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# NON-COMPACT DOUGLAS-PLATEAU PROBLEM BOUNDED BY A LINE AND A JORDAN CURVE

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Abstract. In this article, we prove the existence of a minimal annulus bounded by a Jordan curve and a straight line.

## 1. Introduction

In this paper we consider the Douglas-Plateau problem for surfaces of annular type bounded by a rectifiable Jordan curve and a straight line. Recall that the Douglas-Plateau problem for two contours as follows:

> Let $\Gamma_1$ and $\Gamma_2$ be two disjoint Jordan curves in $\mathbf{R}^3$,
> find a minimal annulus $A$ such that $\partial A = \Gamma_1 \cup \Gamma_2$.

Let $S_1$ and $S_2$ be areas minimizing disks (when we say disks, we mean that they are homeomorphic to the unit disk in $\mathbf{C}$) such that $\partial S_i = \Gamma_i$, $i = 1, 2$, respectively. Let $S$ be the set of all rectifiable annuli $S$ such that $\partial S = \Gamma_1 \cup \Gamma_2$. Then J. Douglas[4] proved that if

$$\inf_{S \in \mathcal{S}} \{\mathrm{Area}(S)\} < \mathrm{Area}(S_1) + \mathrm{Area}(S_2), \tag{1}$$

then there is an area minimizing (therefore minimal) annulus bounded by $\Gamma_1 \cup \Gamma_2$. Now we consider the *non-compact* Douglas-Plateau problem of annular type containing at least one non-compact boundary curve. Recall there are many necessary conditions restricting the solvability of even *compact* Douglas-Plateau problems for two contours, therefore the solvability of the non-compact Douglas-Plateau problem seems should require more hypotheses than the compact case. It is known for more than one hundred years that for some non-compact boundaries we can

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find minimal annuli solving the corresponding "two-contour" Douglas-Plateau problem. A classical example [10] is a minimal annulus bounded by two parallel straight lines, a piece of a *Riemann's minimal example* which is foliated by circles and straight lines along horizontal planes. There are some recent results about the non-compact Douglas-Plateau problem, see [5], [6], [8], [9], etc,....

On the other hand, in 1990 F. Tomi and R. Ye[11] have shown that every rectifiable Jordan curve in $\mathbf{R}^3$ bounds a minimal immersion of the punctured disk which stretches out to infinity. The argument can be outlined as follows: Given a Jordan curve $\Gamma$, we choose a sequence of expanding round circles $\Gamma_k$ and obtain a sequence of expanding the least area annuli which span $\Gamma$ and $\Gamma_k$. Using an area estimate and Courant-Lebesgue type argument, we control the conformal types of these annuli and prove the convergence of their conformal parametrizations. Then we can take the limit surface of this sequence, which is the solution of the exterior Plateau problem for $\Gamma$. In this paper, we use this technique to construct a minimal annulus bounded by a rectifiable Jordan curve and a straight line as following:

THEOREM 1. *Let $\Gamma$ be a rectifiable curve in $\mathbf{R}^3$ and $L$ be a straight line, then there is a minimal annulus $A^+$ bounded by $\Gamma \cup L$.*

## 2. Preliminaries

Let $B := \{z \in \mathbf{C} : |z| < 1\}$ be the unit disk in the plane and let $X : B \to \mathbf{R}^3$ be an immersion which is given in conformal parameters $w = u + iv$, $u, v \in \mathbf{R}$, then the Dirichlet integral of $X$ is defined by

$$D(X, B) := \frac{1}{2} \int\int_B \left(|X_u|^2 + |X_v|^2\right) dudv.$$

LEMMA 1 (Courant-Lebesgue lemma [2]). *Let $X \in C^0(\bar{B}, \mathbf{R}^3) \cap C^1(B, \mathbf{R}^3)$ and $D(X) < \infty$. Let $z_0 \in \partial B$ and*

$$Z(r, \theta) := X(z_0 + re^{i\theta}),$$

*where $r, \theta$ denote polar coordinates about $z_0$. Take a ball $B_R(z_0)$ with radius $0 < R < 1$ and centered at $z_0$, and let*

$$\bar{B} \cap \partial B_R(z_0) = \{z_0 + Re^{i\theta} : \ \theta_1(R) \le \theta \le \theta_2(R)\}.$$

*Then for all* $\delta \in (0, R^2)$, *there is* $\rho \in (\delta, \sqrt{\delta})$ *such that for any angles* $\theta_1$, $\theta_2$ *with* $\theta_1(\rho) \leq \theta_1 \leq \theta_2 \leq \theta_2(\rho)$, *we have*

$$\int_{\theta_1}^{\theta_2} \left| \frac{\partial Z}{\partial \theta}(\rho, \theta) \right| d\theta \leq \eta(\delta, R), \tag{2}$$

*where*

$$\eta(\delta, R) = \left\{ \frac{2}{\log 1/\delta} D(X, B \cap B_R(z_0)) \right\}^{1/2}$$

*and in particular*

$$|Z(\rho, \theta_1) - Z(\rho, \theta_2)| \leq \eta(\delta, R) |\theta_1 - \theta_2|^{1/2}. \tag{3}$$

LEMMA 2 (Reflection and Rotation Theorems [2]). *If a plane geodesic which is not a straight line segment lies on a minimal surface, then reflection in the plane of the geodesic is a congruence of the minimal surface. If a straight line segment lies on a minimal surface, then* $\pi$-*degree rotation around the straight line is a congruence of the minimal surface.*

## 3. Construction

Let us denote horizontal planes by

$$\Pi := \{(x_1, x_2, x_3) \in \mathbf{R}^3 : \ x_3 = 0\},$$
$$\Pi^+ := \{(x_1, x_2, 0) : \ x_1 > 0\}.$$

We assume that the given straight line $L$ is the $x_2$-axis line, and the rectifiable curve $\Gamma$ meets $\Pi^+$ transversely. Take a planar half-disk $D_R^+$ contained in $\Pi^+$ with radius $R > 0$ as

$$D_R^+ := \{(x_1, x_2, 0) : x_1^2 + x_2^2 < R^2, \ x_1 > 0\}$$

and denote its boundary by $\Gamma_R^+ := \partial D_R^+$. We may assume that the projection of $\Gamma$ onto the horizontal plane intersects to the half-disk. Then it is well known that $\Gamma$ and $\Gamma_R^+$ bound the least area disks $S_\Gamma$ and $S_R^+$, respectively. Let us denote a solid cylinder

$$C_T := \{(x_1, x_2, x_3) \in \mathbf{R}^3 : \ x_1^2 + x_2^2 < T^2\}$$

with radius $T > 0$, and take $0 < T_0 < R$ such that $\Gamma \subset C_{T_0}$.

LEMMA 3 (Uniform Douglas conditions [11]). *For every* $R > T_0$, *we have*

$$a_{\Gamma, \Gamma_R^+} < \mathrm{Area}(S_\Gamma) + \mathrm{Area}(S_R^+) - \delta, \tag{4}$$

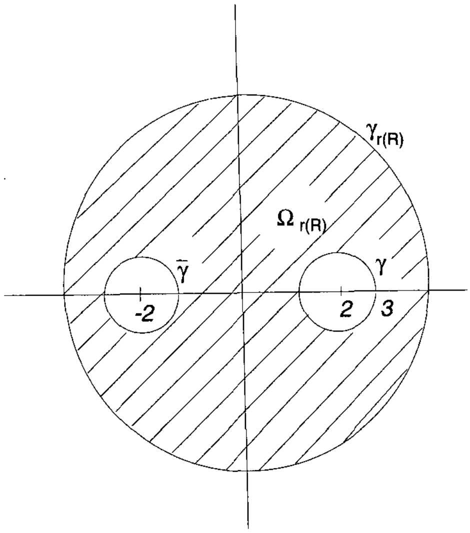


FIGURE 1.

*where $a_{\Gamma,\Gamma_R^+}$ is the infimum of area among all annulus type surfaces spanning $\Gamma$ and $\Gamma_R^+$, and $\delta > 0$ is independent constant of $R$.*

Using this lemma and the Douglas sufficient condition (1), we have an area minimizing annulus $A_R^+ \subset \mathbf{R}^3$ such that

$$\partial A_R^+ = \Gamma \cup \Gamma_R^+.$$

Now by Lemma 2, we can rotate $A_R^+$ by the $\pi$-degree along the straight line $L$. Denote

$$\mathrm{Rot}_L : \mathbf{R}^3 \to \mathbf{R}^3$$

by the $\pi$-degree rotation around the straight line $L$. Then we have a minimal surface $A_R$ which is the union of $A_R^+$ and the rotated one, i.e.,

$$A_R := A_R^+ \cup \mathrm{Rot}_L(A_R^+) \cup (L \cap C_R).$$

Conformally, $A_R$ is equivalent to a 3-fold connected domain $\Omega_{r(R)} \subset \mathbf{C}$, which is also symmetric under a conformal mapping of the plane. So we can take the three boundary curves of it as following:

$$\partial\Omega_{r(R)} := \gamma \cup \bar{\gamma} \cup \gamma_{r(R)},$$
$$\gamma := \{z \in \mathbf{C} : |z-2| = 1\}, \quad \bar{\gamma} := \{-\bar{z} : z \in \gamma\},$$
$$\gamma_{r(R)} := \{z \in \mathbf{C} : |z| < r(R)\}.$$

Let us define a minimal immersion, i.e., a harmonic conformal mapping

$$X_R : \Omega_{r(R)} \to A_R \subset \mathbf{R}^3$$

such that $X_R|_\gamma = \Gamma$ and $X_R|_{\bar{\gamma}} = \bar{\Gamma} := \mathrm{Rot}_L(\Gamma)$.

CLAIM 1. *If $R_k \to \infty$, then there is a subsequence of $X_{R_k}$ which converges to a conformal harmonic map $X : \Omega \to \mathbf{R}^3$ locally smoothly in the interior of the unbounded domain $\Omega$ which is defined by*

$$\Omega := \mathbf{C} \setminus \overline{D_\gamma \cup D_{\bar{\gamma}}},$$

*where $D_\gamma$ and $D_{\bar{\gamma}}$ are the disks in the plane bounded by $\gamma$ and $\bar{\gamma}$, respectively.*

Throughout this article we assume that $T_0 < T$, where $\Gamma \subset C_{T_0}$. Let us denote the preimage of the subset of $A_R$ contained in the solid cylinder with radius $T > 0$ by $\Lambda_{R,T}$, that is,

$$\Lambda_{R,T} := X_R^{-1}(A_R \cap C_T).$$

Denote the connected component of $\Lambda_{R,T}$ whose boundary contains $\gamma \cup \bar{\gamma}$ by $\Lambda^\gamma_{R,T}$. Recall, in a minimal surface, the Dirichlet integral is equal to the area of a surface. Since the orthogonal projection onto the $x_1x_2$-plane does not increase area, we can get

$$D(X_R,\ \Lambda_{R,T}) \geq \pi T^2 - 2\pi d_\Gamma^2, \tag{5}$$

$$D(X_R, \Omega_{r(R)} \setminus \Lambda_{R,T}) \geq \pi R^2 - \pi T^2, \tag{6}$$

where $d_\Gamma$ is the diameter of $\Gamma$. Recall Douglas condition (4) implies that

$$\begin{aligned} D(X_R, \Lambda_{R,T}) &= D(X_R,\ \Omega_{r(R)}) - D(X_R,\ \Omega_{r(R)} \setminus \Lambda_{R,T}) \\ &\leq \pi R^2 + 2a_\Gamma - (\pi R^2 - \pi T^2) \\ &= \pi T^2 + 2a_\Gamma \end{aligned} \tag{7}$$

if $a_\Gamma := \mathrm{Area}(S_\Gamma)$ where $S_\Gamma$ is the least area disk bounded by $\Gamma$. Thus for all $T_0 < T_1 < T_2$, together with (5) and (7), we have,

$$D(X_R,\ \Lambda_{R,T_2} \setminus \Lambda_{R,T_1}) \leq \pi T_2^2 - \pi T_1^2 + 2a_\Gamma + 2\pi d_\Gamma^2. \tag{8}$$

Now consider a point $z_0 \in \partial\Lambda^\gamma_{R,T} \setminus \gamma \cup \bar{\gamma}$ such that

$$\mathrm{dist}_{\mathbf{C}}\ (z_0, \partial\Lambda^\gamma_{R,2T} \setminus \gamma \cup \bar{\gamma}) = r_0,$$

where $r_0$ is the distance between $\partial\Lambda^\gamma_{R,2T} \setminus \gamma \cup \bar{\gamma}$ and $\partial\Lambda^\gamma_{R,T} \setminus \gamma \cup \bar{\gamma}$. Denote

$$Z_R(r,\theta) := X_R(z_0 + re^{i\theta}),$$

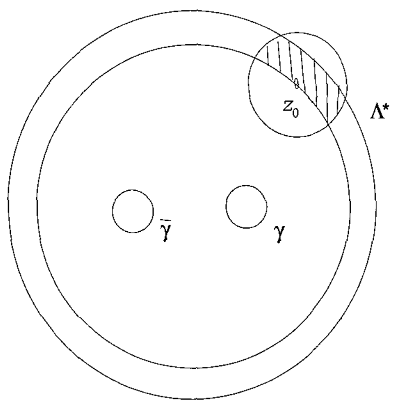


FIGURE 2.

where $r, \theta$ denote polar coordinates about $z_0$. If $r_0 < 1$, then by (2) there exists $\rho \in (r_0, \sqrt{r_0})$ such that

$$\int_{\partial^*} \left| \frac{\partial Z}{\partial \theta}(\rho, \theta) \right| d\theta \leq \left\{ \frac{4\pi}{\log 1/r_0} D(X_R, \Lambda^*) \right\}^{1/2}, \tag{9}$$

where

$$\Lambda^* := B_{\sqrt{r_0}}(z_0) \cap (\Lambda^{\gamma}_{R,2T} \setminus \Lambda_{R,T}),$$
$$\partial^* := \partial B_\rho(z_0) \cap \left(\Lambda^{\gamma}_{R,2T} \setminus \Lambda_{R,T}\right).$$

Notice, since $r_0 < \rho$ we can find an arc $\zeta \subset \partial^*$ connecting $\partial \Lambda^{\gamma}_{R,2T} \setminus \gamma \cup \bar\gamma$ and $\partial \Lambda_{R,T} \setminus \gamma \cup \bar\gamma$. So the length of $X_R(\zeta)$ must be larger than $T$. Since the length of $X_R(\zeta)$ is less than that of $X_R(\partial^*)$, together with (8) and (9), we have

$$T^2 < \frac{4\pi}{\log 1/r_0} \left(4\pi T^2 - \pi T^2 + 2a_\Gamma + 2\pi d_\Gamma^2\right)$$

as well as

$$\begin{aligned} r_0 &\geq \exp\left(-4\pi T^{-2}(3\pi T^2 + 2a_\Gamma + 2\pi d_\Gamma^2)\right) \\ &= \exp\left(-12\pi^2 - 4\pi(2a_\Gamma + 2\pi d_\Gamma^2)T^{-2}\right) \\ &\geq \exp(-8\pi a_\Gamma - 20\pi^2). \end{aligned}$$

Observe this inequality also holds if $r_0 \geq 1$. Take a number $N > 0$ such that

$$2^{-N} T = T_0 + 1 + \epsilon, \quad 0 < \epsilon < 1,$$

then

$$
\begin{aligned}
&\operatorname{dist}_{\mathbf{C}}\ (0,\ \partial\Lambda_{R,T}\setminus\gamma\cup\bar{\gamma})\\
&= \sum_{k=1}^{N}\operatorname{dist}_{\mathbf{C}}\left(\partial\Lambda^{\gamma}_{R,2^{-k}T}\setminus\gamma\cup\bar{\gamma},\ \partial\Lambda^{\gamma}_{R,2^{1-k}T}\setminus\gamma\cup\bar{\gamma}\right)\\
&\geq N\exp(-8\pi a_{\Gamma}-20\pi^2)\\
&\geq c\ \log T
\end{aligned}
\tag{10}
$$

for some constant $c > 0$ satisfying

$$N = \log_2 T - \log_2(T_0 + 1 + \epsilon) \geq\ c\ \exp(8\pi a_{\Gamma} + 20\pi^2)\log T.$$

Notice that we can take $c$ independently with the value $T$. In particular,

$$r(R) = \operatorname{dist}_{\mathbf{C}}\ (0,\ \partial\Lambda_{R,R}\setminus\gamma\cup\bar{\gamma}) \geq\ c\ \log R.$$

Take $3 < r \leq r(R)$, and let $T$ satisfy that

$$\operatorname{dist}_{\mathbf{C}}\ (0,\ \partial\Lambda_{R,T}\setminus\gamma\cup\bar{\gamma}) = r.$$

Then $X_R(\Omega_r) \subset C_T$ clearly, so

$$
\begin{aligned}
D(X_R,\Omega_r) &\leq D(X_R,\Lambda_{R,T})\\
&\leq \pi T^2 + 2a_{\Gamma}\\
&\leq \pi e^{2r/c} + 2a_{\Gamma}
\end{aligned}
\tag{11}
$$

by (10). It leads us to prove that there is a subsequence of $X_{R_k}$ and a conformal harmonic map $X \in C^0(\Omega,\mathbf{R}^3)\cap C^{\omega}(\mathrm{Int}(\Omega),\mathbf{R}^3)$ with

$$\Omega := \mathbf{C}\setminus\overline{D_{\gamma}\cup D_{\bar{\gamma}}}$$

such that

- $X_{R_k}$ converges to $X$ uniformly on every $\Omega_r$.
- $X_{R_k}$ converges to $X$ weakly in each $H^1_2(\Omega_a,\mathbf{R}^3)$.
- $X_{R_k}$ converges to $X$ locally smoothly in $\mathrm{Int}(\Omega)$.

Claim 2. *The sequence of minimal immersions $X_{R_k}$ is equicontinuous on the boundary curves $\partial\Omega = \gamma\cup\bar{\gamma}$.*

Let $M := \pi e^{10/c} + 2a_{\Gamma}$ as in (11), then for all $r_k < 5$ we have

$$D(X_{R_k}, B_{r_k}\cap\Omega_r) \leq M. \tag{12}$$

Recall the set of surfaces $X_{R_k}$ satisfy the *three-point condition* for $\gamma$ if

$$X_{R_k}(w_j) = Q_j,\quad j = 1, 2, 3, \tag{13}$$

for some $w_1, w_2, w_3 \in \gamma$ and $Q_1, Q_2, Q_3 \in \Gamma$. The condition (3) of the Courant-Lebesgue lemma can be applied as follows: Since $\Gamma$ is the topological image of a unit circle $C := \{z \in \mathbf{C} : |z| = 1\}$ for every $\epsilon_k > 0$ there exists a number $\lambda(\epsilon_k) > 0$ with the following property: Any pair of points $P, Q \in \Gamma$ with

$$0 < |P - Q| < \lambda(\epsilon_k)$$

decompose $\Gamma$ into two arcs $\Gamma_1(P, Q)$ and $\Gamma_2(P, Q)$ such that

$$\text{length}(\Gamma_1(P, Q)) < \epsilon_k$$

holds. Hence, if $0 < \epsilon_k < e := \min_{j \neq k} |Q_j - Q_k|$, $j, k = 1, 2, 3$, then $\Gamma_1(P, Q)$ can contain at most one of the points $Q_1, Q_2, Q_3$ appearing in (13). Let $d \in (0, 1)$ be a fixed number with

$$2\sqrt{d} < \min_{j \neq k} |w_j - w_k|,$$

where $w_1, w_2, w_3$ appear in (13). For an arbitrary $\epsilon_k \in (0, e)$, we choose some number $\delta_k = \delta_k(\epsilon_k)$ such that

$$\left\{ \frac{4\pi M}{\log 1/\delta_k} \right\}^{1/2} < \lambda(\epsilon_k)$$

and $\delta_k < d$. Consider an arbitrary point $z_1$ on $\gamma$, and let $\rho \in (\delta_k, \sqrt{\delta_k})$ be some number such that the images $P := X_{R_k}(z)$, $Q := X_{R_k}(z')$ of the two intersection points $z, z' \in \gamma$ and $\partial B_\rho(z_1)$ satisfy

$$|P - Q| \leq \left\{ \frac{4\pi M}{\log 1/\delta_k} \right\}^{1/2}.$$

Then we infer that $|P - Q| < \lambda(\epsilon_k)$, whence $\text{length}(\Gamma_1(P, Q)) < \epsilon_k$ holds. Because of $\epsilon_k < e$ the arc $\Gamma_1(P, Q)$ contains at most one of the points $Q_1, Q_2, Q_3$. On the other hand, if the sequence $X_{R_k}$ satisfy the three-point condition then $X_{R_k}(\gamma \cap \overline{B_\rho(z_1)}$ contains at most one of the points $Q_j$ and must therefore coincide with the arc $\Gamma_1(P, Q)$.

On the contrary, we assume that $X_{R_k}$ are not equicontinuous on $\gamma$. Then, together with the condition (12), we can say that they do not satisfy the three-point condition, see Lemma 3.2 in [1]. Observe then we can take a point $w_0 \in \gamma$ and $r_k \in (\delta_k, \sqrt{\delta_k})$ such that

$$\Gamma_1(P, Q) \neq X_{R_k}(\gamma \cap \overline{B_{r_k}(w_0)})$$

in other words, the complementary arc

$$\Gamma_k^* := \Gamma \setminus X_{R_k}(\gamma \cap \overline{B_{r_k}(z)})$$

has the smaller length $\epsilon_k$. Let $\epsilon_k \to 0$ as $k \to \infty$, then it follows that

$$\lim_{k \to \infty} \text{length}(\Gamma_k^*) = 0.$$

And let $\delta_k \to 0$ as $k \to \infty$, too. Then, via the Courant-Lebesgue lemma again, we have

$$\begin{aligned} \text{length}\,(X_{R_k}(\partial B_{r_k}(z) \cap \Omega_r)) &= \int_{\partial B_{r_k}(z) \cap \Omega_r} \left| \frac{\partial X_{R_k}}{\partial \theta} \right| d\theta \\ &\leq \left\{ \frac{4\pi M}{\log 1/\sqrt{\delta_k}} \right\}^{1/2} \\ &\to 0 \quad \text{as} \quad k \to \infty. \end{aligned}$$

Notice that together with the arc $\partial B_{r_k}(z) \cap \Omega$ and a simple regular arc $\xi_{r_k} \subset \text{Int}(\Omega) \setminus B_{r_k}(z)$ which lies within short distance to $\gamma$ such that

$$\lim_{k \to \infty} \text{length}\ (X_{R_k}(\zeta_{r_k})) = \lim_{k \to \infty} \text{length}\ (\Gamma_k^*) = 0$$

and rounding off corners, one easily constructs a closed regular Jordan curve $\eta_k \subset \text{Int}(\Omega_{r(R)})$ satisfying

1. The length of image's $X_{R_k}(\eta_k)$ tends to zero as $k \to \infty$.
2. $\eta_k$ cuts the domain $\Omega_{r(R_k)}$ into two annular regions.

Now we can apply the cut-paste argument to obtain disk type surfaces $A_k^1$ and $A_k^2$ such that the first surface bounds $\Gamma$ and the last one bounds $\Gamma_{R_k}^+$ such that

$$\lim_{k \to \infty} (\text{Area}(A_k^1) + \text{Area}(A_k^2) - \text{Area}(A_{R_k}^+)) = 0$$

contradicts to Lemma 1, however. It leads us that the sequence $X_{R_k}$ must be equicontinuous on $\gamma$, and so on $\bar{\gamma}$ for the symmetricity. Therefore, the limit curve $X(\gamma)$ is a topological parametrization of $\Gamma$. Let us denote

$$A := X(\Omega).$$

Since $(A_{R_k} \cap \Pi) \setminus (\Gamma \cup \bar{\Gamma})$ tends to the straight line $L$ as $k \to \infty$, the limit surface $A$ also contains $L$ in its interior and then rotational symmetric under the line. Therefore we can take a minimal annulus $A^+ \subset A$ bounded by $\Gamma$ and $L$, it finished the proof of the main theorem of this article.

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