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($\pm$1)-INVARIANT SEQUENCES AND TRUNCATED FIBONACCI SEQUENCES OF THE SECOND KIND

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Abstract. In this paper we present another characterization of ($\pm$1)-invariant sequences. We also introduce truncated Fibonacci and Lucas sequences of the second kind and show that a sequence $\mathbf{x} \in \mathbf{R}^\infty$ is (-1)-invariant(1-invariant resp.) if and only if $D\left[\begin{smallmatrix}0\\ \mathbf{x}\end{smallmatrix}\right]$ is perpendicular to every truncated Fibonacci(truncated Lucas resp.) sequence of the second kind where

$$D = \mathrm{diag}((-1)^0, (-1)^1, (-1)^2, \ldots).$$

1. Introduction

Throughout this paper, let $\mathbf{R}^\infty$ denote the infinite dimensional real vector space consisting of all real sequences $(x_0, x_1, x_2, \ldots)^T$, and let P and D denote the Pascal matrix $[\binom{i}{j}]$, $(i, j = 0, 1, 2, \ldots)$ and the diagonal matrix $\mathrm{diag}((-1)^0, (-1)^1, (-1)^2, \ldots)$ respectively. We also assume that every matrix in this paper is assumed to have infinitely many rows and columns numbered $0, 1, 2, \ldots$ unless otherwise specified. The classical binomial inversion formula states that for $\mathbf{x}, \mathbf{y} \in \mathbf{R}^\infty$, $PD\mathbf{x} = \mathbf{y}$ if and only if $PD\mathbf{y} = \mathbf{x}$. Let $\mathbf{F} = (F_0, F_1, F_2, \ldots)^T = (0, 1, 1, 2, 3, \ldots)^T$ and $\mathbf{L} = (L_0, L_1, L_2, \ldots)^T = (2, 1, 3, 4, \ldots)^T$ be the Fibonacci sequence and the Lucas sequence respectively. Then $PD\mathbf{F} = -\mathbf{F}$ and $PD\mathbf{L} = \mathbf{L}$. Thus, as a linear transformation of $\mathbf{R}^\infty$, PD has eigenvalues -1, 1. In fact,

$$-1 \text{ and } 1 \text{ are the only eigenvalues of } PD$$

because if $\lambda \neq \pm 1$, then $PD\mathbf{x} = \lambda\mathbf{x}$ has no nonzero solution. A sequence $\mathbf{x} \in \mathbf{R}^\infty$ is called a *1-invariant sequence* if $PD\mathbf{x} = \mathbf{x}$, a *$(-1)$-invariant*

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sequence if $PD\mathbf{x} = -\mathbf{x}$ ([1]). Associated with the Pascal matrix P, let $P^{\rightarrow}$ denote the matrix defined by

$$P^{\rightarrow} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 3 & 3 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where the row i, $(i = 0, 1, 2, \ldots)$, is that of the Pascal matrix P preceded by $\mathbf{0}_i^T$, the i-vector of zeros. Let

$$(1.1) \qquad Q = P + \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & P \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 2 & 0 & 0 & \cdots \\ 1 & 4 & 5 & 2 & 0 & \cdots \\ 1 & 5 & 9 & 7 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Associated with Q, let $Q^{\rightarrow}$ denote the matrix defined by

$$Q^{\rightarrow} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 3 & 2 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 4 & 5 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 5 & 9 & 7 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where the row i is that of Q preceded by $\mathbf{0}_i^T$, $(i = 0, 1, 2, \ldots)$. Then $\mathbf{e}^T[\mathbf{0}, P^{\rightarrow}] = \mathbf{F}$ and $\mathbf{e}^T Q^{\rightarrow} = \mathbf{L}$ where $\mathbf{e} = (1, 1, \ldots)^T \in \mathbf{R}^{\infty}$. It is noted in [1] that

$$Q^{\rightarrow} = P^{\rightarrow} + \begin{bmatrix} 1 & 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & P^{\rightarrow} \end{bmatrix}.$$

For a matrix A whose row index set and column index set are J and K respectively, and for $J_0 \subset J$, $K_0 \subset K$, let $A(J_0|K_0)$ denote the matrix obtained from A by deleting rows in J_0 and columns in K_0, and let $A[J_0|K_0]$ denote the matrix $A(\bar{J_0}|\bar{K_0})$ where $\bar{J_0} = J - J_0$, $\bar{K_0} = K - K_0$.

In [4], generating functions of (± 1)-invariant sequences are investigated. In [1], the authors found some characterization of (± 1)-invariant sequences in connection with the matrices P and Q. They also introduced truncated Fibonacci and Lucas sequences and proved that a sequence $\mathbf{x} \in \mathbf{R}^{\infty}$ is (-1)-invariant(1-invariant resp.) if and only if $\mathbf{x}$

is expressible as a linear combination of truncated Fibonacci(truncated Lucas resp.) sequences.

In this paper we present another characterization of (± 1)-invariant sequences. We also introduce truncated Fibonacci and Lucas sequences of the second kind and show that a sequence $\mathbf{x} \in \mathbf{R}^\infty$ is (-1)-invariant(1-invariant resp.) if and only if $D\left[\begin{smallmatrix}0\\ \mathbf{x}\end{smallmatrix}\right]$ is perpendicular to every truncated Fibonacci(truncated Lucas resp.) sequence of the second kind.

2. Relationship between (± 1)-invariant sequences

In this section we first investigate the relationship between (± 1)-invariant sequences.

LEMMA 2.1. [1] *For a sequence* $\mathbf{x} \in \mathbf{R}^\infty$, *the following hold.*

(*a*) $\mathbf{x}$ *is* (-1)*-invariant if and only if* $P^{\rightharpoondown} D\mathbf{x} = \mathbf{0}$,

(*b*) $\mathbf{x}$ *is* 1*-invariant if and only if* $Q^{\rightharpoondown}(0|0)D\mathbf{x} = \mathbf{0}$.

The matrices $P^{\rightharpoondown}$ and $Q^{\rightharpoondown}$ are related as

$$Q^{\rightharpoondown} = P^{\rightharpoondown} + \begin{bmatrix} 1 & 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & P^{\rightharpoondown} \end{bmatrix}.$$

Yet, there is another relation between $P^{\rightharpoondown}$ and $Q^{\rightharpoondown}$. In what follows let $\Delta_{\mathbf{N}}$ denote the matrix $\operatorname{diag}(1, 2, 3, 4, \dots)$.

LEMMA 2.2. *The matrices* $P^{\rightharpoondown}$ *and* $Q^{\rightharpoondown}$ *are related as*

$$Q^{\rightharpoondown} = \begin{bmatrix} 2 & \mathbf{0}^T \\ \mathbf{0} & \Delta_{\mathbf{N}}^{-1} \end{bmatrix} P^{\rightharpoondown} \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \Delta_{\mathbf{N}} \end{bmatrix}.$$

Proof. Let $P = [\beta_{ij}]$, $Q = [\gamma_{ij}]$, $(i, j = 0, 1, 2, \dots)$. Then $\beta_{ij} = \binom{i}{j}$ and, by (1.1),

$$\gamma_{ij} = \binom{i}{j} + \binom{i-1}{j-1} = \frac{i+j}{i}\binom{i}{j} = \frac{i+j}{i}\beta_{ij} \tag{2.1}$$

for $i, j \geq 1$. Thus we see that Q is obtained from P by multiplying the row i by $\frac{1}{i}$, $(i = 1, 2, \dots)$ and multiplying each of the entries in the line $i + j = k$ by k, $(k = 1, 2, \dots)$ and then multiplying the row 0 by 2. Let

$$P^{\rightharpoondown} = [\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \dots], \quad Q^{\rightharpoondown} = [\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \dots].$$

Then

$$\mathbf{p}_k = (\beta_{0,k}, \beta_{1,k-1}, \dots, \beta_{k-1,1}, \beta_{k,0}, 0, 0, \dots)^T,$$

$$\mathbf{q}_k = (\gamma_{0,k}, \gamma_{1,k-1}, \dots, \gamma_{k-1,1}, \gamma_{k,0}, 0, 0, \dots)^T,$$

for $k = 1, 2, \ldots$, from which we see that

$$\mathbf{q}_k = (2k\beta_{0,k}, k\beta_{1,k-1}, \frac{1}{2}k\beta_{2,k-2}, \frac{1}{3}k\beta_{3,k-3}, \ldots, \frac{1}{k}k\beta_{k,0}, 0, 0, \ldots)^T,$$

for $k = 1, 2, \ldots$, by (2.1). Thus we have the equality in the Lemma. □

The following theorem is a characterization of (± 1)-invariant sequences which looks similar to but is different from Lemma 2.1.

THEOREM 2.3. *Let* $\mathbf{x} = (x_0, x_1, x_2, \ldots)^T \in \mathbf{R}^\infty$. *Then*
(*a*) $\mathbf{x}$ *is* (-1)*-invariant if and only if* $Q^{\rightharpoonup}(x_0, -x_1, \frac{1}{2}x_2, -\frac{1}{3}x_3, \ldots)^T = \mathbf{0}$.
(*b*) $\mathbf{x}$ *is* 1*-invariant if and only if*

$$P^{\rightharpoonup}(0|0)(x_0, -2x_1, 3x_2, -4x_3, \ldots)^T = \mathbf{0}.$$

Proof. Since $P^{\rightharpoonup}D\mathbf{x} = \mathbf{0}$ is equivalent to $Q^{\rightharpoonup}\left[\begin{smallmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \Delta_{\mathbf{N}}^{-1}\end{smallmatrix}\right]D\mathbf{x} = \mathbf{0}$ by Lemma 2.2, (*a*) follows from Lemma 2.1(*a*). By Lemma 2.2, we also have $Q^{\rightharpoonup}(0|0) = \Delta_{\mathbf{N}}^{-1}P^{\rightharpoonup}(0|0)\Delta_{\mathbf{N}}$. So, $Q^{\rightharpoonup}(0|0)D\mathbf{x} = \mathbf{0}$ is equivalent to $P^{\rightharpoonup}(0|0)\Delta_{\mathbf{N}}D\mathbf{x} = \mathbf{0}$, and (*b*) follows from Lemma 2.1(*b*). □

We now present the relationship between the (-1)-invariant sequences and the 1-invariant sequences.

LEMMA 2.4. [1] *Let* $(x_0, x_1, x_2, \ldots)^T \in \mathbf{R}^\infty$ *and let* $s_k = \sum_{i=0}^{k} x_i$, $(k = 0, 1, 2, \ldots)$. *Then* $(x_0, x_1, x_2, \ldots)^T$ *is* λ*-invariant if and only if* $(0, 0, s_0, s_1, s_2, \ldots)^T$ *is* λ*-invariant, for* $\lambda \in \{1, -1\}$.

THEOREM 2.5. *Let* $\mathbf{x} = (x_0, x_1, x_2, \ldots)^T \in \mathbf{R}^\infty$. *Then*

(*a*) $\mathbf{x}$ *is* (-1)*-invariant if and only if* $x_0 = 0$ *and* $(x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \ldots)^T$ *is* 1*-invariant.*

(*b*) $\mathbf{x}$ *is* 1*-invariant if and only if* $(0, x_0, 2x_1, 3x_2, \ldots)^T$ *is* (-1)*-invariant.*

Proof. (*a*) Let $\mathbf{y} = (x_1, x_2, \ldots)^T$ so that $\mathbf{x} = \left[\begin{smallmatrix} x_0 \\ \mathbf{y}\end{smallmatrix}\right]$ and $D\mathbf{x} = \left[\begin{smallmatrix} x_0 \\ -D\mathbf{y}\end{smallmatrix}\right]$. Suppose that $\mathbf{x}$ is (-1)-invariant. Then, by Lemma 2.1, $P^{\rightharpoonup}D\mathbf{x} = \mathbf{0}$, i.e.,

$$\begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & P^{\rightharpoonup}(0|0)\end{bmatrix}\begin{bmatrix} x_0 \\ -D\mathbf{y}\end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{0}\end{bmatrix}$$

Thus $x_0 = 0$ and

$$P^{\rightharpoonup}(0|0)D\mathbf{y} = \mathbf{0}. \tag{2.2}$$

Now (2.2) gives us that

$$\Delta_{\mathbf{N}}^{-1}P^{\rightharpoonup}(0|0)\Delta_{\mathbf{N}}\Delta_{\mathbf{N}}^{-1}D\mathbf{y} = \mathbf{0}, \text{ i.e., } Q^{\rightharpoonup}(0|0)\Delta_{\mathbf{N}}^{-1}D\mathbf{y} = \mathbf{0}.$$

So, by Lemma 2.1 again we see that $\Delta_{\mathbf{N}}^{-1}\mathbf{y} = (x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \ldots)^T$ is 1-invariant. Reversing the above argument we can show that if $\Delta_{\mathbf{N}}^{-1}\mathbf{y}$ is 1-invariant, then $\mathbf{x}$ is (-1)-invariant.

(*b*) Suppose that $\mathbf{x}$ is 1-invariant. Then, by Lemma 2.1, $Q^{\rightarrow}(0|0)D\mathbf{x} = \mathbf{0}$ from which we get $P^{\rightarrow}(0|0)\Delta_{\mathbf{N}}D\mathbf{x} = \mathbf{0}$ so that

$$P^{\rightarrow}\begin{bmatrix} 0 \\ \Delta_{\mathbf{N}}D\mathbf{x} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & P^{\rightarrow}(0|0) \end{bmatrix}\begin{bmatrix} 0 \\ \Delta_{\mathbf{N}}D\mathbf{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix}.$$

Since

$$\begin{bmatrix} 0 \\ \Delta_{\mathbf{N}}D\mathbf{x} \end{bmatrix} = -D\begin{bmatrix} 0 \\ \Delta_{\mathbf{N}}\mathbf{x} \end{bmatrix},$$

we see, by Lemma 2.1, that $\left[\begin{smallmatrix} 0 \\ \Delta_{\mathbf{N}}\mathbf{x} \end{smallmatrix}\right] = (0, x_0, 2x_1, 3x_2, 4x_3, \ldots)^T$ is (-1)-invariant. Reversing the above argument, it can be shown that if $\left[\begin{smallmatrix} 0 \\ \Delta_{\mathbf{N}}\mathbf{x} \end{smallmatrix}\right]$ is (-1)-invariant, then $\mathbf{x}$ is 1-invariant. □

3. (±1)-invariant sequences and the truncated Fibonacci and Lucas sequences

In [1], [2], and [3], some generalization of Fibonacci and Lucas sequences are found. In this section we introduce another extension of the notion of Fibonacci and Lucas sequences.

Suppose that we arrange n congruent cubic blocks in such a way that the blocks are arranged in one or two rows and each of the blocks in the top row(top blocks) is placed on a block in the bottom row(base block). We call such an arrangement a *Fibonacci arrangement* of n blocks. Let $\mathcal{F}_n$ denote the set of all Fibonacci arrangements of n blocks. It is well known that $|\mathcal{F}_n|$, the number of elements of $\mathcal{F}_n$, is equal to $\binom{n}{0} + \binom{n-1}{1} + \cdots + \binom{1}{n-1} + \binom{0}{n} = F_{n+1}$, the $(n+1)$st Fibonacci number. For a nonnegative integer r, let $\mathcal{F}_{r,n}$ denote the set of all Fibonacci arrangements of n blocks in such a way that the number of top blocks is $\leq r$. The sequence $\mathbf{F}_r = (f_{r,0}, f_{r,1}, f_{r,2}, \ldots)^T$ defined by $f_{r,0} = 0$, $f_{r,1} = 1$, $f_{r,n} = |\mathcal{F}_{r,n-1}|$, the number of elements of $\mathcal{F}_{r,n-1}$, $(n \geq 2)$ is called rth *truncated Fibonacci sequence* ([1]). It is noted in [1] that $f_{r,n+1} = \sum_{k=0}^{r}\binom{n-k}{k}$ which is the sum of entries of P lying on the line $i + j = n$ within the range $0 \leq j \leq r$.

Consider the set $\mathcal{F}'_{r,n}$ of all Fibonacci arrangements of n blocks such that the number of base blocks is $\leq r$. Let $\mathbf{F}'_r = (f'_{r,0}, f'_{r,1}, f'_{r,2}, \ldots)^T$ be defined by

$$f'_{r,0} = 0,\ f'_{r,1} = 1,\ f'_{r,n} = |\mathcal{F}'_{r,n-1}|,\ (n \geq 2).$$

We call $\mathbf{F}'_r$ the rth *truncated Fibonacci sequence of the second kind.* Let k be the number of base blocks in an arrangement in $\mathcal{F}'_{r,n}$. Then the number of top blocks is $n-k$, and there are $\binom{k}{n-k}$ ways that the $n-k$ top blocks can be placed. Since $0 \le k \le r$, we have

$$f'_{r,n+1} = \binom{0}{n} + \binom{1}{n-1} + \cdots + \binom{r}{n-r},$$

which is the sum of entries of P lying on the line $i+j=n$ within the range $0 \le i \le r$. Therefore we see that $\mathbf{F}'_r$ is the column sum vector of $[\mathbf{0}, P^{\rightarrow}[0,1,\ldots,r|0,1,2,\ldots]]$.

Let $Q = [\gamma_{ij}]$ and let $g_{r,n} = \sum_{k=0}^{r} \gamma_{n-k,k}$ and $\mathbf{L}_r = (g_{r,0}, g_{r,1}, g_{r,2}, \ldots)^T$. Then $g_{r,n}$ is equal to the sum of the entries of Q lying on the line $i+j=n$ within the range $0 \le j \le r$. The sequence $\mathbf{L}_r$ is called the rth *truncated Lucas sequence* ([1]).

Let

$$g'_{r,n} = \sum_{k=0}^{r} \gamma_{k,n-k} \tag{3.1}$$

and let $\mathbf{L}'_r = (g'_{r,0}, g'_{r,1}, g'_{r,2}, \ldots)^T$. We call $\mathbf{L}'_r$ the rth *truncated Lucas sequence of the second kind.* Since the right hand side of (3.1) is the sum of the entries of Q lying on the line $i+j=n$ within the range $0 \le i \le r$, we see that $\mathbf{L}'_r$ is the column sum vector of $Q^{\rightarrow}[0,1,\ldots,r|0,1,2,\ldots]$.

A sequence $\mathbf{x} = (x_0, x_1, x_2, \ldots)^T \in \mathbf{R}^\infty$ is called a *finite sequence* if $x_i = 0$ for all but a finite number of i's. Note that the sequences $\mathbf{F}'_r$ and $\mathbf{L}'_r$ are finite sequences for all r.

It is proved in [1] that a sequence is (-1)-invariant (1-invariant resp.) if and only if it is expressible as a linear combination of truncated Fibonacci(truncated Lucas resp.) sequences. In particular, the truncated Fibonacci(truncated Lucas resp.) sequences are (-1)-invariant (1-invariant resp.) sequences.

Are the truncated Fibonacci and the truncated Lucas sequences of the second kind (± 1)-invariant sequences?

The answer to this question is given in the following

THEOREM 3.1. *No nonzero (± 1)-invariant sequence is a finite sequence.*

Proof. Let $\mathbf{x} = (x_0, x_1, x_2, \ldots)^T \in \mathbf{R}^\infty$ be a finite sequence. Take an integer m such that $x_i = 0$ for all $i > m$. Let $n = 2m$ and let $\mathbf{y} = (x_0, x_1, \ldots, x_m)^T$ so that $\mathbf{x} = (\mathbf{y}^T, x_{m+1}, x_{m+2}, \ldots)^T$. Suppose that $\mathbf{x}$ is (-1)-invariant. Then $P^{\rightarrow} D\mathbf{x} = \mathbf{0}$ by Lemma 2.1 and hence

$$P^{\rightarrow}[0,1,\ldots,m|0,1,\ldots,n]D_n\left[\begin{smallmatrix}\mathbf{y}\\ \mathbf{0}_m\end{smallmatrix}\right]=\mathbf{0}$$

where $D_n = \mathrm{diag}((-1)^0, (-1)^1, \cdots, (-1)^n)$ and $\mathbf{0}_m$ denotes the m-vector of zeros. Since $P^{\rightarrow}[0,1,\ldots,m|0,1,\ldots,n]$ has rank $m+1$, we have that $D_m\mathbf{y} = \mathbf{0}$ and hence that $\mathbf{y} = \mathbf{0}$ so that $\mathbf{x} = \mathbf{0}$. □

Though the truncated Fibonacci and the truncated Lucas sequences of the second kind are not (±1)-invariant sequences, they still retain substantial importance in determining whether a sequence is (±1)-invariant or not.

THEOREM 3.2. *Let* $\mathbf{x} \in \mathbf{R}^{\infty}$. *Then*

(*a*) $\mathbf{x}$ *is* (-1)*-invariant if and only if* $(\mathbf{F}'_r)^T D\left[\begin{smallmatrix}0\\ \mathbf{x}\end{smallmatrix}\right] = \mathbf{0}$ *for all* $r = 0, 1, 2, \ldots$.

(*b*) $\mathbf{x}$ *is* 1*-invariant if and only if* $(\mathbf{L}'_r)^T D\left[\begin{smallmatrix}0\\ \mathbf{x}\end{smallmatrix}\right] = \mathbf{0}$ *for all* $r = 0, 1, 2, \ldots$.

Proof. (*a*) Let

$$\Psi = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & \cdots \\ 1 & 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad F' = \begin{bmatrix} (\mathbf{F}'_0)^T \\ (\mathbf{F}'_1)^T \\ (\mathbf{F}'_2)^T \\ \vdots \end{bmatrix}.$$

Then $F' = \Psi[\mathbf{0}, P^{\rightarrow}]$. Let $\mathbf{x} \in \mathbf{R}^{\infty}$. Suppose that $\mathbf{x}$ is (-1)-invariant. Then $P^{\rightarrow}D\mathbf{x} = \mathbf{0}$ so that $F'D\left[\begin{smallmatrix}0\\ \mathbf{x}\end{smallmatrix}\right] = \mathbf{0}$, proving the 'only if' part of (*a*).

Conversely suppose that $F'D\left[\begin{smallmatrix}0\\ \mathbf{x}\end{smallmatrix}\right] = \mathbf{0}$. Then $\Psi P^{\rightarrow}D\mathbf{x} = \mathbf{0}$. Let

$$\Phi = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then $\Phi\Psi = \mathrm{diag}(1, 1, \ldots)$ and we get $P^{\rightarrow}D\mathbf{x} = \Phi\Psi P^{\rightarrow}D\mathbf{x} = \mathbf{0}$, which tells us that $\mathbf{x}$ is (-1)-invariant by Lemma 2.1(*a*), and the proof of (*a*) is complete.

(b) Let

$$L' = \begin{bmatrix} (\mathbf{L}'_0)^T \\ (\mathbf{L}'_1)^T \\ (\mathbf{L}'_2)^T \\ \vdots \end{bmatrix}.$$

Then $L' = \Psi Q^{\rightharpoonup}$. Let $\mathbf{x} \in \mathbf{R}^{\infty}$. Suppose that $\mathbf{x}$ is 1-invariant. Then $Q^{\rightharpoonup}(0|0)D\mathbf{x} = \mathbf{0}$ so that

$$Q^{\rightharpoonup}\begin{bmatrix} 0 \\ D\mathbf{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix}.$$

But then

$$L'\begin{bmatrix} 0 \\ D\mathbf{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix},$$

which yields that

$$L'D\left[\begin{smallmatrix} 0 \\ \mathbf{x} \end{smallmatrix}\right] = \mathbf{0}$$

and the 'only if' part of (b) is proved. The 'if' part can be proved by reversing the above argument. □

4. Some properties of truncated Fibonacci and Lucas sequences of the second kind

In this section we give a couple of sequential properties of truncated Fibonacci and Lucas sequences of the second kind. In the sequel, we assume, for integers i, j, that $\binom{i}{j} = 0$ if either $i < 0$ or $j < 0$ or $i < j$.

THEOREM 4.1. *Let r be a positive integer. Then*

(a) $f'_{r,n} = f'_{r,n-1} + f'_{r,n-2} - \binom{r+1}{n-r-2}$, $(n \geq 2)$.

(b) $g'_{r,n} = g'_{r,n-1} + g'_{r,n-2} - \frac{n}{r+1}\binom{r+1}{n-r-1}$, $(n \geq 2)$.

Proof. (a) By the definition of the numbers $f'_{r,n}$, we have

$$\begin{aligned} f'_{r,n-1} + f'_{r,n-2} &= \binom{0}{n-2} + \binom{1}{n-3} + \cdots + \binom{r}{n-r-2} \\ &\quad + \binom{0}{n-3} + \binom{1}{n-4} + \cdots + \binom{r}{n-r-3} \\ &= \binom{1}{n-2} + \binom{2}{n-3} + \cdots + \binom{r+1}{n-r-2} \\ &= \sum_{k=0}^{r} \binom{k}{n-k-1} + \binom{r+1}{n-r-2} \\ &= f'_{r,n} + \binom{r+1}{n-r-2}, \end{aligned}$$

since $\binom{0}{n-1} = 0$ for $n \geq 2$.

(*b*) Let $Q = [\gamma_{ij}]$. Then $\gamma_{i,j} + \gamma_{i,j+1} = \gamma_{i+1,j+1}$ for all $i, j = 0, 1, 2, \ldots$, and $\gamma_{ij} = 0$ if $i < j$. Let δ_{ij} be a number defined for every pair (i, j) of integers by

$$\delta_{ij} = \begin{cases} \gamma_{ij} & \text{if } i, j \geq 0, \\ 0 & \text{if either } i < 0 \text{ or } j < 0. \end{cases}$$

Then $\delta_{i,j} + \delta_{i,j+1} = \delta_{i+1,j+1}$ for all i, j, and

$$\begin{aligned} g'_{r,n-1} + g'_{r,n-2} &= \delta_{0,n-1} + \delta_{1,n-2} + \cdots + \delta_{r,n-r-1} \\ &\quad + \delta_{0,n-2} + \delta_{1,n-3} + \cdots + \delta_{r,n-r-2} \\ &= \delta_{1,n-1} + \delta_{2,n-2} + \cdots + \delta_{r+1,n-r-1} \\ &= g'_{r,n} + \delta_{r+1,n-r-1}. \end{aligned}$$

If $n \geq r + 2$, then

$$\begin{aligned} \delta_{r+1,n-r-1} = \gamma_{r+1,n-r-1} &= \binom{r+1}{n-r-1} + \binom{r}{n-r-2} \\ &= \frac{n}{r+1}\binom{r+1}{n-r-1}. \end{aligned}$$

If $n = r + 1$, then

$$\delta_{r+1,n-r-1} = 1 = \frac{n}{r+1}\binom{r+1}{n-r-1}.$$

If $n < r + 1$, then

$$\delta_{r+1,n-r-1} = 0 = \frac{n}{r+1}\binom{r+1}{n-r-1}.$$

Thus (*b*) is proved. □

As finite sequences, the truncated Fibonacci and Lucas sequences of the second kind have finite sums which look very simple as we see in the following

THEOREM 4.2. *Let r be a nonnegative integer. Then*

(*a*) $\sum_{n=0}^{\infty} f'_{r,n} = 2^{r+1} - 1$.
(*b*) $\sum_{n=0}^{\infty} g'_{r,n} = 3 \cdot 2^r - 1$.

Proof. (*a*) $\sum_{n=0}^{\infty} f'_{r,n}$ is equal to the sum of all entries $\binom{i}{j}$ of P in the range $0 \leq i \leq r$. Therefore $\sum_{n=0}^{\infty} f'_{r,n} = \sum_{i=0}^{r} 2^i = 2^{r+1} - 1$.

(*b*) $\sum_{n=0}^{\infty} g'_{r,n}$ is equal to the sum of all entries γ_{ij} of Q in the range $0 \le i \le r$. Since $Q = P + \left[\begin{smallmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & P \end{smallmatrix}\right]$ we see that

$$\sum_{n=0}^{\infty} g'_{r,n} = \sum_{i=0}^{r} 2^i + 1 + \sum_{i=0}^{r-1} 2^i = 2^{r+1} - 1 + 1 + 2^r - 1 = 3 \cdot 2^r - 1. \qquad \square$$

Our last discussion is the relationship between the Fibonacci(Lucas resp.) sequence and the truncated Fibonacci(truncated Lucas resp.) sequences of the second kind.
Since the Fibonacci sequence $\mathbf{F} = (F_0, F_1, F_2, \ldots)^T$ and the Lucas sequence $\mathbf{L} = (L_0, L_1, L_2, \ldots)^T$ are (-1)-invariant and 1-invariant respectively, it follows, from Theorem 3.2, that

$$\sum_{n=1}^{\infty} (-1)^n f'_{r,n} F_{n-1} = 0, \ \ (r = 0, 1, 2, \ldots)$$

and

$$\sum_{n=1}^{\infty} (-1)^n g'_{r,n} L_{n-1} = 0, \ \ (r = 0, 1, 2, \ldots).$$

Since, for $i \ge 1$,

$$\gamma_{i,n-i} = \binom{i}{n-i} + \binom{i-1}{n-i-1} = \frac{n}{i}\binom{i}{n-i},$$

we see, by the definition of $g'_{r,n}$, that for $n \ge 1$,

$$g'_{r,n} = \sum_{i=0}^{r} \gamma_{i,n-i} = \sum_{i=1}^{r} \gamma_{i,n-i} = \sum_{i=1}^{r} \frac{n}{i}\binom{i}{n-i}.$$

So we have the following

THEOREM 4.3. *The Fibonacci and Lucas sequences satisfy the following relations.*

(*a*) $\sum_{n=0}^{\infty} (-1)^{n+1} F_n \sum_{i=0}^{r} \binom{i}{n-i} = 0.$

(*b*) $\sum_{n=0}^{\infty} (-1)^{n+1} L_n (n+1) \sum_{i=1}^{r} \frac{1}{i} \binom{i}{n-i+1} = 0.$

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