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INJECTIVE COVERS OVER COMMUTATIVE NOETHERIAN RINGS OF GLOBAL DIMENSION AT MOST TWO II

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ABSTRACT. In studying injective covers, the modules C such that $\mathrm{Hom}(E, C) = 0$ and $\mathrm{Ext}^1(E, C) = 0$ for all injective module E play an important role because of Wakamatsu's lemma. If C is a module over the ring $k[[x, y]]$ with k a field, the class of these modules C contains the class $\bar{D}$ of all direct summands of products of modules of finite length ([3, Theorem 2.9]). In this paper we show that every module over any commutative ring has a $\bar{D}$-preenvelope.

1. Introduction

Assume throughout the paper that R denotes a commutative ring with identity. Let $\mathcal{X}$ be a class which is closed under isomorphism, direct summands and finite direct sums. An *$\mathcal{X}$-envelope* of a module M is a homomorphism $\phi : M \to X$ with $X \in \mathcal{X}$ such that the following two conditions hold;

(1) $\mathrm{Hom}_R(X, X') \to \mathrm{Hom}_R(M, X') \to 0$ is exact for any $X' \in \mathcal{X}$,

(2) Any $f : X \to X$ with $f \circ \phi = \phi$ is an automorphism of $\mathcal{X}$.

If $\phi : M \to X$ satisfies (1) and perhaps not (2), ϕ is called an *$\mathcal{X}$-preenvelope* of M. Dually, an *$\mathcal{X}$-precover* of M is a homomorphism $\psi : X \to M$ with $X \in \mathcal{X}$ such that $\mathrm{Hom}_R(X', X) \to \mathrm{Hom}_R(X', M) \to 0$ is exact for any $X \in \mathcal{X}$ and if an $\mathcal{X}$-precover $\psi : X \to M$ of M satisfies the condition that any $f : X \to X$ with $\psi \circ f = \psi$ is an automorphism of X, then $\psi : X \to M$ is called an *$\mathcal{X}$-cover* of M.

Note that an $\mathcal{X}$-envelope(an $\mathcal{X}$-cover, respectively) of a module is unique up to isomorphism, if it exists. By convention, (pre)envelopes

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and (pre)covers are named according to the name of the class $\mathcal{X}$. For example, an injective cover is an $\mathcal{X}$-cover where $\mathcal{X}$ is the class of injective modules.

We recall that for a class $\mathcal{X}$ of R-modules, $\mathcal{X}^{\perp}$ consists of all R-modules K such that $\mathrm{Ext}^1_R(X, K) = 0$ for all $X \in \mathcal{X}$ and ${}^{\perp}\mathcal{X}$ consists of all R-modules G such that $\mathrm{Ext}^1_R(G, X) = 0$ for all $X \in \mathcal{X}$. (See [5, p.29]).

Wakamatsu's lemma ([5, Lemma 2.1.1] or [4]) says that if $\psi : X \to M$ is an $\mathcal{X}$-cover of an R-module M and if $\mathcal{X}$ is closed under extensions, then $\mathrm{Ker}\psi \in \mathcal{X}^{\perp}$. Conversely, if $\psi : X \to M$ is a surjection with $X \in \mathcal{X}$ and $\mathrm{Ker}\psi \in \mathcal{X}^{\perp}$, then for any $X' \in \mathcal{X}$, $\mathrm{Hom}_R(X', X) \to \mathrm{Hom}_R(X', M) \to \mathrm{Ext}^1_R(X', \mathrm{Ker}\psi) = 0$ is exact. So $\psi : X \to M$ is an $\mathcal{X}$-precover. Such a precover is called a *special $\mathcal{X}$-precover* of M.

2. Rings with global dimension at most 2

From the results in [2, Proposition 8.1], we know that if $\psi : E \to M$ is an injective cover with kernel K, then $\mathrm{Hom}_R(\bar{E}, K) = 0$ and $\mathrm{Ext}^1_R(\bar{E}, K) = 0$ for all injective module $\bar{E}$. Conversely, given such a module K, if $K \subset E$ is an injective envelope, then the natural map $E \to E/K$ is an injective precover. Moreover, the diagram

$$\begin{array}{ccc} E & \xrightarrow{\psi} & E/K \\ \vdots & \nearrow_{\psi} & \\ E & & \end{array}$$

can only be completed to a commutative diagram by id_E.

Proposition 2.1. [3] *If a module C over $R = k[[x, y]]$ with k a field is a direct summand of a product of modules of finite length, then C has the property that $\mathrm{Hom}_R(E, C) = 0$ and $\mathrm{Ext}^1_R(E, C) = 0$ for all injective R-modules E.*

Let $\mathcal{C}$ be the category of R-modules C such that $\mathrm{Hom}_R(E, C) = 0$ and $\mathrm{Ext}^1_R(E, C) = 0$ for all injective modules E and let $\bar{\mathcal{D}}$ be the category of all direct summands of products of modules of finite length. First, we need the lemma.

Lemma 2.2. *Given two modules S_1, S_2, there exists a set of short exact sequence $0 \to S_1 \to G \to S_2 \to 0$ such that for any such short* *exact sequence* $0 \to S_1 \to H \to S_2 \to 0$, *there is a commutative diagram;*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_1 & \longrightarrow & H & \longrightarrow & S_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & S_1 & \longrightarrow & G & \longrightarrow & S_2 & \longrightarrow & 0 \end{array}$$

PROOF. Let S_1, S_2 be any modules and let $X = S_1 \times S_2$ as a set. Let $0 \longrightarrow S_1 \xrightarrow{g} G \xrightarrow{h} S_2 \longrightarrow 0$ be any short exact sequence. For any section $\sigma : S_2 \to G$ of h as a function of sets, the function $\phi : X \to G$ defined by $\phi(x_1, x_2) = g(x_1) + \sigma(x_2)$ is a bijection. Hence X can be made an R-module in a natural way such that $\phi : X \to G$ is an isomorphism of R-modules. Let this module be G' (so $G' = X$ as sets).

Then there is an exact sequence $0 \longrightarrow S_1 \xrightarrow{g'} G' \xrightarrow{h'} S_2 \longrightarrow 0$ of R-modules such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_1 & \xrightarrow{g} & G & \xrightarrow{h} & S_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & S_1 & \xrightarrow[g']{} & G' & \xrightarrow[h']{} & S_2 & \longrightarrow & 0 \end{array}$$

is commutative. □

THEOREM 2.3. *For any ring R, every R-module has a $\bar{\mathcal{D}}$-preenvelope.*

PROOF. Let $\mathcal{S}_1 = \{S \cong R/\mathcal{M} | \mathcal{M}$ is a maximal ideal of $R\}$ be the set of simple R-modules. Then every simple R-module S' is isomorphic to some R-module $S \in \mathcal{S}_1$. For any two modules $S_1, S_2 \in \mathcal{S}_1$, $G = S_1 \times S_2$ can be made into an R-module and $0 \longrightarrow S_1 \longrightarrow G \longrightarrow S_2 \longrightarrow 0$ is a short exact sequence by Lemma 2.2.

Let $\mathcal{S}_2$ be the set of all R-modules G obtained by the above argument for each $S_1, S_2 \in \mathcal{S}_1$. If M is an R-module of length two, then there exists a simple submodule S of M such that

$$0 \longrightarrow S \longrightarrow M \longrightarrow M/S \longrightarrow 0$$

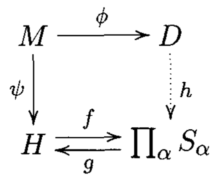
is exact. Since $M/S \in \mathcal{S}_1$, $M \cong T$ for some $T \in \mathcal{S}_2$. Similarly, let $\mathcal{S}_k$ be the set of all R-modules $S_1 \times S_{k-1}$ for each $S_1 \in \mathcal{S}_1$ and $S_{k-1} \in \mathcal{S}_{k-1}$. Then every R-module N of length k is isomorphic to a module $S_k \in \mathcal{S}_k$.

Now set $\mathcal{S} = \bigcup_{i=1}^{\infty} \mathcal{S}_i$. Then every R-module of finite length is isomorphic to some R-module $S \in \mathcal{S}$ of the same length. So every module in the class $\bar{D}$ is a direct summand of products of modules in $\mathcal{S}$. Let

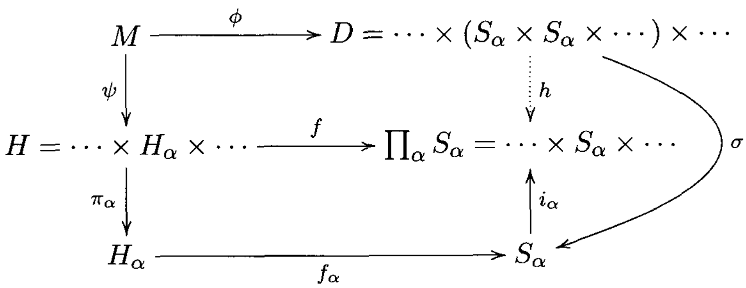
M be an R-module. For any $S \in \mathcal{S}$, let S^* be the direct product of $\mathrm{Hom}_R(M,S)$ of S and let $\mathcal{O}_S : M \to S^*$ be defined by $\mathcal{O}_S(m) = (f(m))$ for $m \in M$ and $f \in \mathrm{Hom}_R(M,S)$.

Let $D = \prod_{S \in \mathcal{S}} S^*$ and $\phi : M \to D$ be a map such that $\phi(m) = \prod_{S \in \mathcal{S}} \mathcal{O}_S(m)$. Then $D \in \bar{\mathcal{D}}$. For any map $\psi : M \to H$ with $H \in \bar{\mathcal{D}}$, there exist $S_\alpha \in \mathcal{S}(\alpha \in I)$ such that H is a direct summand of $\prod_{\alpha \in I} S_\alpha$. So $H = \prod_{\alpha \in I} H_\alpha$ for some direct summands H_α of S_α for each α.

Let $f_\alpha : H_\alpha \to S_\alpha$ and $g_\alpha : S_\alpha \to H_\alpha$ be the canonical maps. Let $f = \prod_{\alpha \in I} f_\alpha$ and $g = \prod_{\alpha \in I} g_\alpha$. To show that ϕ is a $\bar{\mathcal{D}}$-preenvelope of M, it is sufficient to show that there exists a homomorphism $h : D \to \prod_{\alpha \in I} S_\alpha$ such that $h \circ \phi = f \circ \psi$ and so, $(g \circ h) \circ \phi = \psi$.

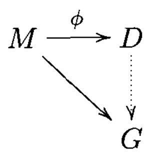


Since for each α, $f_\alpha \circ \pi_\alpha \circ \psi$ is an R-linear map from M to S_α, define $h : D \to \prod_{\alpha \in I} S_\alpha$ by the product of the $\sigma = f_\alpha \circ \pi_\alpha \circ \psi$-th projections for each $\alpha \in I$ and zero maps for $S \neq S_\alpha(\alpha \in I)$, where $\pi_\alpha : H \to H_\alpha$ is the α-th canonical projection. Then $h \circ \phi = f \circ \psi$ and so ϕ is a $\bar{\mathcal{D}}$-preenvelope of M.



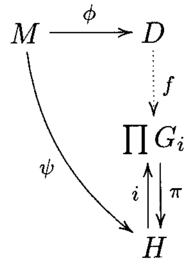
□

PROPOSITION 2.4. *If $D \in \bar{\mathcal{D}}$, then $\phi : M \to D$ is a $\bar{\mathcal{D}}$-preenvelope if and only if any diagram*



can be completed to a commutative diagram whenever G has finite length.

PROOF. Let $H \in \bar{\mathcal{D}}$. Then H is a direct summand of $\prod_{i\in I} G_i$ for some G_i of finite length. Since $\mathrm{Hom}_R(D, G_i) \to \mathrm{Hom}_R(M, G_i) \to 0$ is exact for each $i \in I$, $\prod_{i\in I} \mathrm{Hom}_r(D, G_i) \to \prod_{i\in I} \mathrm{Hom}_R(M, G_i) \to 0$ is exact, that is $\mathrm{Hom}_R(D, \prod_{i\in I} G_i) \to \mathrm{Hom}_R(M, \prod_{i\in I} G_i) \to 0$ is exact. So for any diagram



with the canonical maps i and π, there exists a linear map $f : D \to \prod_{i\in I} G_i$ such that $f\circ\phi = i\circ\psi$. Therefore $(\pi\circ f)\circ\phi = \psi$. Thus $\phi : M \to D$ is a preenvelope. The converse is trivial. □

EXAMPLE 2.5. Let R be a ring, M a finitely generated module and Ω the set of maximal ideals of R. If $f : M \to G$ is a homomorphism with G having finite length. Then there is $\mathcal{M}_i \in \Omega (1 \leq i \leq s)$ such that $\mathcal{M}_1\mathcal{M}_2 \cdots \mathcal{M}_s G = 0$ by [1, Corollary 2.17]. So $\mathrm{Ker} f \supset \mathcal{M}_1\mathcal{M}_2\cdots\mathcal{M}_s M$ for some $\mathcal{M}_1, \mathcal{M}_2, \cdots \mathcal{M}_s \in \Omega$.

But for any such $\mathcal{M}_1, \mathcal{M}_2, \cdots \mathcal{M}_s$, $\frac{M}{\mathcal{M}_1\mathcal{M}_2\cdots\mathcal{M}_s M}$ has finite length, and so $M \to \prod \frac{M}{\mathcal{M}_1\mathcal{M}_2\cdots\mathcal{M}_s M}$ is a $\bar{\mathcal{D}}$-preenvelope (over all finite families $\mathcal{M}_1, \mathcal{M}_2, \cdots \mathcal{M}_s \in \Omega$. For example, with $R = \mathbb{Z}$, $\mathbb{Z} \to \prod_{n=2}^{\infty} \mathbb{Z}/n\mathbb{Z}$ would be a $\bar{\mathcal{D}}$-preenvelope.

REMARK 2.6. For $R = \mathbb{Z}$, $\mathbb{Q} \to 0$ is a $\bar{\mathcal{D}}$-preenvelope.

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