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STRICTLY INFINITESIMALLY GENERATED TOTALLY POSITIVE MATRICES

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ABSTRACT. Let G be a Lie group, let $L(G)$ be its Lie algebra, and let $\exp : L(G) \to G$ denote the exponential mapping. For $S \subseteq G$, we define the *tangent set* of S by $L(S) = \{X \in L(G) : \exp(tX) \in S \text{ for all } t \geq 0\}$. We say that a semigroup S is *strictly infinitesimally generated* if S is the same as the semigroup generated by $\exp(L(S))$. We find a tangent set of the semigroup of all non-singular totally positive matrices and show that the semigroup is strictly infinitesimally generated by the tangent set of the semigroup. This generalizes the familiar relationships between connected Lie subgroups of G and their Lie algebras

1. Introduction

Let G be a Lie group, let $L(G)$ be its Lie algebra, and let exp: $L(G) \to G$ denote the exponential mapping. For $S \subseteq G$, we define the *tangent set* of S by $L(S) = \{X \in L(G) : \exp(tX) \in S \text{ for all } t \geq 0\}$. Finding the tangent sets of given semigroups may be considered as a generalization of taking derivatives. We say that a semigroup S is *strictly infinitesimally generated* if S is the same as the semigroup generated by $\exp(L(S))$. Computing the strictly infinitesimally generated semigroups may be considered as a generalization of finding integrands. It is standard that if S is a connected Lie subgroup of G, then $L(S)$ is a Lie subalgebra of $L(G)$ and every element of S is written as a finite product of exponentials of elements of $L(S)$, that is, S is the same as the group generated by $\exp(L(S))$. Thus characterizing the strictly infinitesimally generated semigroups may be considered as a generalization of

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the relationships between connected Lie subgroups of G and their Lie algebras.

In Section 2, we find a tangent set of the semigroup of all non-singular totally positive matrices. In Section 3, we show that the semigroup of all non-singular totally positive matrices is strictly infinitesimally generated by factorizing $n \times n$ non-singular totally positive matrix as finite products of exponentials of Jacobi matrices with non-negative off-diagonal elements.

2. Tangent cone of the semigroup of non-singular totally positive matrices

Let V denote a Banach space, $gl(V)$ the space of continuous linear transformations from V into V, and $GL(V)$ the space of all invertible continuous linear transformations from V into V. We consider the special case of $G = GL(V)$ and $L(G) = gl(V)$. It is well known that if S is a closed subgroup of $GL(V)$, then its Lie algebra arises as the set $\{X \in gl(V) : \exp(tX) \in S \text{ for all } t \geq 0\}$. This motivates the following definition.

DEFINITION 2.1. For $S \subseteq GL(V)$, let $L(S) = \{X \in gl(V) : \exp(tX) \in S \text{ for all } t \geq 0\}$. $L(S)$ is called the *tangent set* of S.

DEFINITION 2.2. A subset W of a real topological vector space V is called a *wedge* or a *cone* if it satisfies the following conditions:

(1) $W + W \subseteq W$
(2) $\mathbb{R}^+ W \subseteq W$
(3) W is closed in V,

where $\mathbb{R}^+$ denotes the set of non-negative real numbers.

PROPOSITION 2.3. *If S is a closed semigroup of $GL(V)$, then $L(S)$ is a closed cone.*

PROOF. Straightforward from a standard fact(see [5]) that for $A,\ B \in gl(V)$, $\exp(A+B) = \lim_{n\to\infty} \left(\exp(\frac{A}{n})\exp(\frac{B}{n})\right)^n$. □

Given a closed semigroup S, computing $W = L(S)$ may be thought of the generalization of "taking the derivative". Let $\mathbb{R}$ denote the set of all real numbers. In the following we consider matrices with entries from $\mathbb{R}$. In the case V is an n-dimensional vector space over $\mathbb{R}$ with a fixed basis, we identify $gl(V)$ with the set of all $n \times n$ matrices over $\mathbb{R}$.

NOTATION. We will denote the determinant formed from elements of the given matrix $A = \|a_{ik}\|$ $(i = 1, 2, \ldots, m; k = 1, 2, \ldots, n)$ as follows:

$$A\begin{pmatrix} i_1 & i_2 & \ldots & i_p \\ k_1 & k_2 & \ldots & k_p \end{pmatrix} = \det \begin{vmatrix} a_{i_1k_1} & a_{i_1k_2} & \cdots & a_{i_1k_p} \\ a_{i_2k_1} & a_{i_2k_2} & \cdots & a_{i_2k_p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_pk_1} & a_{i_pk_2} & \cdots & a_{i_pk_p} \end{vmatrix}.$$

DEFINITION 2.4. A rectangular matrix $A = \|a_{ik}\|$ $(i = 1, 2, \ldots m : k = 1, 2, \ldots, n)$ is called *totally positive*—hereafter denoted by TP— if all its minors of any order are nonnegative. An $n \times n$ matrix is called totally positive of order m and is denoted by TP_m if all its minors of order $j \leq m$ are non-negative. The square matrix $A = \|a_{ij}\|$ is called a *Jacobi matrix* if all elements outside the main diagonal and the first super-diagonal and sub-diagonal are zero. Let $A = \|a_{ij}\|$ be an $n \times n$ square matrix. The *principal minors* of A are the scalars of the form

$$A\begin{pmatrix} i_1 & i_2 & \ldots & i_p \\ j_1 & j_2 & \ldots & j_p \end{pmatrix} \text{ for } 1 \leq i_1 = j_1 < i_2 = j_2 < \cdots < i_p = j_p \leq n.$$

It is easy to see that the set of all TP matrices forms a closed semigroup from the Binet-Cauchy formula(see [4]).

THEOREM 2.5. *Let S be the semigroup of all invertible TP matrices and W be the set of all Jacobi matrices with nonnegative off-diagonal elements, where off-diagonal element is an element other than the main diagonal elements. Then $L(S) = W$.*

PROOF. By Proposition 2.3, $L(S)$ is a cone. Let $A=\exp(tE_{ij})$ for $|i - j|=1$, where E_{ij} is a matrix such that (ii)-th entry is -1, (ij)-th entry is 1, and the other entries are 0. Then A is a Jacobi matrix such that all its entries outside the main diagonal except for (ij)-th entry (which is $1 - e^{-t}$) are 0 and all its main diagonal entries except for (ii)-th entry (which is e^{-t}) are 1. For $1 \leq i_1 = j_1 < \cdots < i_p = j_p \leq n$,

$$A\begin{pmatrix} i_1 & i_2 & \ldots & i_p \\ j_1 & j_2 & \ldots & j_p \end{pmatrix}$$

is the determinant of the triangular matrix whose main diagonal elements are from the main diagonal elements of A. Since every main diagonal element of A is positive, principal minors of A are positive. Since a

Jacobi matrix is TP if and only if all its elements and principal minors are non-negative(see p.100 of [2, vol.2]), A is TP. Thus $E_{ij} \in L(S)$. Let E_k be the matrix such that all its entries are 0 except for the (kk)-th entry which is equal to 1. Then $\exp(tE_k)$ is the matrix such that all its entries outside the main diagonal are 0. Clearly $\exp(tE_k)$ is TP. Similarly $\exp(t(-E_k))$ is TP. Thus $E_k,\ -E_k \in L(S)$. Since $L(S)$ is a cone,

$$\sum_{1\le i,j\le n|i-j|=1} a_{ij}E_{ij} + \sum_{k=1}^{n} b_k E_k - \sum_{k=1}^{n} c_k E_k \in L(S) \text{ for } a_{ij},\ b_k,\ c_k \ge 0.$$

Thus $W \subseteq L(S)$.

Conversely, let $A \in L(S)$. Then $\exp(tA)$ is TP for all $t \ge 0$. Let

$$H(t) = \exp(tA) = \begin{pmatrix} h_{11}(t) & h_{12}(t) & \dots & h_{1n}(t) \\ h_{21}(t) & h_{22}(t) & \dots & h_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1}(t) & h_{n2}(t) & \dots & h_{nn}(t) \end{pmatrix}.$$

Then

$$A = \frac{d}{dt}e^{tA}\Big|_{t=0} = \lim_{t\to 0^+} \frac{e^{tA}-I}{t} = \begin{pmatrix} h'_{11}(0) & h'_{12}(0) & \dots & h'_{1n}(0) \\ h'_{21}(0) & h'_{22}(0) & \dots & h'_{2n}(0) \\ \vdots & \vdots & \ddots & \vdots \\ h'_{n1}(0) & h'_{n2}(0) & \dots & h'_{nn}(0) \end{pmatrix}.$$

If $j > i+1$,

$$\begin{aligned} A_{ij}(t) &= \begin{vmatrix} h_{i,i+1}(t) & h_{ij}(t) \\ h_{i+1,i+1}(t) & h_{i+1,j}(t) \end{vmatrix} \\ &= h_{i,i+1}(t)h_{i+1,j}(t) - h_{ij}(t)h_{i+1,i+1}(t) \ge 0 \text{ for all } t \ge 0. \end{aligned}$$

Since $h_{pq}(0) = 0$ for $p \neq q$, $A_{ij}(0) = 0$. Thus

$$A'_{ij}(0) = \lim_{t\to 0^+} \frac{A_{ij}(t) - A_{ij}(0)}{t} \ge 0.$$

Hence $A'_{ij}(0) = -h'_{ij}(0) \ge 0$. Since $h_{ij}(t) \ge 0$ and $h_{ij}(0) = 0$, $h'_{ij}(0) \ge 0$. Thus $h'_{ij}(0) = 0$ for $j > i+1$ and $i = 1, 2, \dots, n-2$. Similarly $h'_{ij}(0) = 0$ for $j < i-1$ and i=3, $\dots, n$. Since $h_{pq}(0) = 0$ and $h_{pq}(t) \ge 0$ for $p \neq q$ and $t \ge 0$, $h'_{pq}(0) \ge 0$ for $p \neq q$. Thus A is a Jacobi matrix with nonnegative off-diagonal elements. That is, $L(S) \subseteq W$. □

3. Strictly infinitesimally generated non-singular totally positive matrices

LEMMA 3.1. *If an $n \times n$ nonsingular matrix $A = \|a_{ij}\|$ $(n \geq m)$ has all its minors of order $m-1$ nonnegative and has all its minors of order m which come from consecutive rows nonnegative, then all mth order minors are nonnegative.*

PROOF. See [1]. □

LEMMA 3.2. *Let $V = \|v_{ij}\|$ be an $n \times n$ nonsingular TP matrix. Then $v_{11} > 0,\ v_{21} > 0, \ldots, v_{p1} > 0, v_{p+1,1} = \cdots = v_{n1} = 0$ for some p.*

PROOF. See [6]. □

LEMMA 3.3. *Let $W = \|w_{ij}\|$ be an $n \times n$ non-singular TP matrix and set $l = \max\{k : w_{k1} > 0\}$. Assume*

$$W_{l-1,1} = \begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{l-1,1} & w_{l-1,2} & \cdots & w_{l-1,n} \\ u_1 w_{l1} - u_2 w_{l-1,1} & u_1 w_{l2} - u_2 w_{l-1,2} & \cdots & u_1 w_{ln} - u_2 w_{l-1,n} \\ w_{l+1,1} & w_{l+1,2} & \cdots & w_{l+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{pmatrix}$$

(1) *If $w_{l-1,1} < w_{l1}$, $u_1 = \frac{w_{l-1,1}}{w_{l1} - w_{l-1,1}}$, and $u_2 = \frac{w_{l1}}{w_{l1} - w_{l-1,1}}$, then $W_{l-1,1}$ is TP and non-singular.*
(2) *If $w_{l-1,1} > w_{l1}$, $u_1 = \frac{w_{l-1,1}}{w_{l-1,1} - w_{l1}}$, and $u_2 = \frac{w_{l1}}{w_{l-1,1} - w_{l1}}$, then $W_{l-1,1}$ is TP and non-singular.*
(3) *If $w_{l-1,1} = w_{l1}$ and $u_1 = u_2 = 1$, then $W_{l-1,1}$ is TP and non-singular.*

PROOF. $w_{11} > 0, \ldots, w_{l-1,1} > 0$, and $w_{l1} > 0$ by Lemma 3.2.
(1) Consider the following three cases (a), (b), and (c).

(a) Suppose mth order minor of $W_{l-1,1}$ which comes from consecutive rows contains the lth row of $W_{l-1,1}$ between its first row and its last row.

Suppose $p+k=l<p+m-1\leq n$. Then

$$\begin{aligned}
&W_{l-1,1}\begin{pmatrix} p & p+1 & \dots & p+k-1 & p+k & \dots & p+m-1 \\ j_1 & j_2 & \dots & j_k & j_{k+1} & \dots & j_m \end{pmatrix}\\
&= u_1 W\begin{pmatrix} p & p+1 & \dots & l-1 & l & \dots & p+m-1 \\ j_1 & j_2 & \dots & j_k & j_{k+1} & \dots & j_m \end{pmatrix}\\
&\quad - u_2 W\begin{pmatrix} p & p+1 & \dots & l-1 & l-1 & \dots & p+m-1 \\ j_1 & j_2 & \dots & j_k & j_{k+1} & \dots & j_m \end{pmatrix}\\
&\geq 0,
\end{aligned}$$

since the second determinant equals to 0. Thus all mth order minors of $W_{l-1,1}$, which come from consecutive rows and contain the lth row of $W_{l-1,1}$ between its first row and its last row, are nonnegative.

(b) Suppose mth order minor of $W_{l-1,1}$ which comes from consecutive rows contains the lth row of $W_{l-1,1}$ as its first row.

$$\begin{aligned}
&W_{l-1,1}\begin{pmatrix} l & l+1 & \dots & l+m-1 \\ j_1 & j_2 & \dots & j_m \end{pmatrix}\\
&= u_1 W\begin{pmatrix} l & l+1 & \dots & l+m-1 \\ j_1 & j_2 & \dots & j_m \end{pmatrix}\\
&\quad - u_2 W\begin{pmatrix} l-1 & l+1 & \dots & l+m-1 \\ j_1 & j_2 & \dots & j_m \end{pmatrix}\\
&= \begin{vmatrix} u_1 & w_{l-1,j_1} & w_{l-1,j_2} & \dots & w_{l-1,j_m} \\ u_2 & w_{l,j_1} & w_{l,j_2} & \dots & w_{l,j_m} \\ 0 & w_{l+1,j_1} & w_{l+1,j_2} & \dots & w_{l+1,j_m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & w_{l+m-1,j_1} & w_{l+m-1,j_2} & \dots & w_{l+m-1,j_m} \end{vmatrix}\\
&= \begin{cases} \frac{1}{w_{l1}-w_{l-1,1}} W\begin{pmatrix} l-1 & l & \dots & l+m-1 \\ 1 & j_1 & \dots & j_m \end{pmatrix} \geq 0, & \text{if } j_1>1, \\ 0, & \text{if } j_1=1. \end{cases}
\end{aligned}$$

Thus all mth order minors of $W_{l-1,1}$, which come from consecutive rows and contain the lth row of $W_{l-1,1}$ as its first row, are nonnegative.

(c) Suppose mth order minor of $W_{l-1,1}$ which comes from consecutive rows contains the lth row of $W_{l-1,1}$ as its last row.
Similarly we may show that all mth order minors of $W_{l-1,1}$, which come from consecutive rows and which contain the lth row of $W_{l-1,1}$ as its last row, are nonnegative.

From (a), (b), and (c), $W_{l-1,1} \in TP_m$ based on the consecutive rows. Since

$$w_{l-1,1}w_{li} - w_{l1}w_{l-1,i} = W\begin{pmatrix} l-1 & l \\ 1 & i \end{pmatrix} \geq 0,$$

$$u_1 w_{li} - u_2 w_{l-1,i} = \frac{1}{w_{l1} - w_{l-1,1}}(w_{l-1,1}w_{li} - w_{l1}w_{l-1,i}) \geq 0,$$

and hence $W_{l-1,1}$ is TP_1. Suppose $W_{l-1,1}$ is TP_{m-1}. According to the Lemma 3.1, $W_{l-1,1} \in TP_m$. By induction on the order of its minors, $W_{l-1,1} \in TP$. Since $W_{l-1,1}$ is row equivalent to W and W is non-singular, $W_{l-1,1}$ is non-singular.
(2) The proof is analogous to that of part (1).
(3) The proof is analogous to that of part (1). □

LEMMA 3.4. *Suppose an $n \times n$ matrix*

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ a_{l-1,1} & a_{l-1,2} & \cdots & a_{l-1,l-1} & a_{ll} & \cdots & 0 \\ 0 & a_{l2} & \cdots & a_{l,l-1} & a_{ll} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{n,l-1} & a_{nl} & \cdots & a_{nn} \end{pmatrix}$$

is non-singular and TP, where $l-1 = \max\{k : a_{k1} > 0\}$.
Then

$$A_1 = \begin{pmatrix} a_{11} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ a_{l-1,1} & a_{l-1,2} - a_{l2} & \cdots & a_{l-1,l-1} - a_{l,l-1} & 0 & \cdots & 0 \\ 0 & a_{l2} & \cdots & a_{l,l-1} & a_{ll} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{n,l-1} & a_{nl} & \cdots & a_{nn} \end{pmatrix}$$

is non-singular and TP if all elements of the $l-1$th row of A_1 are nonnegative.

PROOF. Clearly, A_1 is TP_1. Suppose A_1 is TP_{m-1}. Consider the following three cases.

(a) Suppose mth order minor of A_1 contains the $l-1$th row of A_1 between its first row and its last row.

$$\begin{aligned}
&A_1\begin{pmatrix} p & p+1 & \dots & l-1 & l & \dots & p+m-1 \\ j_1 & j_2 & \cdots & j_l & j_{l+1} & \cdots & j_m \end{pmatrix} \\
&= A\begin{pmatrix} p & p+1 & \dots & l-1 & l & \dots & p+m-1 \\ j_1 & j_2 & \cdots & j_l & j_{l+1} & \cdots & j_m \end{pmatrix} \\
&- A\begin{pmatrix} p & p+1 & \dots & l & l & \dots & p+m-1 \\ j_1 & j_2 & \cdots & j_l & j_{l+1} & \cdots & j_m \end{pmatrix} \\
&\geq 0.
\end{aligned}$$

(b) Suppose mth order minor of A_1 contains the $l-1$th row of A_1 as its first row.

$$\begin{aligned}
&A_1\begin{pmatrix} l-1 & l & \dots & l+m-2 \\ j_1 & j_2 & \cdots & j_m \end{pmatrix} \\
&= A\begin{pmatrix} l-1 & l & \dots & l+m-2 \\ j_1 & j_2 & \cdots & j_m \end{pmatrix} - A\begin{pmatrix} l & l & \dots & l+m-2 \\ j_1 & j_2 & \cdots & j_m \end{pmatrix} \\
&\geq 0.
\end{aligned}$$

(c) Suppose mth order minor of A_1 contains the $l-1$th row of A_1 as its last row.

$$\begin{aligned}
&A_1\begin{pmatrix} l-m & \dots & l-1 \\ j_1 & \cdots & j_m \end{pmatrix} \\
&= A\begin{pmatrix} l-m & \dots & l-1 \\ j_1 & \cdots & j_m \end{pmatrix} - A\begin{pmatrix} l-m & \dots & l-2 & l \\ j_1 & \cdots & j_{m-1} & j_m \end{pmatrix} \\
&= \begin{vmatrix} a_{l-m,j_1} & \cdots & a_{l-m,j_m} & 0 \\ \vdots & \dots & \vdots & \vdots \\ a_{l-2,j_1} & \cdots & a_{l-2,j_m} & 0 \\ a_{l-1,j_1} & \cdots & a_{l-1,j_m} & 1 \\ a_{l,j_1} & \cdots & a_{l,j_m} & 1 \end{vmatrix} \\
&= \begin{cases} \frac{1}{a_{ll}} A\begin{pmatrix} l-m & \dots & l-2 & l-1 & l \\ j_1 & \cdots & j_{m-1} & j_m & l \end{pmatrix} \geq 0, & \text{if } j_m < l, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

From (a), (b), and (c), A_1 is TP_m based on the consecutive rows. According to the Lemma 3.1, $A_1 \in TP_m$. By induction on the order of its minors, A_1 is TP. Since the determinant of A_1 is equal to that of A, A_1 is non-singular. $\square$

LEMMA 3.5. *Let $C_1, C_2, \ldots, C_n$ be the columns of $n \times n$ non-singular upper triangular TP matrix $C = \|c_{ij}\| = (C_1, C_2, \ldots, C_n)$. Set $l = \max\{k : c_{1k} > 0\}$ and $C_{1,l-1} = (C_1, \ldots, p_1 C_{l-1}, C_l - p_2 C_{l-1}, C_{l+1}, \ldots, C_n)$ for $p_1 = \frac{c_{1,l-1}+c_{1l}}{c_{1,l-1}}$ and $p_2 = \frac{c_{1l}}{c_{1,l-1}}$. Then $C_{1,l-1}$ is non-singular and TP.*

PROOF. Straightforward from Lemma 3.3 and Lemma 3.4. $\square$

THEOREM 3.6. *If $W = \|w_{ij}\|$ is an $n \times n$ nonsingular TP matrix, then it can be represented as $W = B_1 B_2 \ldots B_{n-1} D C_1 C_2 \ldots C_{n-1}$, where B_i and C_i are of the form*

$$\begin{aligned} B_i &= \exp(J^i_{n,n-1}) \exp(J^i_{n-1,n-2}) \cdots \exp(J^i_{i+1,i}), \\ C_i &= \exp(H^i_{n-i,n-i+1}) \exp(H^i_{n-i+1,n-i+2}) \cdots \exp(H^i_{n-1,n}). \end{aligned}$$

Here D is an exponential of diagonal matrix, J^i_{kp} is the matrix such that its (kk)-th entry is a^i_{kk}, its (kp)-th entry is $a^i_{kp} \geq 0$, and other entries are 0, and H^j_{lq} is the matrix such that its (ll)-th entry is b^j_{ll}, its (lq)-th entry is $b^j_{lq} \geq 0$, and other entries are 0. That is, J^i_{kp} and H^j_{lq} are Jacobi matrices with nonnegative off-diagonal elements.

PROOF. Assume $w_{n1} > 0$. Then $w_{k1} > 0$ for $k = 1, 2, \ldots n-1$ by Lemma 3.2. Let $W_1, W_2, \ldots W_n$ be the rows in order of W. Let

$$W_{n-1,1} = \begin{pmatrix} W_1 \\ \vdots \\ W_{n-1} \\ q_1 W_n - q_2 W_{n-1} \end{pmatrix}, \quad W'_{n-1,1} = \begin{pmatrix} W_1 \\ \vdots \\ W_{n-1} \\ q'_1 W_n - q'_2 W_{n-1} \end{pmatrix},$$

$$W''_{n-1,1} = \begin{pmatrix} W_1 \\ \vdots \\ W_{n-1} \\ W_n - W_{n-1} \end{pmatrix},$$

where

$$\begin{aligned} q_1 &= \frac{w_{n-1,1}}{w_{n1} - w_{n-1,1}}, \quad & q_2 &= \frac{w_{n1}}{w_{n1} - w_{n-1,1}}, \\ q'_1 &= \frac{w_{n-1,1}}{w_{n-1,1} - w_{n1}}, \quad & q'_2 &= \frac{w_{n1}}{w_{n-1,1} - w_{n1}}. \end{aligned}$$

Consider the following three cases

(a) In case of $w_{n-1,1} < w_{n1}$.

Let

$$H_{k,k-1} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \frac{w_{k1}}{w_{k-1,1}} & \frac{w_{k1}}{w_{k-1,1}} - 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 1 \end{pmatrix}$$

for $k = 2, \dots, n$, where $\frac{w_{k1}}{w_{k-1,1}}$ and $\frac{w_{k1}}{w_{k-1,1}} - 1$ are in the kth row and $\frac{w_{k1}}{w_{k-1,1}} - 1$ is on the diagonal. That is, $H_{k,k-1} = E_1 + \cdots + E_{k-1} + \frac{w_{k1}}{w_{k-1,1}} E_{k,k-1} + (\frac{2w_{k1}}{w_{k-1,1}} - 1)E_k + E_{k+1} + \cdots + E_n$, where $E_{k,k-1}, E_k$ are matrices in Theorem 2.5. Then $W = H_{n,n-1}W_{n-1,1}$, where $W_{n-1,1}$ is the matrix in Lemma 3.3. Since $\frac{w_{n1}}{w_{n-1,1}} > 1$ and $\frac{w_{n1}}{w_{n-1,1}} - 1 > 0$, $H_{n,n-1} = \exp(J'_{n,n-1})$ for some Jacobi matrix $J'_{n,n-1}$ with nonnegative off-diagonal elements as denoted in the hypothesis. By Lemma 3.3, $W_{n-1,1}$ is TP and non-singular. Thus $W = \exp(J'_{n,n-1})W_{n-1,1}$, where $W_{n-1,1}$ is TP and non-singular.

(b) In case of $w_{n-1,1} > w_{n1}$.

Let

$$H'_{k,k-1} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \frac{w_{k1}}{w_{k-1,1}} & 1 - \frac{w_{k1}}{w_{k-1,1}} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 1 \end{pmatrix}$$

for $k = 2, \dots, n$, where $\frac{w_{k1}}{w_{k-1,1}}$ and $1 - \frac{w_{k1}}{w_{k-1,1}}$ are in the kth row. That is, $H'_{k,k-1} = E_1 + \cdots + E_{k-1} + \frac{w_{k1}}{w_{k-1,1}} E_{k,k-1} + E_k + E_{k+1} + \cdots + E_n$. Then $W = H'_{n,n-1}W'_{n-1,1}$. Since $0 < \frac{w_{n1}}{w_{n-1,1}} < 1$ and $0 < 1 - \frac{w_{n1}}{w_{n-1,1}}$, $H'_{nn-1} = \exp(J''_{nn-1})$ for some Jacobi matrix J''_{nn-1} with nonnegative off-diagonal elements. By Lemma 3.3, $W'_{n-1,n}$ is TP and non-singular. Thus $W = \exp(J''_{n,n-1})W'_{n-1,1}$, where $W'_{n-1,1}$ is TP and non-singular.

(c) In case of $w_{n-1,1} = w_{n1}$.

Let

$$G_{k,k-1} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 1 \end{pmatrix}$$

for $k = 2, \dots, n$, where 1 in the first sub-diagonal is in the kth row. That is, $G_{k,k-1} = E_1 + \cdots + E_{k-1} + E_{k,k-1} + 2E_k + E_{k+1} + \cdots + E_n$. Then $W = G_{n,n-1}W''_{n-1,1}$. Also, $G_{n,n-1} = \exp(J'''_{n,n-1})$ for some Jacobi matrix $J'''_{n,n-1}$ with nonnegative off-diagonal elements. By Lemma 3.3, $W''_{n-1,1}$ is TP and non-singular. Thus $W = \exp(J'''_{nn-1})W''_{n-1,1}$, where $W''_{n-1,1}$ is TP and non-singular.

From (a), (b), and (c),

$$W = \begin{cases} \exp(J'_{n,n-1})W_{n-1,1}, & \text{if } w_{n-1,1} < w_{n1}, \\ \exp(J''_{n,n-1})W'_{n-1,1}, & \text{if } w_{n-1,1} > w_{n1}, \\ \exp(J'''_{n,n-1})W''_{n-1,1}, & \text{if } w_{n-1,1} = w_{n1}, \end{cases}$$

where $W_{n-1,1}$, $W'_{n-1,1}$, $W''_{n-1,1}$ are TP and nonsingular. Similarly we may factorize any of $W_{n-1,1}$, $W'_{n-1,1}$, $W''_{n-1,1}$ depending on the case. Continuing this process, we can represent W as

$$W = \exp(J^1_{n,n-1}) \exp(J^1_{n-1,n-2}) \dots \exp(J^1_{2,1})V_{11} = B_1V_{11},$$

where V_{11} is TP, non-singular, and of the following form:

$$V_{11} = \begin{pmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ 0 & v_{22} & \dots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & v_{2n} & \dots & v_{nn} \end{pmatrix}.$$

Since V_{11} is TP and non-singular, we can factorize V_{11} and represent $V_{11} = \exp(J^2_{n,n-1}) \exp(J^2_{n-1,n-2}) \dots \exp(J^2_{3,2})U_{22} = B_2U_{22}$, where U_{22} is TP, non-singular, and of the following form:

$$U_{22} = \begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots & u_{1n} \\ 0 & u_{22} & u_{23} & \dots & u_{2n} \\ 0 & 0 & u_{33} & \dots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & u_{n3} & \dots & u_{nn} \end{pmatrix}.$$

Continuing this process, we can represent W as $W = B_1 B_2 \dots B_{n-1} K$, where $B_i = \exp(J^i_{n,n-1}) \exp(J^i_{n-1,n-2}) \dots \exp(J^i_{i+1,i})$ and K is TP, non-singular, upper triangular, and of the following form:

$$K = \begin{pmatrix} k_{11} & k_{12} & \dots & k_{1n} \\ 0 & k_{22} & \dots & k_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & k_{nn} \end{pmatrix}.$$

Let $K_1, \dots, K_n$ be the columns of K such that $K = \|k_{ij}\| = (K_1, K_2, \dots, K_n)$ and let $l = \max\{m : k_{1m} > 0\}$ and $K_{1,l-1} = (K_1, \dots, u_1 K_{l-1}, K_l - u_2 K_{l-1}, K_{l+1}, \dots, K_n)$ for $u_1 = \frac{k_{1,l-1}+k_{1l}}{k_{1,l-1}}$ and $u_2 = \frac{k_{1l}}{k_{1,l-1}}$. Then $K = K_{1,l-1} M_{l-1,l}$ for $l = 2, \dots, n$, where

$$M_{l-1,l} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \frac{k_{1,l-1}}{k_{1,l-1}+k_{1l}} & \frac{k_{1l}}{k_{1,l-1}+k_{1l}} & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Here $\frac{k_{1,l-1}}{k_{1,l-1}+k_{1l}}$ and $\frac{k_{1l}}{k_{1,l-1}+k_{1l}}$ are in the $l-1$th row of $M_{l-1,l}$. That is, $M_{l-1,l} = E_1 + \dots + E_{l-2} + \frac{k_{1l}}{k_{1,l-1}+k_{1l}} E_{l-1,l} + E_{l-1} + E_l + \dots + E_n$. Since $M_{l-1,l} = \exp(H^{l-1}_{l-1,l})$ for some Jacobi matrix $H^{l-1}_{l-1,l}$ with nonnegative off-diagonal elements, $K = K_{1,l-1} \exp(H^{l-1}_{l-1,l})$. According to the Lemma 3.5, $K_{1,l-1}$ is TP and non-singular. Continuing this process, we can represent K as $K = F_{11} \exp(H^{n-1}_{1,2}) \dots \exp(H^{n-1}_{n-1,n}) = F_{11} C_{n-1}$, where F_{11} is TP, non-singular, upper triangular, and of the following form:

$$F_{11} = \begin{pmatrix} f_{11} & 0 & \dots & 0 \\ 0 & f_{22} & \dots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f_{nn} \end{pmatrix}.$$

Since F_{11} is TP and non-singular, we can factorize F_{11} again as above and represent $F_{11} = E_{22} C_{n-2}$, where E_{22} is TP, non-singular, upper triangular, and of the following form:

$$E_{22} = \begin{pmatrix} e_{11} & 0 & 0 & \dots & 0 \\ 0 & e_{22} & 0 & \dots & 0 \\ 0 & 0 & e_{33} & \dots & e_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & e_{nn} \end{pmatrix}.$$

Continuing this process, we can represent K as $K = DC_1C_2\dots C_{n-1}$, where D is an exponential of diagonal matrix. Thus $W = B_1B_2\dots B_{n-1} \times DC_1C_2\dots C_{n-1}$. □

For each cone $W \subseteq L(G)$, there is a local subsemigroup of Lie group G for which the given cone is the set of subtangent vectors, but in general there need not exist a globally defined semigroup for which this is true (see [3]). This motivates the following definition.

DEFINITION 3.7. A semigroup S is said to be *strictly infinitesimally generated by* $L(S)$ if S is the same as the semigroup generated by $\exp(L(S))$.

Given $L(S)$, computing the strictly infinitesimally generated semigroups may be considered as a generalization of finding integrands.

THEOREM 3.8. *The semigroup S of $n \times n$ non-singular TP matrices is strictly infinitesimally generated by the set W of $n \times n$ Jacobi matrices with non-negative off-diagonal entries which forms a tangent cone of the semigroup.*

PROOF. Let $Sg(\exp(W))$ denote a semigroup generated by $\exp(W)$. Then $\exp(W) \subseteq S$, $L(S) = W$, and $Sg(\exp(W)) \subseteq S$ by Theorem 2.5. By Theorem 3.6, $S \subseteq Sg(\exp(W))$. □

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