# Eigenvalue Analysis of Double-span Timoshenko Beams by Pseudospectral Method

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The pseudospectral method is applied to the free vibration analysis of double-span Timoshenko beams. The analysis is based on the Chebyshev polynomials. Each section of the double-span beam has its own basis functions, and the continuity conditions at the intermediate support as well as the boundary conditions are treated separately as the constraints of the basis functions. Natural frequencies are provided for different thickness-to-length ratios and for different span ratios, which agree with those of Euler-Bernoulli beams when the thickness-to-length ratio is small but deviate considerably as the thickness-to-length ratio grows larger.

**Key Words :** Eigenvalue Analysis, Double-span Timoshenko Beam, Pseudospectral Method, Chebyshev Polynomials

## 1. Introduction

The Euler-Bernoulli beam theory neglects the effect of the transverse shear strain of beam bending because of the assumption that the plane cross-sections perpendicular to the axis of the beam remain plane and perpendicular after deformation. The Euler-Bernoulli beam theory can give excellent solutions to the vibration analysis of slender beams. Beams in real practice, however, may have appreciable thickness where the transverse shear and rotary inertia are not negligible as assumed in the Euler-Bernoulli beam theory. As the result the Timoshenko beam theory that takes the transverse shear and the rotary inertia into consideration has gained more popularity.

Research on beam vibration can be divided into three categories. Firstly, there exist exact solutions only for a restricted number of simple cases. Secondly, studies of semi-analytic solutions are available. Finally, there are the most widely used discretization methods such as the finite element method and the finite difference method. As it is more useful to have analytical results than to resort to numerical methods, most efforts focus on developing efficient semi-analytic solutions.

Multi-span beams are frequently used in many mechanical and civil engineering applications such as the rail systems and the bridges. Gorman computed the natural frequencies of double-span Euler-Bernoulli beams by proposing local solutions for each span and by matching the continuity conditions at the intermediate support (Gorman, 1974). The study on the free vibration of multi-span Euler-Bernoulli beams also has been carried out by various methods such as the finite element method (Hayashikawa and Watanabe, 1985), the Green function method (Kukla, 1991), and the transfer matrix method (Hosking et al., 2004). The free vibration analysis of multi-span beams based on the Timoshenko theory has been investigated using various methods such as Rayleigh-Ritz method (Zhou, 2001)

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and the transfer matrix method (Lin and Chang, 2005). Also the response of multi-span beams subjected to moving loads or masses is studied extensively (Cai et al., 1988; Chatterjee et al., 1994; Wang, 1997).

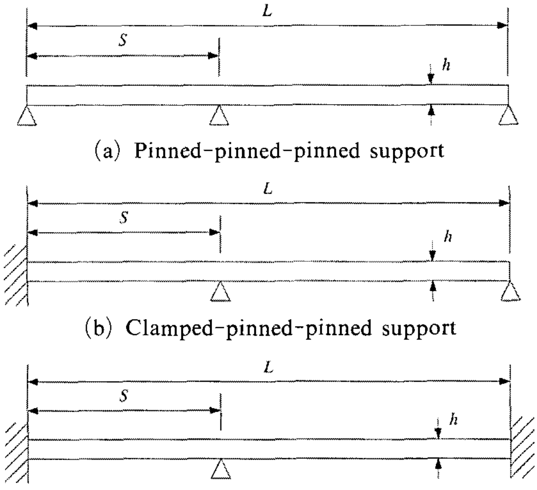
The pseudospectral method can be considered to be a spectral method that performs a collocation process. As the formulation is straightforward and powerful enough to produce approximate solutions close to exact solutions, this method has been highly successful in many areas such as turbulence modeling, weather prediction and non-linear waves (Boyd, 1989). Even though this method can be used for the solution of structural mechanics problems, it has been largely unnoticed by the structural mechanics community, and few articles are available where the pseudospectral method has been applied. Recently it has been successfully applied to the eigenvalue problems of Timoshenko beams and Mindlin plates (Lee, 1998; 2002; 2003a; 2003b; 2003c; 2004; Lee and Schultz, 2004). In the present work, the pseudospectral method is applied to the free vibration analysis of double-span Timoshenko beams.

## 2. Formulations of Double-span Timoshenko Beams

Consider a uniform beam of length $L$, which is either pinned or clamped at the ends and has a roller support at an intermediate location $x=S$ as depicted in Fig. 1. The equations of motion of the beam in the intervals of $0<x<S$ and $S<x<L$ are given by

$$EI\frac{d^2\theta}{dx^2}+\alpha hG\left(\frac{dw}{dx}-\theta\right)=-\omega^2\rho I\theta$$
$$\alpha hG\frac{d}{dx}\left(\frac{dw}{dx}-\theta\right)=-\omega^2\rho hw \tag{1}$$

where $\theta$, $w$ and $\omega$ are the lateral deflection, the rotation of the normal line and the natural frequency, respectively. $E$ and $G$ are Young's modulus and the shear modulus, $\alpha$ is the shear correction factor, $h$ is the thickness of the beam,




(a) Pinned-pinned-pinned support

(b) Clamped-pinned-pinned support

(c) Clamped-pinned-clamped support

**Fig. 1** Beam geometry and support conditions

$I$ is the second moment of area, and $\rho$ is the density.

The boundary conditions are either

$$\textit{clamped}: \theta=0,\ w=0$$
or
$$\textit{pinned}: \frac{d\theta}{dx}=0,\ w=0 \tag{2}$$

at $x=0$ and $x=L$. The continuity conditions at $x=S$ are represented as follows:

$$w(x=S^-)=0$$
$$w(x=S^+)=0$$
$$\theta(x=S^-)=\theta(x=S^+) \tag{3}$$
$$\frac{d\theta}{dx}(x=S^-)=\frac{d\theta}{dx}(x=S^+)$$

It is convenient to introduce normalized variables $z_1$ and $z_2$ such that each of the section between the supports is represented by

$$z_1=\frac{2x-S}{S}\in[-1,\ 1] \text{ for } (0\le x\le S)$$
$$z_2=\frac{2x-S-L}{L-S}\in[-1,\ 1] \text{ for } (S\le x\le L) \tag{4}$$

The governing equations (1) can be rewritten as

$$EI\left(\frac{2}{S}\right)^2\theta''-ahG\theta+ahG\frac{2}{S}\theta'=-\omega^2\rho I\theta$$
$$-ahG\frac{2}{S}\theta'+ahG\left(\frac{2}{S}\right)^2 w''=-\omega^2\rho hw \qquad (5)$$
$$(-1<z_1<1)$$

and

$$EI\left(\frac{2}{L-S}\right)^2\theta^{\dagger\dagger}-ahG\theta+ahG\frac{2}{L-S}\theta^{\dagger}=-\omega^2\rho I\theta$$
$$-ahG\frac{2}{L-S}\theta^{\dagger}+ahG\left(\frac{2}{L-S}\right)^2 w^{\dagger\dagger}=-\omega^2\rho hw \qquad (6)$$
$$(-1<z_2<1)$$

where $'$ and $\dagger$ represent the differentiations with respect to $z_1$ and $z_2$, respectively. The series expansions of the exact solutions $\theta(x)$ and $w(x)$ have infinite numbers of terms. In this study, however, the dependent variables are approximated by the partial sums as follows:

$$\theta(x)\cong\bar{\theta}(z_1)=\sum_{k=1}^{K+2}a_k T_{k-1}(z_1)$$
$$w(x)\cong\bar{w}(z_1)=\sum_{k=1}^{K+2}b_k T_{k-1}(z_1) \qquad (-1<z_1<1) \quad (7)$$

and

$$\theta(x)\cong\bar{\theta}(z_2)=\sum_{n=1}^{N+2}c_n T_{n-1}(z_2)$$
$$w(x)\cong\bar{w}(z_2)=\sum_{n=1}^{N+2}d_n T_{n-1}(z_2) \qquad (-1<z_2<1) \quad (8)$$

where $a_k$, $b_k$, $c_n$ and $d_n$ are the expansion coefficients and $T_{n-1}$ is the Chebyshev polynomial of the first kind of degree of $n-1$. Mikami and Yoshimura suggested an efficient way to handle the boundary conditions by adopting less collocation points than the number of expansion terms (Mikami and Yoshimura, 1984).

Expansions (7) and (8) are substituted into Eqs. (5) and (6) and are collocated at the Gauss-Lobatto collocation points

$$\xi_i=-\cos\frac{\pi(2i-1)}{2K} \ (i=1, 2, \cdots, K) \text{ for } (-1<z_1<1)$$
$$\eta_j=-\cos\frac{\pi(2j-1)}{2N} \ (j=1, 2, \cdots, N) \text{ for } (-1<z_2<1) \qquad (9)$$

to yield

$$\sum_{k=1}^{K+2}\left[a_k\left\{\frac{4EI}{S^2}T''_{k-1}(\xi_i)-ahGT_{k-1}(\xi_i)\right\}+b_k\frac{2ahG}{S}T'_{k-1}(\xi_i)\right]$$
$$=-\omega^2\rho I\sum_{k=1}^{K+2}a_k T_{k-1}(\xi_i)$$
$$\sum_{k=1}^{K+2}\left\{-a_k\frac{2ahG}{S}T'_{k-1}(\xi_i)+b_k\frac{4ahG}{S^2}T''_{k-1}(\xi_i)\right\} \qquad (10)$$
$$=-\omega^2\rho h\sum_{k=1}^{K+2}b_k T_{k-1}(\xi_i)$$
$$(i=1, 2, \cdots, K)$$

and

$$\sum_{n=1}^{N+2}\left[c_n\left\{\frac{4EI}{(L-S)^2}T^{\dagger\dagger}_{n-1}(\eta_j)-ahGT_{n-1}(\eta_j)\right\}+d_n\frac{2ahG}{L-S}T^{\dagger}_{n-1}(\eta_j)\right]$$
$$=-\omega^2\rho I\sum_{n=1}^{N+2}c_n T_{n-1}(\eta_j)$$
$$\sum_{n=1}^{N+2}\left\{-c_n\frac{2ahG}{L-S}T^{\dagger}_{n-1}(\eta_j)+d_n\frac{4ahG}{(L-S)^2}T^{\dagger\dagger}_{n-1}(\eta_j)\right\} \qquad (11)$$
$$=-\omega^2\rho h\sum_{n=1}^{N+2}d_n T_{n-1}(\eta_j)$$
$$(j=1, 2, \cdots, N)$$

Eqs. (10) and (11) can be rearranged in the matrix form

$$[H]\{\delta\}+[H^*]\{\delta^*\}$$
$$=\omega^2([F]\{\delta\}+[F^*]\{\delta^*\}) \qquad (12)$$

where the vectors in Eq. (12) are defined by

$$\{\delta\}=\{a_1\ a_2\cdots a_K\ b_1\ b_2\cdots b_K\ c_1\ c_2\cdots c_N\ d_1\ d_2\cdots d_N\}^T$$
$$\{\delta^*\}=\{a_{K+1}\ a_{K+2}\ b_{K+1}\ b_{K+2}\ c_{N+1}\ c_{N+2}\ d_{N+1}\ d_{N+2}\}^T \qquad (13)$$

The total number of equations in Eq. (12) is $2(K+N)$ whereas the total number of unknowns in Eq. (13) is $2(K+N+4)$. The remaining eight equations are obtained from the continuity conditions and the boundary conditions.

Using the expansions (7) and (8), the continuity conditions (3) can be rewritten as

$$\sum_{k=1}^{N+2}b_k T_{k-1}(1)=0$$
$$\sum_{n=1}^{K+2}d_n T_{n-1}(-1)=0$$
$$\sum_{k=1}^{K+2}a_k T_{k-1}(1)=\sum_{n=1}^{N+2}c_n T_{n-1}(-1) \qquad (14)$$
$$\frac{1}{S}\sum_{k=1}^{K+2}a_k T'_{k-1}(1)=\frac{1}{L-S}\sum_{n=1}^{N+2}c_n T^{\dagger}_{n-1}(-1)$$

The boundary conditions are either

$$\textit{clamped}: \sum_{k=1}^{K+2} a_k T_{k-1}(-1)=0$$
$$\sum_{k=1}^{K+2} b_k T_{k-1}(-1)=0$$

or (15)

$$\textit{pinned}: \sum_{k=1}^{K+2} a_k T'_{k-1}(-1)=0$$
$$\sum_{k=1}^{K+2} b_k T_{k-1}(-1)=0$$

at $x=0$, and either

$$\textit{clamped}: \sum_{n=1}^{N+2} c_n T_{n-1}(1)=0$$
$$\sum_{n=1}^{N+2} d_n T_{n-1}(1)=0$$

or (16)

$$\textit{pinned}: \sum_{n=1}^{N+2} c_n T^{\dagger}_{n-1}(1)=0$$
$$\sum_{k=1}^{N+2} d_k T_{k-1}(1)=0$$

at $x=L$.

The continuity conditions (14) and boundary conditions (15)-(16) can be rearranged in the matrix form

$$[U]\{\delta\}+[V]\{\delta^*\}=\{0\} \tag{17}$$

where $\{0\}$ is a zero vector. Since $\{\delta^*\}$ in Eq. (17) can be expressed as

$$\{\delta^*\}=-[V]^{-1}[U]\{\delta\} \tag{18}$$

the set of equations (12) can be reformulated as

$$([H]-[H^*][V]^{-1}[U])\{\delta\} \\ =\omega^2([F]-[F^*][V]^{-1}[U])\{\delta\} \tag{19}$$

The solution of (19) yields the estimate for the natural frequencies and the corresponding mode shapes.

## 3. Numerical Examples

A preliminary run for the convergence check of the eigenvalues of a double-span Timoshenko beam which has a clamped-pinned-pinned support is carried out for $h/L=0.01$ and $S/L=0.5$, and the results are given in Table 1. The numbers of collocation points which determines the size of the problem change from $K=M=3$ to $K=M=20$. The total number of equations in (10) and (11) is $2(K+N)$, and the size of matrices in equation (19) becomes $80\times80$ for $K=M=20$. Table 1 clearly shows the rapid convergence nature of the pseudospectral method, where it is readily shown that it requires less than $K=M=10$ for the 4 lowest eigenvalues to converge to 6 significant digits, and less than $K=M=15$ for eigenvalues of the 10 lowest modes to 6 significant digits. The numbers given in Tables 1~4 are the non-dimensionalized frequency parameters $\beta$ defined as

$$\beta=\sqrt[4]{\rho A\omega^2/EI} \tag{20}$$

where $A$ is the cross sectional area of the beam.

**Table 1** Convergence test of the non-dimensionalized frequency parameter $\beta$ of the double span Timoshenko beam as the number of the collocation points increase (clamped-pinned-pinned support, $\nu=0.3$, $\alpha=5/6$, $h/L=0.01$, $S/L=0.5$)

| Mode | $K=N=3$ | $K=N=5$ | $K=N=10$ | $K=N=15$ | $K=N=20$ |
|---|---|---|---|---|---|
| 1 | 6.92346 | 6.78556 | 6.78306 | 6.78306 | 6.78306 |
| 2 | 9.20854 | 8.93385 | 8.91641 | 8.91641 | 8.91641 |
| 3 | | 13.8677 | 13.0692 | 13.0692 | 13.0692 |
| 4 | | 16.6498 | 15.1408 | 15.1408 | 15.1408 |
| 5 | | 21.3141 | 19.3074 | 19.3059 | 19.3059 |
| 6 | | 24.3702 | 21.3626 | 21.3590 | 21.3590 |
| 7 | | | 25.5133 | 25.5046 | 25.5046 |
| 8 | | | 27.5497 | 27.5302 | 27.5302 |
| 9 | | | 32.2921 | 31.6532 | 31.6532 |
| 10 | | | 34.9092 | 33.6454 | 33.6454 |

Through out the paper, Poisson's ratio and the shear coefficient of the beam are $\nu=0.3$ and $\alpha=5/6$, respectively.

Computational results for the collocation points $K=M=20$ with pinned-pinned-pinned, clamped-pinned-pinned, and clamped-pinned-clamped supports are given in Tables 2~4, respectively. The natural frequencies are calculated for different thickness-to-length ratios ranging from $h/L=0.005$ to $h/L=0.1$. It is well known that the static and dynamic characteristics of Timoshenko beams approach those of Euler-Bernoulli beams when the thickness of the beams is very small, and the eigenvalues based on the Euler-Bernoulli theory (Gorman, 1974) are given in Tables 2~4 for the purpose of comparison. The results of Tables 2~4 show that the Timoshenko beam results are very close to the Euler-Bernoulli beam results when the thickness-to-length ratio $h/L$ is small, showing that at least three significant digits are identical with the Euler-Bernoulli results in most cases when the thickness-to-length ratio is 0.005. As $h/L$ grows larger, however, the computed eigenvalues show some quantitative differences from those of Euler-Bernoulli beams. The natural frequencies $\omega$ in Tables 2~4 increase as $h/L$ increases, even though the frequency parameters $\beta$ in Tables 2~4 tend to decrease because the second moment of area $I$ grows faster than $\omega^2$ as $h/L$ increases.

It is possible that there might be optimal combinations of $K$ and $M$, the numbers of the Gauss-Labotto collocation points, when the size of one span is different from the other, however, they are assumed to be the same for the sake of simplicity. It is also shown that the computed

**Table 2** Non-dimensionalized frequency parameter $\beta$ of the double span Timoshenko beam (pinned-pinned-pinned support, $\nu=0.3$, $\alpha=5/6$, $K=N=20$)

| $S/L$ | Mode | Classical theory | $h/L$ | | | | |
|---|---|---|---|---|---|---|---|
| | | | 0.005 | 0.01 | 0.02 | 0.05 | 0.1 |
| 0.1 | 1 | 4.22637 | 4.22591 | 4.22455 | 4.21913 | 4.18246 | 4.06718 |
| | 2 | 7.63130 | 7.62983 | 7.62542 | 7.60798 | 7.49282 | 7.15781 |
| | 3 | 11.0505 | 11.0469 | 11.0361 | 10.9934 | 10.7217 | 10.0034 |
| | 4 | 14.4793 | 14.4718 | 14.4497 | 14.3633 | 13.8364 | 12.5864 |
| | 5 | 17.9123 | 17.8990 | 17.8592 | 17.7057 | 16.8169 | 14.9262 |
| 0.2 | 1 | 4.61832 | 4.61794 | 4.61680 | 4.61224 | 4.58106 | 4.47920 |
| | 2 | 8.39155 | 8.39000 | 8.38536 | 8.36697 | 8.24478 | 7.88141 |
| | 3 | 12.1617 | 12.1576 | 12.1452 | 12.0966 | 11.7863 | 10.9647 |
| | 4 | 15.7080 | 15.6997 | 15.6749 | 15.5784 | 14.9926 | 13.6132 |
| | 5 | 17.8725 | 17.8574 | 17.8127 | 17.6395 | 16.6350 | 14.5092 |
| 0.3 | 1 | 5.13179 | 5.13136 | 5.13010 | 5.12506 | 5.09060 | 4.97761 |
| | 2 | 9.27693 | 9.27513 | 9.26976 | 9.24847 | 9.10743 | 8.69103 |
| | 3 | 11.7804 | 11.7760 | 11.7630 | 11.7119 | 11.3851 | 10.5183 |
| | 4 | 14.2845 | 14.2769 | 14.2544 | 14.1666 | 13.6316 | 12.3683 |
| | 5 | 18.4048 | 18.3907 | 18.3488 | 18.1870 | 17.2526 | 15.2730 |
| 0.4 | 1 | 5.78261 | 5.78210 | 5.78058 | 5.77451 | 5.73309 | 5.59796 |
| | 2 | 8.76786 | 8.76607 | 8.76073 | 8.73954 | 8.59896 | 8.18261 |
| | 3 | 11.3129 | 11.3091 | 11.2976 | 11.2522 | 10.9627 | 10.1941 |
| | 4 | 15.7080 | 15.6997 | 15.6749 | 15.5784 | 14.9926 | 13.6132 |
| | 5 | 17.3296 | 17.3158 | 17.2749 | 17.1165 | 16.1984 | 14.2547 |
| 0.5 | 1 | 6.28319 | 6.28265 | 6.28106 | 6.27471 | 6.23136 | 6.09066 |
| | 2 | 7.85321 | 7.85163 | 7.84690 | 7.82817 | 7.70352 | 7.33122 |
| | 3 | 12.5664 | 12.5621 | 12.5494 | 12.4994 | 12.1813 | 11.3431 |
| | 4 | 14.1372 | 14.1294 | 14.1062 | 14.0154 | 13.4611 | 12.1454 |
| | 5 | 18.8496 | 18.8352 | 18.7926 | 18.6282 | 17.6810 | 15.6790 |

**Table 3** Non-dimensionalized frequency parameter $\beta$ of the double span Timoshenko beam (clamped-pinned-pinned support, $\nu=0.3$, $\alpha=5/6$, $K=N=20$)

| $S/L$ | Mode | Classical theory | $h/L$ | | | | |
|---|---|---|---|---|---|---|---|
| | | | 0.005 | 0.01 | 0.02 | 0.05 | 0.1 |
| 0.1 | 1 | 4.25636 | 4.25557 | 4.25324 | 4.24418 | 4.19078 | 4.07085 |
| | 2 | 7.67648 | 7.67446 | 7.66845 | 7.64507 | 7.50420 | 7.16181 |
| | 3 | 11.1062 | 11.1019 | 11.0889 | 11.0383 | 10.7343 | 10.0068 |
| | 4 | 14.5436 | 14.5352 | 14.5103 | 14.4141 | 13.8498 | 12.5890 |
| | 5 | 17.9862 | 17.9716 | 17.9285 | 17.7629 | 16.8312 | 14.9281 |
| 0.2 | 1 | 4.67394 | 4.67338 | 4.67170 | 4.66504 | 4.62108 | 4.49240 |
| | 2 | 8.46945 | 8.46759 | 8.46204 | 8.44010 | 8.29731 | 7.89667 |
| | 3 | 12.2832 | 12.2785 | 12.2645 | 12.2095 | 11.8646 | 10.9876 |
| | 4 | 16.0717 | 16.0620 | 16.0331 | 15.9209 | 15.2512 | 13.7356 |
| | 5 | 19.6346 | 19.6168 | 19.5639 | 19.3604 | 18.1949 | 15.6416 |
| 0.3 | 1 | 5.21414 | 5.21357 | 5.21189 | 5.20518 | 5.15993 | 5.01886 |
| | 2 | 9.47849 | 9.47628 | 9.46967 | 9.44352 | 9.27248 | 8.78674 |
| | 3 | 13.3430 | 13.3370 | 13.3190 | 13.2484 | 12.8034 | 11.6438 |
| | 4 | 15.0778 | 15.0667 | 15.0338 | 14.9062 | 14.1554 | 12.5533 |
| | 5 | 18.5948 | 18.5792 | 18.5329 | 18.3548 | 17.3471 | 15.2952 |
| 0.4 | 1 | 5.92267 | 5.92200 | 5.92000 | 5.91202 | 5.85800 | 5.68760 |
| | 2 | 10.1680 | 10.1651 | 10.1564 | 10.1223 | 9.89863 | 9.25957 |
| | 3 | 11.7988 | 11.7936 | 11.7778 | 11.7159 | 11.3304 | 10.3772 |
| | 4 | 16.1768 | 16.1668 | 16.1372 | 16.0220 | 15.3361 | 13.7909 |
| | 5 | 18.7193 | 18.7003 | 18.6437 | 18.4263 | 17.2019 | 14.7687 |
| 0.5 | 1 | 6.78646 | 6.78561 | 6.78306 | 6.77291 | 6.70440 | 6.48964 |
| | 2 | 8.92665 | 8.92408 | 8.91641 | 8.88607 | 8.68790 | 8.12874 |
| | 3 | 13.0908 | 13.0854 | 13.0692 | 13.0056 | 12.6079 | 11.6079 |
| | 4 | 15.1832 | 15.1726 | 15.1408 | 15.0173 | 14.2814 | 12.6323 |
| | 5 | 19.3731 | 19.3562 | 19.3059 | 19.1128 | 18.0245 | 15.8265 |
| 0.6 | 1 | 6.92042 | 6.91939 | 6.91630 | 6.90403 | 6.82124 | 6.56241 |
| | 2 | 9.05288 | 9.05058 | 9.04371 | 9.01651 | 8.83870 | 8.33524 |
| | 3 | 12.4682 | 12.4624 | 12.4450 | 12.3769 | 11.9505 | 10.8798 |
| | 4 | 16.2966 | 16.2863 | 16.2556 | 16.1364 | 15.4279 | 13.8400 |
| | 5 | 18.1094 | 18.0924 | 18.0418 | 17.8475 | 16.7477 | 14.5324 |
| 0.7 | 1 | 6.20547 | 6.20461 | 6.20201 | 6.19167 | 6.12179 | 5.90227 |
| | 2 | 10.1984 | 10.1957 | 10.1874 | 10.1545 | 9.94041 | 9.33654 |
| | 3 | 12.1248 | 12.1194 | 12.1032 | 12.0400 | 11.6436 | 10.6444 |
| | 4 | 15.2865 | 15.2764 | 15.2462 | 15.1287 | 14.4279 | 12.8534 |
| | 5 | 19.4211 | 19.4028 | 19.3487 | 19.1409 | 17.9699 | 15.6205 |
| 0.8 | 1 | 5.57754 | 5.57682 | 5.57465 | 5.56604 | 5.50770 | 5.32326 |
| | 2 | 9.33697 | 9.33447 | 9.32698 | 9.29737 | 9.10373 | 8.55533 |
| | 3 | 13.0892 | 13.0832 | 13.0656 | 12.9963 | 12.5637 | 11.4789 |
| | 4 | 16.4345 | 16.4239 | 16.3922 | 16.2690 | 15.5378 | 13.9044 |
| | 5 | 18.3252 | 18.3075 | 18.2552 | 18.0538 | 16.9146 | 14.6229 |
| 0.9 | 1 | 5.09491 | 5.09416 | 5.09193 | 5.08306 | 5.02342 | 4.84002 |
| | 2 | 8.48413 | 8.48191 | 8.47525 | 8.44894 | 8.27728 | 7.79470 |
| | 3 | 11.9070 | 11.9020 | 11.8868 | 11.8274 | 11.4550 | 10.5143 |
| | 4 | 15.3373 | 15.3275 | 15.2983 | 15.1848 | 14.5071 | 12.9787 |
| | 5 | 18.7706 | 18.7537 | 18.7035 | 18.5105 | 17.4200 | 15.2193 |

**Table 4** Non-dimensionalized frequency parameter $\beta$ of the double span Timoshenko beam (clamped-pinned-clamped support, $\nu=0.3$, $\alpha=5/6$, $K=N=20$)

| $S/L$ | Mode | Classical theory | $h/L$ | | | | |
|---|---|---|---|---|---|---|---|
| | | | 0.005 | 0.01 | 0.02 | 0.05 | 0.1 |
| 0.1 | 1 | 5.12956 | 5.12842 | 5.12504 | 5.11187 | 5.03279 | 4.84391 |
| | 2 | 8.53225 | 8.52942 | 8.52100 | 8.48819 | 8.28899 | 7.79851 |
| | 3 | 11.9650 | 11.9591 | 11.9416 | 11.8737 | 11.4677 | 10.5175 |
| | 4 | 15.4038 | 15.3930 | 15.3609 | 15.2369 | 14.5204 | 12.9812 |
| | 5 | 18.8474 | 18.8292 | 18.7753 | 18.5695 | 17.4344 | 15.2211 |
| 0.2 | 1 | 5.63992 | 5.63898 | 5.63617 | 5.62505 | 5.55167 | 5.33712 |
| | 2 | 9.42152 | 9.41866 | 9.41010 | 9.37634 | 9.15918 | 8.57073 |
| | 3 | 13.2354 | 13.2288 | 13.2090 | 13.1315 | 12.6552 | 11.5048 |
| | 4 | 17.0022 | 16.9894 | 16.9512 | 16.8036 | 15.9445 | 14.1065 |
| | 5 | 20.3723 | 20.3497 | 20.2827 | 20.0261 | 18.5905 | 15.7041 |
| 0.3 | 1 | 6.30151 | 6.30046 | 6.29731 | 6.28479 | 6.20116 | 5.94770 |
| | 2 | 10.5280 | 10.5245 | 10.5139 | 10.4722 | 10.2044 | 9.48322 |
| | 3 | 13.9338 | 13.9255 | 13.9010 | 13.8051 | 13.2161 | 11.8024 |
| | 4 | 15.7260 | 15.7134 | 15.6762 | 15.5321 | 14.6974 | 12.9468 |
| | 5 | 19.6366 | 19.6165 | 19.5569 | 19.3291 | 18.0715 | 15.6424 |
| 0.4 | 1 | 7.14942 | 7.14808 | 7.14405 | 7.12808 | 7.02152 | 6.70052 |
| | 2 | 10.6107 | 10.6067 | 10.5949 | 10.5484 | 10.2503 | 9.45384 |
| | 3 | 12.7332 | 12.7264 | 12.7062 | 12.6269 | 12.1398 | 10.9681 |
| | 4 | 17.2339 | 17.2204 | 17.1803 | 17.0251 | 16.1249 | 14.2145 |
| | 5 | 19.0450 | 19.0238 | 18.9608 | 18.7199 | 17.3909 | 14.8441 |
| 0.5 | 1 | 7.85321 | 7.85163 | 7.84690 | 7.82817 | 7.70352 | 7.33122 |
| | 2 | 9.46008 | 9.45680 | 9.44699 | 9.40829 | 9.15909 | 8.48403 |
| | 3 | 14.1372 | 14.1294 | 14.1062 | 14.0154 | 13.4611 | 12.1454 |
| | 4 | 15.7064 | 15.6938 | 15.6563 | 15.5112 | 14.6624 | 12.8359 |
| | 5 | 20.4204 | 20.3985 | 20.3338 | 20.0868 | 18.7318 | 16.1487 |

natural frequencies tend to approach those of the single-span beam results as the span ratio $S/L$ approaches either the unity or zero.

## 4. Conclusions

The pseudospectral method is applied to the free vibration analysis of double-span Timoshenko beams. Although the Rayleigh-Ritz method and the differential quadrature method have been successful in the vibration analysis of Timoshenko beams, there are some drawbacks inherent in these methods. For example, they require a process of constructing either weighting coefficients or characteristic polynomials since there are no readily available formulas. The pseudospectral method, on the other hand, uses simple series expansions such as the Chebyshev polynomials as basis functions. The formulation as well as coding for computation is straightforward because the pseudospectral method undergoes the simple collocation process instead of integration.

Basis functions are assumed for each section of the double-span beam. The continuity conditions at the intermediate support and the boundary conditions are considered as the side constraints, and the set of algebraic equations is condensed so that the number of degrees of freedom of the title problem matches the number of the pseudospectral expansion coefficients.

Numerical examples are provided for various thickness-to-length ratios and span ratios. The results from this method agree with those of Euler-Bernoulli beams when the thickness-to-length ratio is very small, however, deviate considerably as the thickness-to-length ratio grows larger.

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