Optimization of a Thermally Asymmetric Rectangular Fin: Based on Fixed Fin Height

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Key words: Thermally asymmetric rectangular fin, Heat loss, Biot number, Optimization

ABSTRACT: A thermally asymmetric straight rectangular fin is analysed and optimized using the two-dimensional separation of variables method. The optimum heat loss is presented as a function of bottom to top Biot number ratio, fin base length and top Biot number. Decreasing rate of the optimum fin length with the increase of the fin base length is listed. The optimum fin tip length is shown as a function of bottom to top Biot number ratio, fin base length and tip to top Biot number ratio. One of the results shows that the optimum heat loss and the actual optimum fin length decrease while the optimum fin tip length increases as the fin base length increases.

Nomenclature

Bi_j : Biot number of the each fin surface ($=(h_j l')/k$, $j=1, 2, e$)
h_j : heat transfer coefficient of the each fin surface [W/m^2℃]
k : thermal conductivity [W/m℃]
l' : one half fin base height [m]
L_b : dimensionless fin base length ($=L_b'/l'$)
L_b' : fin base length [m]
L_e : dimensionless fin tip length ($=L_e'/l'$)
L_e' : fin tip length [m]
Q : dimensionless heat loss from the fin, $q/(k\phi_i)$
q : heat loss from the fin per unit width [W/m]
T : temperature [℃]
T_b : fin base temperature [℃]
T_i : temperature at the inside of the wall [℃]
T_∞ : ambient temperature [℃]
x : dimensionless coordinate along the fin length ($=x'/l'$)
x' : coordinate along the fin length [m]
y : dimensionless coordinate along the fin height ($=y'/l'$)
y' : coordinate along the fin height [m]

Greek symbols

α : bottom to top Biot number ratio ($=Bi_2/Bi_1$)
β : tip to top Biot number ratio ($=Bi_e/Bi_1$)
θ : dimensionless temperature, $(T-T_\infty)/(T_i-T_\infty)$
λ_n : eigenvalues ($n=1, 2, 3, \cdots$)
ϕ_i : adjusted fin base temperature ($=T_i-T_\infty$) [℃]

Subscripts

1 : top side of a rectangular fin
2 : bottom side of a rectangular fin
b : fin base

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e : tip side of a rectangular fin
i : inside of the wall
∞ : ambient

Superscripts

$'$: dimensional quantity
$*$: optimum

1. Introduction

Extended surfaces or fins are designed to increase the rate of heat transfer to a surrounding fluid in many engineering applications such as the cooling of combustion engines, many kind of heat exchangers, air conditioning equipments and so on. Fins are usually analysed for the performance or for the optimization. For the performance of the fin, Abrate and Newnham[(1)] present the effectiveness of triangular fin as a function of dimensionless fin length. Kang[(2)] analyzed the performance of a thermally asymmetric rectangular fin using three dimensional analytical method. Also, optimization procedures of various shapes of fins have been studied. For a given heat dissipation rate, optimization procedure finds the geometric shape that minimizes the material volume. For this kind of optimization, Hrymak et al.[(3)] presented an efficient numerical method to discover the optimal shape for a fin subjected to both convective and radiative heat loss. One of the alternative ways is to fix a suitable simple profile, and then determine the dimensions of the fin that yields the maximum heat dissipation for a given fin volume or mass. For example, Ullmann and Kalman[(4)] considered the problem of increasing the heat dissipation of annular fins (four different cross-section shapes) at a defined magnitude of mass. Chung et al.[(5)] deals with the optimum design of convective longitudinal fins of a trapezoidal profile. Yeh[(6)] determines the optimum dimensions of a one-dimensional longitudinal rectangular fin and a cylindrical pin fin. Casarosa and Franco[(7)] investigated the optimum design of single longitudinal fins with constant thickness by means of an accurate mathematical method yielding the solution of constrained minimization (maximization) problems considering different uniform heat transfer coefficients on the fin faces and on the tip. Razelos and Satyaprakash[(8)] also present an analysis of trapezoidal profile longitudinal fins that delineates their thermal performance and an improved solution of the optimal problem. Another alternative way is to fix a fin height and choose the 98% or 99% of the maximum heat loss as the optimum heat loss. For this optimum procedure, Kang and Look[(9)] showed the optimum fin length as a function of bottom to top Biot number ratio for a thermally asymmetric rectangular fin. Recently, Kang and Look[(10)] present the optimum heat loss and dimensions for a thermally and geometrically asymmetric trapezoidal fin.

In all these papers for the optimization, fin base temperature is given as a constant for the boundary condition and the effect of fin base thickness is not considered. In this study, for a thermally asymmetric straight rectangular fin, inside wall temperature is given and the effect of the fin base thickness on the fin optimization is considered. The optimum heat loss is taken as 98% of the maximum heat loss for given conditions. For this optimum criterion, the optimum heat loss is presented as a function of bottom to top Biot numbers ratio, fin base length and top Biot number. Also, the fin dimensions corresponding to the optimum heat loss are shown as a function of bottom to top Biot numbers ratio, fin base length and tip to top Biot numbers ratio.

2. Two-dimensional analysis

For a thermally asymmetric rectangular fin, illustrated in Fig. 1, the dimensionless governing differential equation is given by Eq. (1).

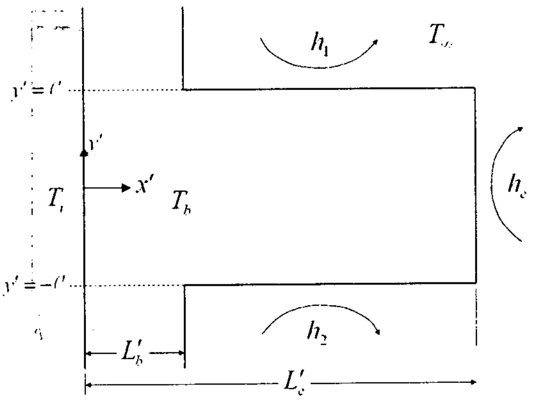

Fig. 1 Geometry of a thermally asymmetric rectangular fin.

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0 \tag{1}$$

Four boundary conditions serve as the required problem formulation.

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=L_b} = \frac{1}{L_b}(1-\theta|_{x=L_b}) \tag{2}$$

$$\left.\frac{\partial\theta}{\partial y}\right|_{y=1} + \mathrm{Bi}_1\cdot\theta|_{y=1} = 0 \tag{3}$$

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=L_e} + \mathrm{Bi}_e\cdot\theta|_{x=L_e} = 0 \tag{4}$$

$$\left.\frac{\partial\theta}{\partial y}\right|_{y=-1} - \mathrm{Bi}_2\cdot\theta|_{y=-1} = 0 \tag{5}$$

By solving Eq. (1) with three boundary conditions listed as Eqs. (2) through (4), the dimensionless temperature can be obtained by the separation of variables procedure.

$$\theta(x, y) = \sum_{n=1}^{\infty}\frac{4\sin(\lambda_n)\cdot f(x)\cdot f(y)}{(A_n+B_n)(C_n+D_n)} \tag{6}$$

where,

$$f(x) = \cosh(\lambda_n x) - f_n\cdot\sinh(\lambda_n x) \tag{7}$$

$$f(y) = \cos(\lambda_n y) + g_n\cdot\sin(\lambda_n y) \tag{8}$$

$$A_n = (f_n - \lambda_n L_b)\cdot\sinh(\lambda_n L_b) \tag{9}$$

$$B_n = (1 - f_n\lambda_n L_b)\cdot\cosh(\lambda_n L_b) \tag{10}$$

$$C_n = 2\lambda_n + \sin(2\lambda_n) \tag{11}$$

$$D_n = g_n^2\cdot\{2\lambda_n - \sin(2\lambda_n)\} \tag{12}$$

f_n and g_n are expressed by

$$f_n = \frac{\lambda_n\cdot\tanh(\lambda_n L_e) + \mathrm{Bi}_e}{\lambda_n + \mathrm{Bi}_e\cdot\tanh(\lambda_n L_e)} \tag{13}$$

$$g_n = \frac{\lambda_n\cdot\tan(\lambda_n) - \mathrm{Bi}_1}{\lambda_n + \mathrm{Bi}_1\cdot\tan(\lambda_n)} \tag{14}$$

Eigenvalues λ_n are obtained using Eq. (15), which comes from top and bottom boundary conditions (i.e. Eqs. (3) and (5)).

$$\frac{\lambda_n\cdot\tan(\lambda_n) - \mathrm{Bi}_1}{\lambda_n + \mathrm{Bi}_1\cdot\tan(\lambda_n)} = \frac{\mathrm{Bi}_2 - \lambda_n\cdot\tan(\lambda_n)}{\lambda_n + \mathrm{Bi}_2\cdot\tan(\lambda_n)} \tag{15}$$

The value of the heat loss per unit width from the thermally asymmetric rectangular fin can be calculated with Eq. (16).

$$q = -k\int_{-l'}^{l'}\left.\frac{\partial T}{\partial x'}\right|_{x'=L_b'} dy' \tag{16}$$

Then, the dimensionless heat loss from the fin can be expressed by

$$Q = q/(k\phi_i) = \sum_{n=1}^{\infty}\frac{8\sin^2(\lambda_n)\cdot f_n}{(A_n+B_n)(C_n+D_n)} \tag{17}$$

3. Results

Figure 2 presents the dimensionless temperature profiles along the fin height at $x=(L_b+L_e)/2$ in the case of Biot numbers ratios of $\alpha=0.8, 0.9$ and 1. As expected, the temperature profile for $\alpha=1$ is symmetric while the temperature at the top surface is lower than that

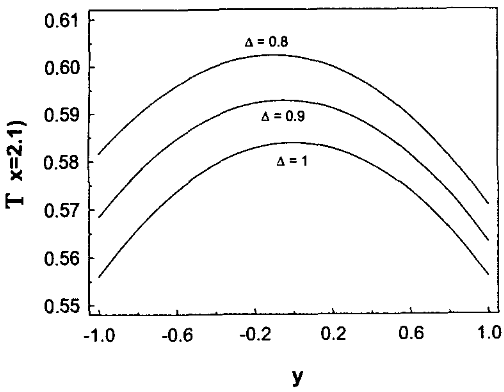

Fig. 2 Dimensionless temperature profile along the fin height with $Bi_1=0.1$, $\beta=1$, $L_b=0.1$ and $L_e=4.1$.

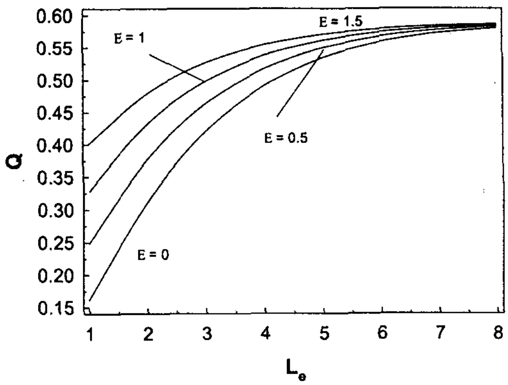

Fig. 4 Heat loss as a function of dimension-less fin tip length for $Bi_1=0.1$, $\alpha=0.9$ and $L_b=0.1$.

at the bottom surface for $\alpha=0.8$ and 0.9. It also indicates that temperature decreases as α increases at the same y-coordinate position; this statement is based on the fact that the average of all the surface Biot numbers increases as α increases.

Figure 3 represents the dimensionless heat loss from the fin as a function of dimensionless fin tip length for three values of α. It can be noted that the heat loss increases rapidly first and then increases slowly as fin tip length increases for all three values of α. This

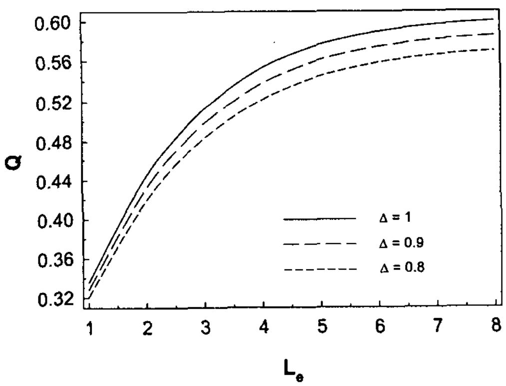

Fig. 3 Heat loss as a function of dimensionless fin tip length for $Bi_1=0.1$, $\beta=1$ and $L_b=0.1$.

is why the optimum heat loss is chosen as the 98% of the maximum heat loss. For example, the dimensionless fin tip length must be elongated from 6.5 for 98% of the maximum heat loss to 22.2 for the maximum heat loss just to get the rest 2% of the maximum heat loss in the case of $\alpha=0.9$. This figure also shows that the difference of heat loss between different α becomes a little larger as fin tip length increases.

The dimensionless heat loss from the fin as a function of dimensionless fin tip length for several values of β in the case of $Bi_1=0.1$, $\alpha=0.9$ and $L_b=0.1$ is shown in Fig. 4. The effect of β on the heat loss becomes smaller as fin tip length increases. From this phenomenon, it can be guessed that the optimum heat loss is independent of β since the heat loss becomes 98% of the maximum heat loss when the fin tip length becomes long enough.

Figure 5 represents the optimum heat loss as a function of the fin base length for $\alpha=0.9$ and $\beta=1$. It is noted that the optimum heat loss decreases almost linearly as the fin base length increases. It is because that the resistance between the inside wall and the fin base increases as the fin base length increases and the temperature at the fin base decreases. As

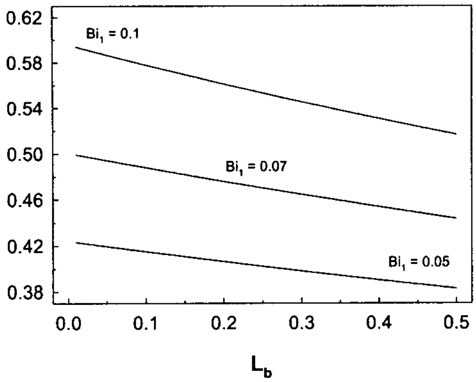

Fig. 5 Optimum heat loss versus the fin base length for α=0.9 and β=1.

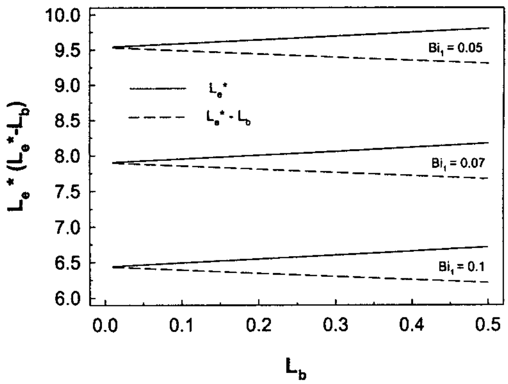

Fig. 6 Optimum length versus the fin base length for α=0.9 and β=1.

expected, it also shows that the optimum heat loss increases as top Biot number increases at the same fin base length.

The optimum length versus the fin base length for the same conditions as in Fig. 5 is depicted in Fig. 6. Both the optimum fin tip length (L_e^*) and the optimum fin length ($L_e^* - L_b$) can be referred as the optimum length. The optimum fin tip length is defined as the fin tip length which produces 98% of the maximum heat loss and the optimum fin length is defined as the fin length which produces 98% of the maximum heat loss for given conditions. For given top Biot numbers, the optimum fin tip length increases while the optimum fin length decreases as the fin base length increases. It means physically that the actual fin length becomes shorter even though the fin tip length increases for the optimum heat loss as the fin base length increases.

Table 1 lists the decreasing rate of the optimum fin length with the increase of fin base length. It can be known that the decreasing rate of the optimum fin length with the increase of the fin base length becomes larger as top Biot number increases even though this decreasing rate is not much. As already shown in Fig. 6, this table also indicates the decreasing rate of the optimum fin length varies almost linearly as the fin base length increases for all three values of top Biot number.

Table 1 Decreasing rate of the optimum fin length (β=1, α=0.9)

L_b	D. R. of ($L_e^* - L_b$) (%)		
	Bi_1=0.05	Bi_1=0.1	Bi_1=0.15
0.05 → 0.2	0.76	1.10	1.38
0.2 → 0.5	1.46	2.11	2.63

Figure 7 presents the variation of the optimum heat loss as a function of α in the case of β=1 and L_b=0.1. It shows that the optimum heat loss increases linearly as α increases. Note that the increasing rates of heat loss with the increase of α are almost the

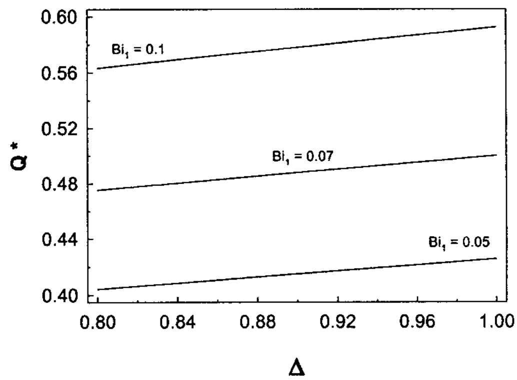

Fig. 7 Optimum heat loss as a function of α for β=1 and L_b=0.1.

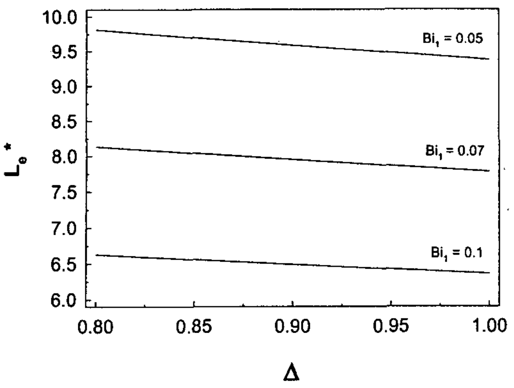

Fig. 8 Optimum fin tip length as a function of α for $\beta=1$ and $L_b=0.1$.

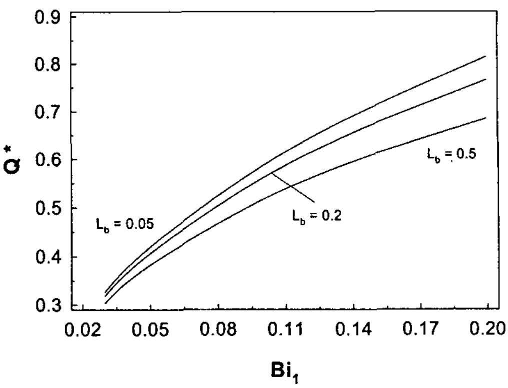

Fig. 10 Optimum heat loss versus Biot number for $\alpha=0.9$ and $\beta=1$.

same for given top Biot numbers. For example, the increasing rate of heat loss for Bi=0.07 is 5.21% while that for Bi=0.05 is 5.24% as α increases from 0.8 to 1.0.

The optimum dimensionless fin tip length versus α under the same condition given in Fig. 7 is shown in Fig. 8. The optimum fin tip length decreases linearly as α increases for given top Biot numbers. It also can be assumed from Figs. 7 and 8 that the optimum heat loss increases almost linearly while the optimum fin tip length decreases a little rapidly first and then decreases relatively slowly as top Biot number increases for fixed values of α.

Figure 9 depicts the optimum fin tip length as a function of β in the case of $\alpha=0.9$ and $L_b=0.1$. It is noticed from this figure that the optimum fin tip length decreases as β increases for all given values of top Biot number and the optimum fin tip length increases as top Biot number decreases for the same value of β. It also can be known that the effect of β on the optimum fin tip length is much greater than the effect of α on that by comparing this figure with Fig. 8.

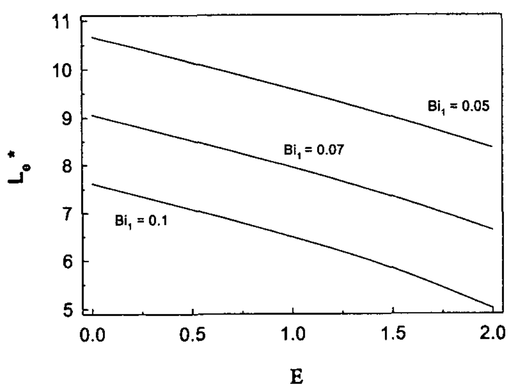

Fig. 9 Optimum fin tip length versus β for $\alpha=0.9$ and $L_b=0.1$.

Finally, the optimum heat loss versus top Biot number for three different fin base length in the case of $\alpha=0.9$ and $\beta=1$ is shown in Fig. 10. As expected, the optimum heat loss increases considerably as top Biot number increases. It shows that the ratio of the optimum heat losses for different value of the fin base length becomes larger as top Biot number increases. For example, the optimum heat loss for $L_b=0.05$ is 1.097 times of that for $L_b=0.5$ in the case of $Bi_1=0.05$ while the optimum heat loss for $L_b=0.05$ is 1.189 times of that for $L_b=0.5$ in the case of $Bi_1=0.2$.

4. Conclusions

From the two-dimensional analysis of a thermally asymmetric straight rectangular fin presented here, the following conclusions about the optimization can be drawn.

(1) The optimum heat loss and the actual optimum fin length decrease while the optimum fin tip length increases as the fin base length increases.

(2) The optimum heat loss increases and the optimum fin length decreases as α increases.

(3) Even though the optimum heat loss is independent of β, the optimum fin length decreases considerably as β increases.

(4) The effect of the fin base length variation on the optimum heat loss becomes larger as top Biot number increases for fixed other variables.

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