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BICYCLIC *BSEC* OF BLOCK SIZE 3

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ABSTRACT. A k-sized balanced sampling plan excluding contiguous units of order v and index λ, denoted by $BSEC(v, k, \lambda)$, is said to be *bicyclic* if it admits an automorphism consisting of two disjoint cycles of length $\frac{v}{2}$. In this paper, we obtain a necessary and sufficient condition for the existence of bicyclic $BSEC(v, 3, 2)$s.

1. Introduction

A pair $\{x_i, x_j\}$ of a cyclically ordered set $X = \{x_0, x_1, \ldots, x_{v-1}\}$ is said to be *contiguous* if $j = i + 1$ for $0 \leq i \leq v - 2$ or $\{i, j\} = \{0, v - 1\}$. Otherwise, it is *non-contiguous*. A *k-sized balanced sampling plan excluding contiguous points of order v and index λ*, denoted by $BSEC(v, k, \lambda)$, is a pair $(X, \mathfrak{B})$ where X is a v-set of points (units) in cyclic ordering and $\mathfrak{B}$ is a collection of k-subsets of X, called *blocks*, such that any contiguous pair of X does not appear in any block while any non-contiguous pair of distinct points in X appears in exactly λ blocks. Balanced sampling plans excluding contiguous units can be used for survey sampling when the units are arranged in a one-dimensional ordering and the contiguous units in this ordering provide similar information, such as estimates of population characteristics. When $k = 3$, the existence of a $BSEC(v, k, \lambda)$ is settled by Colbourn and Ling[1].

THEOREM 1.1. [1] *There exists a $BSEC(v, 3, \lambda)$ if and only if $v \in \{0, 3\}$ or $\lambda(v - 3) \equiv 0 (mod\ 6)$, $v \geq 9$.*

A $BSEC(v, k, \lambda)$ is said to be *cyclic* if it admits an automorphism consisting of a single cycle of length v. In 2002, Wei[3] establishes the existence of cyclic $BSEC(v, 3, \lambda)$ with $\lambda = 1, 2$.

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THEOREM 1.2. [3] *There exists a cyclic $BSEC(v,3,1)$ if and only if $v \equiv 3$(mod 6); there exists a cyclic $BSEC(v,3,2)$ if and only if $v \equiv 0$,3 or 9 (mod 12).*

There does not exist a cyclic $BSEC(v,3,2)$ for $v \equiv 6$ (mod 12). However, Colbourn and Ling[1] defined bicyclic $BSEC(v,3,2)$ and gave some small examples. Whether there exists a bicyclic $BSEC(v,3,2)$ for all $v \equiv 6$ (mod 12) still remains as an interesting problem. A $BSEC(v,k,\lambda)$ is said to be *bicyclic* if it admits an automorphism consisting of two disjoint cycles of length $\frac{v}{2}$. In this paper, we obtain a necessary and sufficient condition for the existence of bicyclic $BSEC(v,3,2)$s.

2. Existence of bicyclic $BSEC(v,3,2)$s

A necessary condition for the existence of bicyclic $BSEC(v,3,2)$s is easily obtained.

LEMMA 2.1. *If there exists a bicyclic $BSEC(v,3,2)$, then $v \equiv 0$ or 6 (mod 12) and $v \neq 6$.*

If $v \equiv 0$ (mod 12), the existence of a cyclic $BSEC(v,3,2)$ gives rise to the existence of a bicyclic $BSEC(v,3,2)$.

LEMMA 2.2. *If $v \equiv 0$ (mod 12), then there exists a bicyclic $BSEC(v,3,2)$.*

Let $\mathbb{Z}_v = \{0,1,\ldots,v-1\}$ denote the cyclic additive group of order v. If $(x,i) \in \mathbb{Z}_v \times \{1,2\}$ is an element, we write briefly x_i for it. We consider $\mathbb{Z}_v \times \{1,2\}$ as a cyclically ordered set

$$(0_1,0_2,1_1,1_2,\ldots,(v-1)_1,(v-1)_2).$$

In this cyclic ordering, note that a pair $\{a,b\}$ of distinct points is contiguous if and only if $(a,b) = (i_1,i_2)$ or $(i_2,(i+1)_1)$ where $i \in \mathbb{Z}_v$.

It remains to construct a bicyclic $BSEC(v,3,2)$ for $v \equiv 6$ (mod 12) and $v \neq 6$. We will construct our bicyclic $BSEC(v,3,2)$ with point set $V = \mathbb{Z}_{\frac{v}{2}} \times \{1,2\}$ and the corresponding bicyclic automorphism is

$$\alpha = \left(0_1,1_1,\ldots,\left(\frac{v}{2}-1\right)_1\right)\left(0_2,1_2,\ldots,\left(\frac{v}{2}-1\right)_2\right).$$

Let $< \alpha >$ be the group generated by α. If $v \equiv 6$ (mod 12) and if there exists a collection of 3-subsets

$$B_1,B_2,\ldots,B_{\frac{2(v-3)}{3}}$$

of $V = \mathbb{Z}_{\frac{v}{2}} \times \{1,2\}$, which produce under the bicyclic automorphism α each of the pairs

$$\begin{aligned} &\{0_1, i_1\}, \quad i = 1, 2, \ldots, \frac{v}{2} - 1, \\ &\{0_1, i_2\}, \quad i = 1, 2, \ldots, \frac{v}{2} - 2, \\ &\{0_2, i_1\}, \quad i = 2, 3, \ldots, \frac{v}{2} - 1, \\ &\{0_2, i_2\}, \quad i = 1, 2, \ldots, \frac{v}{2} - 1 \end{aligned}$$

exactly twice, then the orbits $\mathcal{O}(B_i) = \{\beta(B_i) | \beta \in <\alpha>\}$, $i = 1, 2, \ldots, \frac{2(v-3)}{3}$, form the blocks for a bicyclic $BSEC(v,3,2)$. Such a collection of 3-subsets is called a collection of *base blocks* for the bicyclic $BSEC(v,3,2)$. If $v = 12t + 6$, we will construct $8t + 2$ base blocks consisting of

(i) $2t$ of the form $\{0_1, a_1, b_1\}$,
(ii) 2 of the form $\{c_1, d_1, 0_2\}$,
(iii) $6t - 1$ of the form $\{0_2, r_2, s_2\}$,
(iv) one of the form $\{0_2, x_2, y_2\}$,

which give rise to a bicyclic $BSEC(12t+6,3,2)$. $2t$ blocks with type (i) will be taken each twice of t base blocks for a cyclic $STS(6t+3)$ (a cyclic Steiner triple system $STS(v)$ exists for all $v \equiv 1$ or $3 \pmod 6$, $v \neq 9$, [2]) based on $\mathbb{Z}_{6t+3} \times \{1\}$, except the base block $\{0_1, (2t+1)_1, (4t+2)_1\}$, 2 blocks with type (ii) are $\{(2t+1)_1, (4t+2)_1, 0_2\}$ and $\{(6t+2)_1, (2t)_1, 0_2\}$, and one block with (iv) will be taken $\{0_2, 1_2, (3t+2)_2\}$. We need some observations for blocks with types (iii). If S is a set, $2S$ denotes the multiset with each object repeated twice.

Definition 2.3. A $(6t+2)$-*system* is a set of ordered pairs

$$\{(a_r, b_r) | r = 1, 2, \ldots, 3t\} \cup \{(c_r, d_r) | r = 2, 3, \ldots, 3t\} \cup \{(a,b), (c,d)\}$$

such that

$$\begin{aligned} &\{a_r, b_r | r = 1, 2, \ldots, 3t\} \cup \{c_r, d_r | r = 2, 3, \ldots, 3t\} \cup \{a, b, c, d\} \\ &= 2\{2, 3, \ldots, 6t+2\} \end{aligned}$$

and

$$\begin{aligned} b_r - a_r &= r, \ r = 1, 2, \ldots, 3t, \\ d_r - c_r &= r, \ r = 2, 3, \ldots, 3t, \end{aligned}$$

$$(a,b) = (2t+1, 4t+2),$$
$$(c,d) = (2t, 6t+2).$$

LEMMA 2.4. *If there exists a $(6t+2)$-system and $t \geq 2$, then there exists a bicyclic $BSEC(12t+6,3,2)$.*

PROOF. Let

$$\{(a_r, b_r) | r = 1, 2, \ldots, 3t\} \cup \{(c_r, d_r) | r = 2, 3, \ldots, 3t\} \cup \{(a,b), (c,d)\}$$

be a $(6t+2)$-system. Then the following triples:

base blocks for a cyclic $STS(6t+3)$ based on $\mathbb{Z}_{6t+3} \times \{1\}$ each twice, except its base block $\{0_1, (2t+1)_1, (4t+2)_1\}$, and

$\{(2t+1)_1, (4t+2)_1, 0_2\}, \{(6t+2)_1, (2t)_1, 0_2\}$,

$\{0_2, r_2, (b_r)_1\}, \quad r = 1, 2, \ldots, 3t$,

$\{0_2, r_2, (d_r)_1\}, \quad r = 2, 3, \ldots, 3t$,

$\{0_2, 1_2, (3t+2)_2\}$

form base blocks for a bicyclic $BSEC(12t+6,3,2)$. □

LEMMA 2.5. [1] *There exists a bicyclic $BSEC(v,3,2)$ for $v = 18$, $30, 42$.*

It remains to construct $(6t+2)$-system for all $t \geq 2$.

LEMMA 2.6. *If $t \equiv 2 \pmod 4$ and $t \geq 2$, then there exists a $(6t+2)$-system.*

PROOF. If $t = 2$, then

$$(4,14),\ (5,10),\ (3,6),\ (2,7),\ (11,12),\ (3,6),\ (2,7),$$
$$(10,12),\ (9,13),\ (8,14),\ (9,11),\ (4,8),\ (5,11)$$

form a 14-system.

If $t \equiv 2 \pmod 4$ and $t \geq 6$, then the following ordered pairs form a $(6t+2)$-system:

$$(a,b) = (2t, 6t+2),\ (c,d) = (2t+1, 4t+1),$$
$$(1+r, 3t+3-r),\ r = 1, 2, \ldots, t+1,$$
$$(3t+3+r, 6t+3-r),\ r = 1, 2, \ldots, \frac{3t-2}{2},$$
$$(t+2+r, 2t-r),\ r = 1, 2, \ldots, \frac{t-4}{2},$$
$$(1+r, 3t+2-r),\ r = 1, 2, \ldots, \frac{3t}{2},$$

$$(3t+1+r, 6t+2-r),\ r=1,2,\ldots,\frac{3t-2}{4},$$
$$\left(\frac{3t+2}{2},\frac{9t+2}{2}\right),\ \left(3t+3,\frac{9t+6}{2}\right),$$

and we divide into two cases:

Case 1. $t \equiv 6 \pmod 8$. If $t=6$, then

$$(30,33),\ (24,29),\ (25,32),\ (27,31).$$

If $t \geq 14$, then

$$(4t+2-r, 5t+1-r),\ r=1,2,\ldots,\frac{t-2}{4},$$
$$(4t+3+2r, 5t+2-2r),\ r=1,2,\ldots,\frac{t-6}{4},$$
$$(4t+2+2r, 5t-1-2r),\ r=1,2,\ldots,\frac{t-6}{8},$$
$$\left(\frac{9t+2}{2}-2r, \frac{9t}{2}+2r\right),\ r=1,2,\ldots,\frac{t-14}{8},\ (t>14),$$
$$(4t+3, 5t+1),\ \left(\frac{17t+10}{4},\frac{21t+6}{4}\right),$$
$$\left(\frac{15t+6}{4},\frac{19t-6}{4}\right),\ \left(\frac{9t+6}{2}, 5t-1\right).$$

Case 2. $t \equiv 2 \pmod 8$.

$$\left(\frac{15t+6}{4}+r, \frac{21t+6}{4}-r\right),\ r=1,2,\ldots,\frac{t-2}{4},$$
$$(4t+2+r, 5t+1-r),\ r=1,2,\ldots,\frac{t-2}{4},$$
$$\left(\frac{17t+10}{4}+2r, \frac{19t+10}{4}-2r\right),\ r=1,2,\ldots,\frac{t-10}{8}, (t>10),$$
$$\left(\frac{17t+6}{4}+2r, \frac{19t-2}{4}-2r\right),\ r=1,2,\ldots,\frac{t-10}{8}, (t>10),$$
$$\left(\frac{17t+10}{4},\frac{21t+6}{4}\right),\ \left(\frac{9t+6}{2}, 5t+1\right),\ \left(\frac{15t+6}{4},\frac{19t-2}{4}\right). \quad \square$$

LEMMA 2.7. *If $t \equiv 0 \pmod 4$ and $t \geq 4$, then there exists a $(6t+2)$-system*

PROOF. If $t \equiv 0 \pmod 4$ and $t \geq 4$, then the following ordered pairs form a $(6t+2)$-system:

$$(a,b) = (2t, 6t+2),\ (c,d) = (2t+1, 4t+2),$$
$$(1+r, 3t+3-r),\ r = 1,2,\ldots,t+1,$$
$$(t+2+r, 2t-r),\ r = 1,2,\ldots,\frac{t-4}{2},\ (t>4),$$
$$(3t+2+r, 6t+3-r),\ r = 1,2,\ldots,\frac{3t}{2},$$
$$(1+r, 3t+3-r),\ r = 1,2,\ldots,\frac{3t}{2},$$
$$(3t+2+r, 6t+1-r),\ r = 1,2,\ldots,\frac{t-4}{2},\ (t>4),$$
$$(4t+2-r, 5t+3+r),\ r = 1,2,\ldots,\frac{t-2}{2},$$
$$\left(\frac{3t+2}{2}, \frac{7t+4}{2}\right),\ \left(\frac{3t+4}{2}, \frac{9t+2}{2}\right),$$
$$(4t+3, 5t+1),\ (t>4),$$

and we divide into two cases:

Case 1. $t \equiv 0 \pmod 8$.

$$(4t+5+r, 5t+4-r),\ r = 1,2,\ 5,6,\ 9,10,\ldots,\frac{t-4}{2},$$
$$(4t+3+r, 5t-2-r),\ r = 1,2,\ 5,6,\ 9,10,\ldots,\frac{t-12}{2},\ (t>8),$$

Case 2. $t \equiv 4 \pmod 8$.

$$\left(\frac{7t+2}{2}, \frac{9t+4}{2}\right),\ (5t+2, 6t+1),$$

and we distinguish into three subcases:

Subcase 1. $t \equiv 4 \pmod{24}$. If $t = 4$, an ordered pair $(21, 23)$ is added. If $t > 4$,

$$(4t+3+3r, 5t+6-3r),\ r = 1,2,\ldots,\frac{t-4}{8},$$
$$(4t+3+r, 5t-r),\ r = 1,2,\ 4,5,\ 7,8,\ldots,\frac{t-18}{2},$$

$$\left(\frac{9t+6}{2}, \frac{9t+12}{2}\right), \left(\frac{9t-2}{2}, \frac{9t+8}{2}\right), \left(\frac{9t}{2}, \frac{9t+14}{2}\right),$$
$$\left(\frac{9t-8}{2}, \frac{9t+10}{2}\right), \left(\frac{9t-6}{2}, \frac{9t+16}{2}\right).$$

Subcase 2. $t \equiv 12 \pmod{24}$.

$$(4t+3+3r, 5t+6-3r),\ r = 1, 2, \ldots, \frac{t}{6},$$
$$(4t+3+r, 5t-r),\ r = 1, 2,\ 4, 5,\ 7, 8, \ldots, \frac{t-8}{2}.$$

Subcase 3. $t \equiv 20 \pmod{24}$.

$$(4t+3+3r, 5t+6-3r),\ r = 1, 2, \ldots, \frac{t-4}{8},$$
$$(4t+3+r, 5t-r),\ r = 1, 2,\ 4, 5,\ 7, 8, \ldots, \frac{t-16}{2},$$
$$\left(\frac{9t+6}{2}, \frac{9t+12}{2}\right), \left(\frac{9t}{2}, \frac{9t+10}{2}\right), \left(\frac{9t-6}{2}, \frac{9t+8}{2}\right),$$
$$\left(\frac{9t-4}{2}, \frac{9t+14}{2}\right), \left(\frac{9t-2}{2}, \frac{9t+20}{2}\right). \qquad \square$$

LEMMA 2.8. *If* $t \equiv 1 \pmod 2$ *and* $t \geq 3$, *then there exists a* $(6t+2)$-*system*

PROOF. If $t = 3$, then the following ordered pairs form a 20-system:

$$(a, b) = (6, 20),\ (c, d) = (7, 14),$$
$$(10+r, 21-r),\ r = 1, 2, 3, 4, 5,$$
$$(1+r, 11-r),\ r = 1, 2, 3,$$
$$(2,4),\ (3,5),\ (6,9),\ (13,17),\ (5,10),$$
$$(12,18),\ (8,15),\ (11,19),\ (7,16).$$

If $t \equiv 1 \pmod 2$ and $t \geq 5$, then the following ordered pairs form a $(6t+2)$-system:

$$(a, b) = (2t, 6t+2),\ (c, d) = (2t+1, 4t+2),$$
$$(3t+1+r, 6t+3-r),\ r = 1, 2, \ldots, \frac{3t+1}{2}$$
$$(1+r, 3t+2-r),\ r = 1, 2, \ldots, t,$$
$$(t+r, 2t-r),\ r = 1, 2, \ldots, \frac{t-3}{2},$$

$$(1+r, 3t+2-r),\ r = 1, 2, \ldots, \frac{3t-1}{2},$$
$$(3t+1+r, 6t+1-r),\ r = 1, 2, \ldots, \frac{t-3}{2},$$
$$\left(\frac{7t+3}{2}+r, \frac{11t+5}{2}-r\right),\ r = 1, 2, \ldots, \frac{t-3}{2},$$
$$(4t+3+r, 5t+1-r),\ r = 1, 2, \ldots, \frac{t-5}{2},\ (t > 5),$$
$$\left(\frac{3t+1}{2}, \frac{7t+3}{2}\right), \left(\frac{7t+1}{2}, \frac{9t+5}{2}\right), \left(\frac{3t+3}{2}, \frac{9t+3}{2}\right),$$
$$(4t+1, 5t+1),\ (4t+3, 5t+2),\ (5t+3, 6t+1).$$

THEOREM 2.9. *If $t \geq 2$ is an integer, then there exists a $(6t+2)$-system.*

Now, we conclude the following theorem.

THEOREM 2.10. *There exists a bicyclic $BSEC(v, 3, 2)$ if and only if $v \equiv 0$ or 6 (mod 12), $v \neq 6$.*

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