Conclusions

In this paper the determination of a position by two free gyros was analyzed by the transformation of the coordinate frames. And the errors involved in the FPS were defined. Finally the error of a position was theoretically investigated.

However there are still many problems to be solved. First the error of a position was experimentally verified. Especially the sensor errors will have to be investigated. In addition, the additional drift needs to be investigated, which occurs when a free gyro is suppressed by the additional device mounted on to prevent gimbals from being locked and tumbled. All these will be dealt with in the next papers.

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