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# ON THE PRODUCT OF $t$ AND BESSEL RANDOM VARIABLES

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ABSTRACT. The distribution of products of random variables is of interest in many areas of the sciences, engineering and medicine. This has increased the need to have available the widest possible range of statistical results on products of random variables. In this note, the distribution of the product $|XY|$ is derived when $X$ and $Y$ are Student's $t$ and Bessel function random variables distributed independently of each other.

## 1. Introduction

For given random variables $X$ and $Y$, the distribution of the product $XY$ arises explicitly in many areas of the sciences, engineering and medicine. We discuss some examples from econometrics, social sciences, biology and hydrology.

In traditional portfolio selection models certain cases involve the product of random variables. The best examples of this are in the case of investment in a number of different overseas markets. In portfolio diversification models(see, for example, Grubel[10]) not only are prices of shares in local markets uncertain but also the exchange rates are uncertain so that the value of the portfolio in domestic currency is related to a product of random variables. Similarly in models of diversified production by multinationals(see, for example, Rugman[24]) there is local production uncertainty and exchange rate uncertainty so that profits in home currency are again related to a product of random variables. An entirely different example is drawn from the econometric literature. In making a forecast from an estimated equation Feldstein[7] pointed out that both the parameter and the value of the exogenous variable in the

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forecast period could be considered as random variables. Hence the forecast was proportional to a product of random variables.

Products of random variables constitute an important class of variables in the social sciences. They are most important because of the regularity with which they are used to capture interaction effects in regression contexts. But they are significant also in several theoretical statements in the social sciences. Perhaps the best-known statement of this type is the Cobb-Douglas production function, which is used regularly in economics to relate outputs to inputs. In psychology the best known is the Atkinson[2] model for the prediction of behavior. In this model, a behavioral response is multiplicatively related to drive strength, habit strength, the incentive value of the anticipated reinforcement, and the intensity of the stimulus. In theorizing about the relationship between attitudes and behavior, social psychologists Rokeach and Kliejunas[23] argue that behavior can best be predicted from the weighted product of one's attitude toward an object and one's attitude toward situation. Sociologists Palmore and Hammond[19] argue that deviation from norms is equal to the product of barriers to legitimate opportunities and the degree of exposure to illegitimate ones. Researchers studying job satisfaction have used product of variables in two ways in their research. First, studies by Schaffer[26], Decker[5], Ewen[6] and Waters[33] have all examined the importance of building overall satisfaction measures as the sum of separate aspects of job satisfaction multiplied by the importance of this aspect of the job for the individual. That is, the measure is a sum of products of variables. Second, job satisfaction is seen as a function of both the properties the worker perceives in a job and the value he or she attaches to each of these properties(Goldthorpe et al.[8], Kalleberg[12, 13]) tests several versions of this general theoretical statement; some of the models explored view job satisfaction as the sum of weighted products of variables. The variables were formed by multiplying one's perception of whether or not a job has a given property by the value one attaches to that property. In tumor biology, the total number of cancer cell nuclei and of mitoses in the primary lesion are potentially important indicators. Ladekarl *et al*[16] obtained such estimates on breast cancers by an unbiased stereologic method. The total number estimates were defined as the product of two variables: the volume of tumor estimated by the Cavalieri principle, and the densities of cancer cell nuclei and of mitoses obtained in small, three-dimensional samples(i.e., optical disectors) of 40-mu m-thick methacrylate sections, which were selected systematically at random from the whole specimen. Products of random variables also arise in hydrology stream flow is often defined as a product of two or more variables, representing, for example, the periodic and the stochastic components, respectively(Cigizoglu and Bayazit[4]). The distribution of $|XY|$ has been studied by several authors especially when $X$ and $Y$ are independent random variables and come from the same family. For instance, see Sakamoto[25] for uniform family, Harter[11] and Wallgren[32] for Student's $t$ family, Springer and Thompson[27] for normal family, Stuart[29] and Podolski[20] for gamma family, Steece[28], Bhargava and Khatri[3] and Tang and Gupta[30] for beta family, Abu-Salih[1] for power function family, and Malik and Trudel[17] for exponential family(see also Rathie and Rohrer[22] for a comprehensive review of known results).

However, there is relatively little work of this kind when $X$ and $Y$ belong to different families. In the applications mentioned above, it is quite possible that $X$ and $Y$ could arise from different but similar distributions. In this note, we study the distribution of $|XY|$ when $X$ and $Y$ are independent random variables having the Student's $t$ and Bessel function distributions with pdfs

$$f(x) = \frac{1}{\sqrt{\nu} B(\nu/2, 1/2)} \left(1 + \frac{x^2}{\nu}\right)^{-(1+\nu)/2} \tag{1}$$

$$f(y) = \frac{| y |^m}{\sqrt{\pi} 2^m b^{m+1} \Gamma(m + 1/2)} K_m\left(\left|\frac{y}{b}\right|\right) \tag{2}$$

respectively, for $-\infty < x < \infty$, $-\infty < y < \infty$, $\nu > 0$, $b > 0$ and $m > 1$, where

$$K_m(x) = \frac{\sqrt{\pi} x^m}{2^m \Gamma(m + 1/2)} \int_1^\infty \left(t^2 - 1\right)^{m-1/2} \exp(-xt)\, dt$$

is the modified Bessel function of the second kind. The Student's $t$ distributions have received much recent popularity:

- $t$ distributions are of central importance in statistical inference.
- Applications of $t$ distributions are a very promising path to take. Classical analysis is soundly bend on the normal distribution while $t$ distributions offer a more viable alternative with respect to real-world data particularly because its tails are more realistic. Already we have seen unexpected applications in novel areas such as cluster analysis, discriminant analysis, multiple regression, robust projection indices and missing data imputation.

- $t$ distributions for the past twenty to thirty years have played a crucial role in Bayesian analysis. They serve as the most popular prior distribution(because elicitation of prior information in various physical, engineering and financial phenomena is closely associated with $t$ distributions) and generate meaningful posterior distributions.

It is fair to say that $t$ distributions have been perhaps unjustly overshadowed–for at least seventy years–by the normal distribution. For further discussion of applications, the reader is referred to Kotz and Nadarajah[15].

Bessel function distributions have found applications in a variety of areas that range from image and speech recognition and ocean engineering to finance. They are rapidly becoming distributions of first choice whenever "something" with heavier than Gaussian tails is observed in the data. Some examples are:

- in communication theory, $Y$ could represent the random noise corresponding to signals.
- in ocean engineering, $Y$ could represent distributions of navigation errors.
- in finance, $Y$ could represent distributions of log-returns of different commodities.
- in image and speech recognition, $Y$ could represent "input" distributions.

For further discussion of applications, the reader is referred to Kotz et al.[14]. Both $t$ and Bessel function distributions arise together in many situations, especially with regard to modelling of non-null distributions of test statistics based on data from thick-tailed populations as well as data from heterogeneous populations(see, for example, Thabane and Drekic[31]). They also arise in model behavior of test statistics in a variety of situations where underlying samples are dependent but uncorrelated, or samples come from symmetric, non-normal populations. We hope that the distributions of the product $XY$ derived in this note will help to enhance the applications of the two distributions.

Nadarajah and Kotz[18] have shown that the cdf corresponding to (1) can be expressed as

$$
\text{(3)}\quad F(x) = \begin{cases} \dfrac{1}{2} + \dfrac{1}{\pi} \arctan\left(\dfrac{x}{\sqrt{\nu}}\right) + \dfrac{1}{2\pi} \displaystyle\sum_{l=1}^{(\nu-1)/2} B\left(l, \dfrac{1}{2}\right) \dfrac{\nu^{l-1/2} x}{\left(\nu + x^2\right)^l}, & \text{if } \nu \text{ is odd}, \\ \dfrac{1}{2} + \dfrac{1}{2\pi} \displaystyle\sum_{l=1}^{\nu/2} B\left(l - \dfrac{1}{2}, \dfrac{1}{2}\right) \dfrac{\nu^{l-1} x}{\left(\nu + x^2\right)^{l-1/2}}, & \text{if } \nu \text{ is even}. \end{cases}
$$

This result will be crucial for the calculations of this note. The calculations involve several special functions, including the Euler psi function defined by

$$
\Psi(x) = \frac{d \log \Gamma(x)}{dx},
$$

the Struve function defined by

$$
H_\nu(x) = \frac{2x^{\nu+1}}{\sqrt{\pi} 2^{\nu+1} \Gamma(\nu + 3/2)} \sum_{k=0}^{\infty} \frac{1}{(3/2)_k (\nu + 3/2)_k} \left(-\frac{x^2}{4}\right)^k,
$$

the Bessel function of the first kind defined by

$$
J_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu + 1)} \sum_{k=0}^{\infty} \frac{1}{(\nu + 1)_k k!} \left(-\frac{x^2}{4}\right)^k,
$$

and the hypergeometric function defined by

$$
G(a; b, c; x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k (c)_k} \frac{x^k}{k!},
$$

where $(e)_k = e(e+1) \cdots (e+k-1)$ denotes the ascending factorial. We also need the following important lemma.

LEMMA 1 (Equation (2.16, 3.13), Prudnikov et al.[21], vol. 2) *For* $c > 0$, $z > 0$ *and* $\alpha > \nu$,

$$\begin{aligned}
&\int_0^\infty \frac{x^{\alpha-1}}{\left(x^2+z^2\right)^\rho} K_\nu(cx) dx \\
&= 2^{\nu-2} c^{-\nu} z^{\alpha-2\rho-\nu} \Gamma(\nu) B\left(\rho + \frac{\nu-\alpha}{2}, \frac{\alpha-\nu}{2}\right) \\
&\times G\left(\frac{\alpha-\nu}{2}; 1-\nu, 1-\rho+\frac{\alpha-\nu}{2}; -\frac{c^2z^2}{4}\right) \\
&+ 2^{-(\nu+2)} c^\nu z^{\alpha-2\rho+\nu} \Gamma(-\nu) B\left(\rho - \frac{\nu+\alpha}{2}, \frac{\alpha+\nu}{2}\right) \\
&\times G\left(\frac{\alpha+\nu}{2}; 1+\nu, 1-\rho+\frac{\alpha+\nu}{2}; -\frac{c^2z^2}{4}\right) \\
&+ 2^{\alpha-2\rho-2} c^{2\rho-\alpha} \Gamma\left(\frac{\alpha+\nu}{2}-\rho\right) \Gamma\left(\frac{\alpha-\nu}{2}-\rho\right) \\
&\times G\left(\rho; 1+\rho-\frac{\alpha+\nu}{2}; 1+\rho-\frac{\alpha-\nu}{2}; -\frac{c^2z^2}{4}\right).
\end{aligned}$$

Further properties of the above special functions can be found in Prudnikov et al.[21] and Gradshteyn and Ryzhik[9].

## 2. CDF

Theorem 1 derives an explicit expression for the cdf of $|XY|$ in terms of the hypergeometric function.

THEOREM 1. *Suppose* $X$ *and* $Y$ *are distributed according to* (1) *and* (2), *respectively. If* $\nu$ *is an odd integer then the cdf of* $Z = |XY|$ *can be expressed as*

$$\begin{aligned}
&F(z) \\
&= I(\nu) + \frac{1}{\pi^{3/2}\sqrt{\nu}2^{m-1}b^{m+1}\Gamma(m+1/2)} \sum_{k=1}^{(\nu-1)/2} B\left(k, \frac{1}{2}\right) A(k),
\end{aligned} \tag{4}$$

*where* $I(\cdot)$ *denotes the integral*

$$\begin{aligned}
&I(a) \\
&= \frac{1}{\pi^{3/2}2^{m-2}b^{m+1}\Gamma(m+1/2)} \int_0^\infty \arctan\left(\frac{z}{\sqrt{a}y}\right) y^m K_m\left(\frac{y}{b}\right) dy,
\end{aligned} \tag{5}$$

$$
\begin{aligned}
A(k) = {} & 2^{-(m+2)}(\nu b)^{-m} z^{2m}\Gamma(-m)B(-m, m+k) \\
& \times G\left(m+k; 1+m, 1+m; -\frac{z^2}{4\nu b^2}\right) \\
& - \left\{2C + \Psi(k)\right\} 2^{m-2} b^m \Gamma(m) G\left(k; 1-m, 1; -\frac{z^2}{4\nu b^2}\right),
\end{aligned}
$$

*and $C$ denotes the Euler's constant.*

PROOF. The cdf $F(z) = \Pr(\mid XY \mid \leq z)$ can be expressed as

$$
\begin{aligned}
F(z) = {} & \frac{1}{\sqrt{\pi} 2^m b^{m+1} \Gamma(m+1/2)} \\
& \times \int_{-\infty}^{\infty} \left\{ F\left(\frac{z}{\mid y \mid}\right) - F\left(-\frac{z}{\mid y \mid}\right) \right\} \mid y \mid^m K_m\left(\left|\frac{y}{b}\right|\right) dy \\
= {} & \frac{1}{\sqrt{\pi} 2^{m-1} b^{m+1} \Gamma(m+1/2)} \\
& \times \int_{0}^{\infty} \left\{ F\left(\frac{z}{y}\right) - F\left(-\frac{z}{y}\right) \right\} y^m K_m\left(\frac{y}{b}\right) dy,
\end{aligned}
$$

where $F(\cdot)$ inside the integrals denotes the cdf of a Student's $t$ random variable with degrees of freedom $\nu$. Substituting the form for $F$ given by (3) for odd degrees of freedom, (6) can be reduced to

$$
(6) \quad F(z) = I(\nu) + \frac{1}{\pi^{3/2}\sqrt{\nu} 2^{m-1} b^{m+1} \Gamma(m+1/2)} \sum_{k=1}^{(\nu-1)/2} B\left(k, \frac{1}{2}\right) J(k),
$$

where $J(k)$ denotes the integral

$$
J(k) = \int_0^{\infty} \frac{y^{m+2k-1} K_m(y/b)}{\left(y^2 + z^2/\nu\right)^k} dy.
$$

By direct application of Lemma 1, one can easily see that $J(k) = A(k)$, where $A(k)$ is given by (6). The result of the theorem follows by substituting this form for $J(k)$ into (6). □

Theorem 2 is the analogue of Theorem 1 for the case when the degrees of freedom $\nu$ is an even integer.

THEOREM 2. *Suppose $X$ and $Y$ are distributed according to (1) and (2), respectively. If $\nu$ is an even integer then the cdf of $Z = |XY|$ can be expressed as*

$$
(7) \qquad F(z) = \frac{1}{\pi^{3/2}\sqrt{\nu} 2^{m-1} b^{m+1} \Gamma(m+1/2)} \sum_{k=1}^{\nu/2} B\left(k - \frac{1}{2}, \frac{1}{2}\right) A(k),
$$

*where*

$$
\begin{aligned}
A(k) &= 2^{-(m+2)}(\nu b)^{-m} z^{2m}\Gamma(-m)B\left(-m, m+k-\frac{1}{2}\right) \\
&\quad \times G\left(m+k-\frac{1}{2}; 1+m, 1+m; -\frac{z^2}{4\nu b^2}\right) \\
&\quad - \left\{2C + \Psi\left(k-\frac{1}{2}\right)\right\} 2^{m-2} b^m \Gamma(m) \\
&\quad \times G\left(k-\frac{1}{2}; 1-m, 1; -\frac{z^2}{4\nu b^2}\right), \qquad (8)
\end{aligned}
$$

*and $C$ denotes the Euler's constant.*

*Proof.* Substituting the form for $F$ given by (3) for even degrees of freedom, (6) can be reduced to

$$
F(z) = \frac{1}{\pi^{3/2}\sqrt{\nu}2^{m-1}b^{m+1}\Gamma\left(m+1/2\right)} \sum_{k=1}^{\nu/2} B\left(k-\frac{1}{2}, \frac{1}{2}\right) J(k), \qquad (9)
$$

where $J(k)$ denotes the integral

$$
J(k) = \int_0^\infty \frac{y^{m+2k-2} K_m\left(y/b\right)}{\left(y^2 + z^2/\nu\right)^{k-1/2}} dy.
$$

By direct application of Lemma 1, one can easily see that $J(k) = A(k)$, where $A(k)$ is given by (8). The result of the theorem follows by substituting this form for $J(k)$ into (9). □

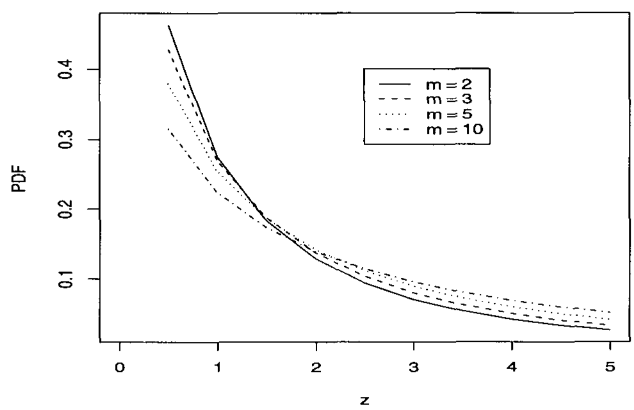


**Figure 1.** Plots of the pdf of (4) and (7) for $\nu = 5$ and $m = 2, 3, 5, 10$.

Figure 1 illustrates possible shapes of the pdf of $|XY|$ for $\nu = 5$ and a range of values of $m$. Note that the shapes are unimodal and that the value of $m$ largely dictates the behavior of the pdf near $z = 0$.

## 3. Particular cases

Corollaries 1 to 5 below derive particular forms of (4) and (7) for certain integer values of $\nu$ and half integer values of $m$. In our calculations, we have used various special properties of the hypergeometric function (see, for example, Chapter 7 of Prudnikov et al.(1986, volume 3)[21]). Note that when $\nu$ is even (7) reduces to forms involving the Struve function and the Bessel function of the first kind. On the other hand, when $\nu$ is odd (7) reduces to elementary forms except for the integral given by (5).

COROLLARY 1. *Suppose $X$ and $Y$ are distributed according to (1) and (2), respectively. For $\nu = 2$ and $m = 3/2, 5/2, \ldots, 11/2$, the cdf of $Z =| XY |$ can be expressed as*

$$\begin{aligned}
F(z) = 1/4\pi^{3/2}\sqrt{b}\Big\{ &- 2z + \sqrt{2}b\pi H_0(w) + \pi H_1(w)\, z - \sqrt{2}bCJ_0(w) - CzJ_1(w) \\
&+ 2\sqrt{2}b\log 2J_0(w) + 2\log 2zJ_1(w)\Big\},
\end{aligned}$$

$$\begin{aligned}
F(z) = 1/8\pi^{3/2}\sqrt{b}\Big\{ &- 12bz - \sqrt{2}\pi H_0(w)\, z^2 + 6\sqrt{2}\pi H_0(w)\, b^2 + 8b\pi H_1(w)\, z \\
&+ \sqrt{2}CJ_0(w)\, z^2 - 6\sqrt{2}b^2 CJ_0(w) - 8bCzJ_1(w) - 2\sqrt{2}\log 2J_0(w)\, z^2 \\
&+ 12\sqrt{2}b^2\log 2J_0(w) + 16b\log 2zJ_1(w)\Big\},
\end{aligned}$$

$$\begin{aligned}
F(z) = 1/8\pi^{3/2}\sqrt{b}\Big\{ &- 60zb^2 + 2z^3 - 7b\sqrt{2}\pi H_0(w)\, z^2 + 30b^3\sqrt{2}\pi H_0(w) \\
&- \pi H_1(w)\, z^3 + 46\pi H_1(w)\, zb^2 + 7\sqrt{2}bCJ_0(w)\, z^2 - 30\sqrt{2}b^3 CJ_0(w) \\
&+ Cz^3 J_1(w) - 46b^2 CzJ_1(w) - 14\sqrt{2}b\log 2J_0(w)\, z^2 + 60\sqrt{2}b^3\log 2J_0(w) \\
&- 2\log 2z^3 J_1(w) + 92b^2\log 2zJ_1(w)\Big\},
\end{aligned}$$

$$\begin{aligned}
F(z) = 1/16\pi^{3/2}\sqrt{b}\Big\{ &44bz^3 - 840b^3 z + \sqrt{2}\pi H_0(w)\, z^4 - 116\sqrt{2}\pi H_0(w)\, z^2 b^2 \\
&+ 420\sqrt{2}\pi H_0(w)\, b^4 - 24b\pi H_1(w)\, z^3 + 704b^3\pi H_1(w)\, z - \sqrt{2}CJ_0(w)\, z^4 \\
&+ 116\sqrt{2}b^2 CJ_0(w)\, z^2 - 420\sqrt{2}b^4 CJ_0(w) + 24bCz^3 J_1(w) - 704b^3 CzJ_1(w) \\
&+ 2\sqrt{2}\log 2J_0(w)\, z^4 7 - 232\sqrt{2}b^2\log 2J_0(w)\, z^2 + 840\sqrt{2}b^4\log 2J_0(w) \\
&- 48b\log 2z^3 J_1(w) + 1408b^3\log 2zJ_1(w)\Big\},
\end{aligned}$$

$$F(z) = -1/16\pi^{3/2}\sqrt{b}\Big\{ -496z^3b^2 + 2z^5 + 7560zb^4 - 17b\sqrt{2\pi}H_0(w)\,z^4 + 1164b^3\sqrt{2\pi}H_0(w)\,z^2 - 3780b^5\sqrt{2\pi}H_0(w) - \pi H_1(w)\,z^5 + 284\pi H_1(w)\,z^3b^2 - 6756\pi H_1(w)\,zb^4 + 17\sqrt{2}bCJ_0(w)\,z^4 - 1164\sqrt{2}b^3CJ_0(w)\,z^2 + 3780\sqrt{2}b^5CJ_0(w) + Cz^5J_1(w) - 284b^2Cz^3J_1(w) + 6756b^4CzJ_1(w) - 34\sqrt{2}b\log 2J_0(w)\,z^4 + 2328\sqrt{2}b^3\log 2J_0(w)\,z^2 - 7560\sqrt{2}b^5\log 2J_0(w) - 2\log 2z^5J_1(w) + 568b^2\log 2z^3J_1(w) - 13512b^4\log 2zJ_1(w)\Big\},$$

*where* $w = z/(\sqrt{2}b)$.

COROLLARY 2. *Suppose* $X$ *and* $Y$ *are distributed according to* (1) *and* (2), *respectively. For* $\nu = 3$ *and* $m = 3/2, 5/2, \ldots, 11/2$, *the cdf of* $Z = |XY|$ *can be expressed as*

$$F(z) = I(3) - 1/6\sqrt{2}\sqrt{\pi}\sqrt{b}\Big\{\sqrt{3}\pi\cos(w)\,z - 3\pi b\sin(w) + 3Cb\cos(w) + C\sqrt{3}z\sin(w)\Big\},$$

$$F(z) = I(3) - 1/6\sqrt{2}\sqrt{\pi}\sqrt{b}\Big\{3\sqrt{3}\pi b\cos(w)\,z - 9\pi b^2\sin(w) + \pi\sin(w)\,z^2 + 9Cb^2\cos(w) - C\cos(w)\,z^2 + 3Cb\sqrt{3}z\sin(w)\Big\},$$

$$F(z) = I(3) - 1/18\sqrt{2}\sqrt{\pi}\sqrt{b}\Big\{45\sqrt{3}\pi b^2\cos(w)\,z - \sqrt{3}\pi\cos(w)\,z^3 + 18\pi b\sin(w)\,z^2 - 135\pi b^3\sin(w) - 18Cb\cos(w)\,z^2 + 135Cb^3\cos(w) + 45Cb^2\sqrt{3}z\sin(w) - C\sqrt{3}z^3\sin(w)\Big\},$$

$$F(z) = I(3) - 1/18\sqrt{2}\sqrt{\pi}\sqrt{b}\Big\{ -10\sqrt{3}\pi b\cos(w)\,z^3 + 315\sqrt{3}\pi b^3\cos(w)\,z + 135\pi b^2\sin(w)\,z^2 - 945\pi b^4\sin(w) - \pi\sin(w)\,z^4 - 135Cb^2\cos(w)\,z^2 + 945Cb^4\cos(w) + C\cos(w)\,z^4 - 10Cb\sqrt{3}z^3\sin(w) + 315Cb^3\sqrt{3}z\sin(w)\Big\},$$

$$F(z) = I(3) - 1/54\sqrt{2}\sqrt{\pi}\sqrt{b}\Big\{ -315\sqrt{3}\pi b^2\cos(w)\,z^3 + 8505\sqrt{3}\pi b^4\cos(w)\,z + \sqrt{3}\pi\cos(w)\,z^5 + 3780\pi b^3\sin(w)\,z^2 - 45\pi b\sin(w)\,z^4 - 25515\pi b^5\sin(w) - 3780Cb^3\cos(w)\,z^2 + 45Cb\cos(w)\,z^4 + 25515Cb^5\cos(w) - 315Cb^2\sqrt{3}z^3\sin(w) + 8505Cb^4\sqrt{3}z\sin(w) + C\sqrt{3}z^5\sin(w)\Big\},$$

*where* $w = z/(\sqrt{3}b)$.

COROLLARY 3. *Suppose* $X$ *and* $Y$ *are distributed according to* (1) *and* (2), *respectively. For* $\nu = 4$ *and* $m = 3/2, 5/2, \ldots, 11/2$, *the cdf of*

$Z = | XY |$ *can be expressed as*

$$F(z) = 1/(32\sqrt{b})\pi^{3/2}\sqrt{2}\Big\{ -12bz + 12\pi H_0(w)b^2 + 4b\pi H_1(w)z - 12b^2CJ_0(w) - 4bCzJ_1(w) + 24b^2\log 2J_0(w) + 8b\log 2zJ_1(w) + \pi H_0(w)z^2 - J_0(w)Cz^2 - 8J_0(w)b^2 - 2J_0(w)z^2 + 2J_0(w)\log 2z^2\Big\},$$

$$F(z) = 1/(64\sqrt{b})\pi^{3/2}\sqrt{2}\Big\{ -2b\pi H_0(w)z^2 + 36\pi H_1(w)zb^2 + \pi H_1(w)z^3 - 72CJ_0(w)b^3 - Cz^3J_1(w) - 4J_0(w)bz^2 - 8zJ_1(w)b^2 + 144\log 2J_0(w)b^3 + 2\log 2z^3J_1(w) + 2CJ_0(w)bz^2 - 36CzJ_1(w)b^2 - 4\log 2J_0(w)bz^2 + 72\log 2zJ_1(w)b^2 + 72b^3\pi H_0(w) - 48J_0(w)b^3 - 2z^3J_1(w) - 72zb^2 - 2z^3\Big\},$$

$$F(z) = 1/(128\sqrt{b})\pi^{3/2}\sqrt{2}\Big\{ -48\pi H_0(w)z^2b^2 - \pi H_0(w)z^4 + 432b^3\pi H_1(w)z + 4b\pi H_1(w)z^3 - 4Cz^3J_1(w)b - 96\log 2J_0(w)z^2b^2 + 864\log 2zJ_1(w)b^3 + 8\log 2z^3J_1(w)b - 432CzJ_1(w)b^3 + 48CJ_0(w)z^2b^2 - 720CJ_0(w)b^4 + CJ_0(w)z^4 - 16J_0(w)z^2b^2 - 128zJ_1(w)b^3 - 16z^3J_1(w)b + 1440\log 2J_0(w)b^4 - 2\log 2J_0(w)z^4 + 720\pi H_0(w)b^4 - 480J_0(w)b^4 + 2J_0(w)z^4 - 4bz^3 - 720b^3z\Big\},$$

$$F(z) = 1/(256\sqrt{b})\pi^{3/2}\sqrt{2}\Big\{2z^5 - 10080zb^4 + 64z^3b^2 + 6768\pi H_1(w)zb^4 - 8\pi H_1(w)z^3b^2 - 32J_0(w)b^3z^2 + 10080b^5\pi H_0(w) - \pi H_1(w)z^5 - 2272zJ_1(w)b^4 - 10080CJ_0(w)b^5 - 176z^3J_1(w)b^2 + 28J_0(w)bz^4 - 10b\pi H_0(w)z^4 - 6768CzJ_1(w)b^4 - 20\log 2J_0(w)bz^4 - 1824\log 2J_0(w)b^3z^2 - 16\log 2z^3J_1(w)b^2 + 10CJ_0(w)bz^4 + 912CJ_0(w)b^3z^2 + 8Cz^3J_1(w)b^2 - 912b^3\pi H_0(w)z^2 - 6720J_0(w)b^5 + 2z^5J_1(w) + 20160\log 2J_0(w)b^5 + Cz^5J_1(w) - 2\log 2z^5J_1(w) + 13536\log 2zJ_1(w)b^4\Big\},$$

$$F(z) = 1/(512\sqrt{b})\pi^{3/2}\sqrt{2}\Big\{36bz^5 + 2592b^3z^3 - 181440b^5z - 120960J_0(w)b^6 - 92\pi H_0(w)z^4b^2 - 19536\pi H_0(w)z^2b^4 - 20b\pi H_1(w)z^5 - 1024b^3\pi H_1(w)z^3 + 131904b^5\pi H_1(w)z + 1024Cz^3J_1(w)b^3 - 131904CzJ_1(w)b^5 - 184\log 2J_0(w)z^4b^2 - 39072\log 2J_0(w)z^2b^4 - 40\log 2z^5J_1(w)b - 2048\log 2z^3J_1(w)b^3 + 263808\log 2zJ_1(w)b^5 + 92CJ_0(w)z^4b^2 + 19536CJ_0(w)z^2b^4 + 20Cz^5J_1(w)b + \pi H_0(w)z^6 + 181440\pi H_0(w)b^6 + 1824J_0(w)z^2b^4 + 48z^5J_1(w)b - 2496z^3J_1(w)b^3 - 47616zJ_1(w)b^5 - CJ_0(w)z^6 - 181440CJ_0(w)b^6 + 2\log 2J_0(w)z^6 + 362880\log 2J_0(w)b^6 + 456J_0(w)z^4b^2 - 2J_0(w)z^6\Big\},$$

*where* $w = z/(2b)$.

COROLLARY 4. *Suppose $X$ and $Y$ are distributed according to (1) and (2), respectively. For $\nu = 5$ and $m = 3/2, 5/2, \ldots, 11/2$, the cdf of $Z = |XY|$ can be expressed as*

$$\begin{aligned}
F(z) = I(5) - 1/(30\sqrt{b})\sqrt{2}\sqrt{\pi}\Big\{ & 5\sqrt{5}\pi b\cos(w)\, z - 25\pi b^2 \sin(w) + 25Cb^2\cos(w) \\
& + 5C\sqrt{5}z\sin(w)\, b - \pi\sin(w)\, z^2 + C\cos(w)\, z^2 + 10b^2\cos(w) + \cos(w)\, z^2 \\
& + 2\sqrt{5}z\sin(w)\, b\Big\},
\end{aligned}$$

$$\begin{aligned}
F(z) = I(5) - 1/(150\sqrt{b})\sqrt{2}\sqrt{\pi}\Big\{ & 75\sqrt{5}\pi b^2\cos(w)\, z - 375\pi b^3\sin(w) \\
& + 20\pi b\sin(w)\, z^2 + 375C\cos(w)\, b^3 - 20C\cos(w)\, bz^2 + 75C\sqrt{5}z\sin(w)\, b^2 \\
& + \sqrt{5}\pi\cos(w)\, z^3 + C\sqrt{5}z^3\sin(w) - 5\cos(w)\, bz^2 + 150\cos(w)\, b^3 \\
& + 30\sqrt{5}z\sin(w)\, b^2 + \sqrt{5}z^3\sin(w)\Big\},
\end{aligned}$$

$$\begin{aligned}
F(z) = I(5) - 1/(150\sqrt{b})\sqrt{2}\sqrt{\pi}\Big\{ & 375\sqrt{5}\pi b^3\cos(w)\, z - 2\sqrt{5}\pi b\cos(w)\, z^3 \\
& + 135\pi b^2\sin(w)\, z^2 - 1875\pi b^4\sin(w) - 135C\cos(w)\, z^2 b^2 \\
& + 1875C\cos(w)\, b^4 + 375C\sqrt{5}z\sin(w)\, b^3 - 2C\sqrt{5}z^3\sin(w)\, b + \pi\sin(w)\, z^4 \\
& - C\cos(w)\, z^4 - 45\cos(w)\, z^2 b^2 - \cos(w)\, z^4 + 750\cos(w)\, b^4 \\
& + \sqrt{5}z^3\sin(w)\, b + 150\sqrt{5}z\sin(w)\, b^3\Big\},
\end{aligned}$$

$$\begin{aligned}
F(z) = I(5) + 1/(750\sqrt{b})\sqrt{2}\sqrt{\pi}\Big\{ & 175\sqrt{5}\pi b^2\cos(w)\, z^3 - 13125\sqrt{5}\pi b^4\cos(w)\, z \\
& - 5250\pi b^3\sin(w)\, z^2 - 13125C\sqrt{5}z\sin(w)\, b^4 + \sqrt{5}\pi\cos(w)\, z^5 \\
& + 5250C\cos(w)\, b^3 z^2 + 5C\cos(w)\, bz^4 - 5\pi b\sin(w)\, z^4 \\
& + 175C\sqrt{5}z^3\sin(w)\, b^2 + C\sqrt{5}z^5\sin(w) + 25\sqrt{5}z^3\sin(w)\, b^2 \\
& - 5250\sqrt{5}z\sin(w)\, b^4 - 65625C\cos(w)\, b^5 + 65625\pi b^5\sin(w) \\
& + 20\cos(w)\, bz^4 + 1875\cos(w)\, b^3 z^2 + \sqrt{5}z^5\sin(w) - 26250\cos(w)\, b^5\Big\},
\end{aligned}$$

$$\begin{aligned}
F(z) = I(5) - 1/(750\sqrt{b})\sqrt{2}\sqrt{\pi}\Big\{ & -525\sqrt{5}z^3\sin(w)\, b^3 + 47250\sqrt{5}z\sin(w)\, b^5 \\
& - 49875C\cos(w)\, z^2 b^4 + 150C\cos(w)\, z^4 b^2 - 2100C\sqrt{5}z^3\sin(w)\, b^3 \\
& + 118125C\sqrt{5}z\sin(w)\, b^5 - 5C\sqrt{5}z^5\sin(w)\, b - \pi\sin(w)\, z^6 \\
& - 2100\sqrt{5}\pi b^3\cos(w)\, z^3 + 118125\sqrt{5}\pi b^5\cos(w)\, z - 5\sqrt{5}\pi b\cos(w)\, z^5 \\
& + 49875\pi b^4\sin(w)\, z^2 - 150\pi b^2\sin(w)\, z^4 \\
& - 590625\pi b^6\sin(w) + 590625C\cos(w)\, b^6 + C\cos(w)\, z^6 - 75\cos(w)\, z^4 b^2 \\
& - 18375\cos(w)\, z^2 b^4 - 8\sqrt{5}z^5\sin(w)\, b + \cos(w)\, z^6 + 236250\cos(w)\, b^6\Big\},
\end{aligned}$$

*where $w = z/(\sqrt{5}b)$.*

COROLLARY 5. *Suppose $X$ and $Y$ are distributed according to (1) and (2), respectively. For $\nu = 6$ and $m = 3/2, 5/2, \ldots, 11/2$, the cdf of $Z = |XY|$ can be expressed as*

$$\begin{aligned}
F(z) = {} & 1/(1728b^{3/2})\pi^{3/2}\Big\{ - 810b^3\sqrt{2}J_0(w)\,C - 81b\sqrt{2}J_0(w)\,Cz^2 \\
& - 192b\sqrt{2}J_0(w)\,z^2 + 1620b^3\sqrt{2}J_0(w)\log 2 + 162b\sqrt{2}J_0(w)\log 2z^2 \\
& - 864b^3\sqrt{2}J_0(w) + 810b^3\sqrt{2}\pi H_0(w) \\
& + 81b\sqrt{2}\pi H_0(w)\,z^2 + 126\sqrt{3}b^2\pi H_1(w)\,z - 3\sqrt{3}\pi H_1(w)\,z^3 \\
& - 126b^2C\sqrt{3}zJ_1(w) + 3C\sqrt{3}z^3J_1(w) + 48b^2\sqrt{3}zJ_1(w) + 8\sqrt{3}z^3J_1(w) \\
& + 252b^2\log 2\sqrt{3}zJ_1(w) - 6\log 2\sqrt{3}z^3J_1(w) - 540\sqrt{3}b^2z + 6\sqrt{3}z^3\Big\},
\end{aligned}$$

$$\begin{aligned}
F(z) = {} & 1/(3456b^{3/2})\pi^{3/2}\Big\{ - 8\sqrt{2}J_0(w)\,z^4 + 1296b^3\sqrt{3}\pi H_1(w)\,z \\
& - 1296b^3C\sqrt{3}zJ_1(w) + 2592b^3\log 2\sqrt{3}zJ_1(w) + 72\sqrt{2}b^2\log 2J_0(w)\,z^2 \\
& + 9720\sqrt{2}b^4\log 2J_0(w) + 96b\log 2\sqrt{3}z^3J_1(w) - 36\sqrt{2}b^2CJ_0(w)\,z^2 \\
& - 3\sqrt{2}CJ_0(w)\,z^4 - 4860\sqrt{2}b^4CJ_0(w) - 48bC\sqrt{3}z^3J_1(w) \\
& - 432\sqrt{2}b^2J_0(w)\,z^2 - 5184\sqrt{2}b^4J_0(w) + 6\sqrt{2}\log 2J_0(w)\,z^4 \\
& - 112b\sqrt{3}z^3J_1(w) - 288b^3\sqrt{3}zJ_1(w) + 36b^2\sqrt{2}\pi H_0(w)\,z^2 \\
& + 3\sqrt{2}\pi H_0(w)\,z^4 + 4860b^4\sqrt{2}\pi H_0(w) + 48\sqrt{3}b\pi H_1(w)\,z^3 - 108\sqrt{3}bz^3 \\
& - 3240\sqrt{3}b^3z\Big\},
\end{aligned}$$

$$\begin{aligned}
F(z) = {} & 1/(10368b^{3/2})\pi^{3/2}\Big\{3\sqrt{3}\pi H_1(w)\,z^5 + 6\log 2\sqrt{3}z^5J_1(w) - 3C\sqrt{3}z^5J_1(w) \\
& - 3456\sqrt{2}b^3J_0(w)\,z^2 + 145800\sqrt{2}b^5\log 2J_0(w) - 72900\sqrt{2}b^5CJ_0(w) \\
& + 144\sqrt{2}bJ_0(w)\,z^4 - 1440b^2\sqrt{3}z^3J_1(w) - 9504b^4\sqrt{3}zJ_1(w) \\
& + 72900b^5\sqrt{2}\pi H_0(w) + 468\sqrt{3}b^2\pi H_1(w)\,z^3 + 24300\sqrt{3}b^4\pi H_1(w)\,z \\
& - 1620b^3\sqrt{2}\pi H_0(w)\,z^2 - 3240\sqrt{2}b^3\log 2J_0(w)\,z^2 - 468b^2C\sqrt{3}z^3J_1(w) \\
& - 24300b^4C\sqrt{3}zJ_1(w) + 63\sqrt{2}bCJ_0(w)\,z^4 + 1620\sqrt{2}b^3CJ_0(w)\,z^2 \\
& + 936b^2\log 2\sqrt{3}z^3J_1(w) + 48600b^4\log 2\sqrt{3}zJ_1(w) - 63b\sqrt{2}\pi H_0(w)\,z^4 \\
& - 126\sqrt{2}b\log 2J_0(w)\,z^4 - 6\sqrt{3}z^5 - 8\sqrt{3}z^5J_1(w) - 720\sqrt{3}b^2z^3 \\
& - 48600\sqrt{3}b^4z - 77760\sqrt{2}b^5J_0(w)\Big\},
\end{aligned}$$

$$\begin{aligned}
F(z) = {} & 1/(20736b^{3/2})\pi^{3/2}\Big\{ - 184896b^5\sqrt{3}zJ_1(w) - 16704b^3\sqrt{3}z^3J_1(w) \\
& + 1020600b^6\sqrt{2}\pi H_0(w) + 2041200\sqrt{2}b^6\log 2J_0(w) \\
& - 1020600\sqrt{2}b^6CJ_0(w) - 25056\sqrt{2}b^4J_0(w)\,z^2 + 2304\sqrt{2}b^2J_0(w)\,z^4 \\
& + 48b\sqrt{3}z^5J_1(w) + 3\sqrt{2}CJ_0(w)\,z^6 - 6\sqrt{2}\log 2J_0(w)\,z^6 - 3\sqrt{2}\pi H_0(w)\,z^6 \\
& - 24\sqrt{3}b\pi H_1(w)\,z^5 + 388800\sqrt{3}b^5\pi H_1(w)\,z + 3600\sqrt{3}b^3\pi H_1(w)\,z^3
\end{aligned}$$

$$
\begin{aligned}
& + 24bC\sqrt{3}z^5 J_1(w) + 40500\sqrt{2}b^4 C J_0(w) z^2 + 846\sqrt{2}b^2 C J_0(w) z^4 \\
& - 81000\sqrt{2}b^4 \log 2 J_0(w) z^2 - 1692\sqrt{2}b^2 \log 2 J_0(w) z^4 \\
& + 777600 b^5 \log 2\sqrt{3} z J_1(w) + 7200 b^3 \log 2\sqrt{3} z^3 J_1(w) \\
& - 388800 b^5 C\sqrt{3} z J_1(w) - 3600 b^3 C\sqrt{3} z^3 J_1(w) - 40500 b^4 \sqrt{2}\pi H_0(w) z^2 \\
& - 846 b^2 \sqrt{2}\pi H_0(w) z^4 + 60\sqrt{3} b z^5 - 3600\sqrt{3} b^3 z^3 - 680400\sqrt{3} b^5 z \\
& - 1088640\sqrt{2} b^6 J_0(w) + 8\sqrt{2} J_0(w) z^6 - 48 b \log 2\sqrt{3} z^5 J_1(w) \Big\},
\end{aligned}
$$

$$
\begin{aligned}
F(z) = & -1/(62208 b^{3/2})\pi^{3/2} \Big\{ -6\sqrt{3} z^7 - 2412\sqrt{3} b^2 z^5 - 16200\sqrt{3} b^4 z^3 \\
& + 18370800\sqrt{3} b^6 z + 29393280\sqrt{2} b^7 J_0(w) - 8\sqrt{3} z^7 J_1(w) \\
& - 1206 b^2 C\sqrt{3} z^5 J_1(w) + 35100 b^4 C\sqrt{3} z^3 J_1(w) \\
& + 115182000 b^6 C\sqrt{3} z J_1(w) + 9\sqrt{2} b C J_0(w) z^6 + 2412 b^2 \log 2\sqrt{3} z^5 J_1(w) \\
& - 70200 b^4 \log 2\sqrt{3} z^3 J_1(w) - 230364000 b^6 \log 2\sqrt{3} z J_1(w) \\
& - 18090\sqrt{2} b^3 C J_0(w) z^4 + 36180\sqrt{2} b^3 \log 2 J_0(w) z^4 \\
& + 2867400\sqrt{2} b^5 \log 2 J_0(w) z^2 - 9 b\sqrt{2}\pi H_0(w) z^6 \\
& + 18090 b^3 \sqrt{2}\pi H_0(w) z^4 + 1433700 b^5 \sqrt{2}\pi H_0(w) z^2 \\
& - 1433700\sqrt{2} b^5 C J_0(w) z^2 + 1206\sqrt{3} b^2 \pi H_1(w) z^5 \\
& - 35100\sqrt{3} b^4 \pi H_1(w) z^3 - 11518200\sqrt{3} b^6 \pi H_1(w) z \\
& - 18\sqrt{2} b \log 2 J_0(w) z^6 + 3\sqrt{3}\pi H_1(w) z^7 + 6 \log 2\sqrt{3} z^7 J_1(w) \\
& - 3C\sqrt{3} z^7 J_1(w) + 27556200\sqrt{2} b^7 C J_0(w) - 59616\sqrt{2} b^3 J_0(w) z^4 \\
& + 248832\sqrt{2} b^5 J_0(w) z^2 - 55112400\sqrt{2} b^7 \log 2 J_0(w) - 3024 b^2 \sqrt{3} z^5 J_1(w) \\
& + 375840 b^4 \sqrt{3} z^3 J_1(w) + 6080832 b^6 \sqrt{3} z J_1(w) - 27556200 b^7 \sqrt{2}\pi H_0(w) \Big\},
\end{aligned}
$$

*where* $w = z/(\sqrt{6}b)$.

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