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CONTINUITY OF AN APPROXIMATE JORDAN MAPPING

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Abstract. We show that every ε-approximate Jordan functional on a Banach algebra A is continuous. From this result we obtain that every ε-approximate Jordan mapping from A into a continuous function space $C(S)$ is continuous and it's norm less than or equal $1+\varepsilon$ where S is a compact Hausdorff space. This is a generalization of Jarosz's result [3, Proposition 5.5].

1. Introduction

A linear mapping f from a normed algebra A into a normed algebra B is an *ε-homomorphism* if for every a, b in A

$$\|f(ab) - f(a)f(b)\| \leq \varepsilon\|a\|\|b\|.$$

In [3, Proposition 5.5], Jarosz proved that every ε-homomorphism from a Banach algebra into a continuous function space $C(S)$ is necessarily continuous, where S is a compact Hausdorff space. A Jordan functional on a Banach algebra A is a nonzero linear functional ϕ such that $\phi(a^2) = \phi(a)^2$ for every a in A. It is well known that every Jordan functional ϕ on A is multiplicative ([2]). Consider the linear mapping f on Banach algebras which are approximately Jordan mappings in the sense that $\|f(a^2) - f(a)^2\|$ is small in norm. A linear mapping f from a normed algebra A into a normed algebra B is called an *ε-approximate Jordan mapping* if for all a in A

$$\|f(a^2) - f(a)^2\| \leq \varepsilon\|a\|^2.$$

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If B is the complex field, then f is called an *ε-approximate Jordan functional.* Also a continuous linear mapping f between normed algebras is an *ε-near Jordan mapping* if $\|f - J\| < \varepsilon$ for some continuous Jordan mapping J as in [4]. It is similar to stability problem ([1]). We show that every approximate Jordan functional on a Banach algebra is continuous and it has norm less than or equal $1 + \varepsilon$ from it.

2. Main results

The author and G. H. Kim[5] showed that if f is an approximate Jordan functional on a Banach algebra A with the multiplicative norm, then f is continuous. In this paper we prove the continuity of f without this condition.

THEOREM 2.1. *Let $0 < \varepsilon < \frac{1}{2}$ be given. Then an ε-approximate Jordan functional f on a Banach algebra A is continuous and $\|f\| \leq 1 + \varepsilon$.*

PROOF. If A does not possess a unit, then we can extended f to $A \oplus (\lambda e)$ by putting, $f(a + \lambda e) = f(a) + \lambda$ and the extended f is still an ε-approximate Jordan functional. Thus without loss of generality we may assume that A has a unit. Suppose that f is discontinuous. Then the kernel Ker(f) of f is a dense subset of A. Since the unit element 1 is the closure of Ker(f) , we can choose $c \in$ Ker(f) such that $\|c - 1\| < \frac{1}{3}$. Then c is invertible, and $c^{-1} = 1 + \sum_{n=1}^{\infty}(1 - c)^n$. And so $\|c^{-1}\| \leq \frac{1}{1-\|c-1\|} \leq \frac{3}{2}$. Let $b = \frac{c}{\|c\|} \in$ Ker(f). Then $b^{-1} = \|c\|c^{-1}$ and $\|b^{-1}\| \leq 2$. Put $|f(b^{-1})| = \alpha$. Note that for every $x, y \in A$

$$|f(xy + yx) - 2f(x)f(y)| \leq 2\varepsilon(\|x\|^2 + \|x\|\|y\| + \|y\|^2).$$

If b^{-1} is not in Ker(f), then for every a in A with $\|a\| = 1$,

$$\begin{aligned}
|f(a)| &= \frac{1}{2\alpha}|2f(a)f(b^{-1})| \\
&\leq \frac{1}{2\alpha}(|2f(a)f(b^{-1}) - f(ab^{-1} + b^{-1}a)| \\
&\quad + |f(bb^{-1}ab^{-1} + b^{-1}ab^{-1}b) - 2f(b^{-1}ab^{-1})f(b)|) \\
&\leq \frac{28\varepsilon}{\alpha}.
\end{aligned}$$

Thus f is bounded. Note that

$$\begin{aligned}2|f(bab)| &\le |f(bab+b^2a)| + |f(bab+ab^2)| + |f(b^2a+ab^2)| \\ &\le |f(bab+b^2a) - 2f(ba)f(b)| \\ &\quad + |f(bab+ab^2) - 2f(b)f(ab)| \\ &\quad + |f(b^2a+ab^2) - 2f(a)f(b^2)| + 2|f(a)||f(b^2)|,\end{aligned}$$

and

$$|f(b^2)| = |f(b^2) - f(b)^2| \le \varepsilon\|b\|^2 = \varepsilon.$$

If b^{-1} is in $\operatorname{Ker}(f)$, then

$$|f(b^{-2})| = |f(b^{-2}) - f(b^{-1})^2| \le \varepsilon\|b^{-1}\|^2 \le 4\varepsilon$$

and for every a in A with $\|a\| = 1$,

$$\begin{aligned}|f(a)| &\le \frac{1}{2}(|f(a+b^{-1}ab)| + |f(a+bab^{-1})| + |f(b^{-1}ab+bab^{-1})|) \\ &\le \frac{1}{2}(|f(a+b^{-1}ab) - 2f(b^{-1}a)f(b)| \\ &\quad + |f(a+bab^{-1}) - 2f(ab^{-1})f(b)| \\ &\quad + |f(b^{-1}ab+bab^{-1}) - 2f(bab)f(b^{-2})|) + |f(bab)||f(b^{-2})| \\ &\le 35\varepsilon + |f(bab)||f(b^{-2})| \\ &\le 35\varepsilon + (9\varepsilon + |f(a)|\varepsilon)4\varepsilon \\ &\le 35\varepsilon + 36\varepsilon^2 + 4|f(a)|\varepsilon^2.\end{aligned}$$

Since $\varepsilon < \frac{1}{2}$, we have

$$|f(a)| \le \frac{35\varepsilon + 36\varepsilon^2}{1 - 4\varepsilon^2}$$

for all $a \in A$ with $\|a\| = 1$. Thus f is bounded. This shows that f is bounded at any cases either $b^{-1} \in Ker(f)$ or not. Therefore f is continuous.

Now for every x in A with $\|x\| \le 1$ we have

$$|f(x^2) - f(x)^2| < \varepsilon.$$

Thus

$$|f(x)|^2 - \varepsilon \leq |f(x^2)| \leq \|f\|$$

and consequently

$$\|f\| \geq \|f\|^2 - \varepsilon.$$

This proves $\|f\| \leq 1 + \varepsilon$. □

COROLLARY 2.2. *Let S be a compact Hausdorff space and $C(S)$ the set of all continuous complete valued functions and $0 < \varepsilon < \frac{1}{2}$ be given. If f is an ε-approximate Jordan mapping from a Banach algebra A into $C(S)$, then f is continuous and $\|f\| \leq 1 + \varepsilon$.*

PROOF. For each x in S we define a linear mapping $f_x : A \to C$ by $f_x(a) = f(a)(x)$. Then f_x is ε-approximate Jordan functional on A. By Theorem 2.1, $\|f_x\| \leq 1 + \varepsilon$. Thus

$$\|f(a)\| = \sup_{x \in S} \|f(a)(x)\| = \sup_{x \in S} \|f_x(a)\| \leq (1 + \varepsilon)\|a\|$$

and so $\|f\| \leq 1 + \varepsilon$. □

COROLLARY 2.3. [3, PROPOSITION 5.5] *Let $0 < \varepsilon < \frac{1}{2}$ be given. If f is an ε-homomorphism from a Banach algebra A into $C(S)$, then f is continuous and $\|f\| \leq 1 + \varepsilon$.*

PROOF. Since every ε-homomorphism is an ε-approximate Jordan mapping, we complete the proof from Corollary 2.2. □

COROLLARY 2.4. *Let $0 < \varepsilon < \frac{1}{2}$ be given. Every ε-approximate Jordan mapping from a Banach algebra A into $C(S)$ is a $(1+\varepsilon)$-near Jordan mapping.*

PROOF. Let f be a ε-Jordan mapping from a Banach algebra A into $C(S)$. By Corollary 2.2, f is continuous. If f is a Jordan mapping let $f = h$, and if not let $h = 0$. Then h is a continuous Jordan mapping such that $\| f - h \| \leq 1 + \varepsilon$. □

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