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POSITIVE PERIODIC SOLUTIONS OF SYSTEMS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

YOUSSEF N. RAFFOUL

ABSTRACT. We apply a cone theoretic fixed point theorem and obtain conditions for the existence of positive periodic solutions of the system of functional differential equations

$$x'(t) = A(t)x(t) + \lambda f(t, x(t - \tau(t))).$$

1. Introduction

In this paper, we are concerned with determining values for λ so that the system of functional differential equations

$$x'(t) = A(t)x(t) + \lambda f(t, x(t - \tau(t))) \tag{1.1}$$

has a positive periodic solution. The matrix $A(t) = \text{diag}[a_1(t), a_2(t), \ldots, a_n(t)]$, $a_j \in C(\mathbb{R}, \mathbb{R}), \tau : \mathbb{R} \to \mathbb{R}$, are continuous and ω-periodic, $j = 1, 2 \ldots, n$ with $\omega > 0$. The function $f : \mathbb{R} \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is continuous, where $\mathbb{R}^n = (x_1, x_2, \ldots, x_n)^T$ and $\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n : x_j > 0,\ j = 1, 2, \ldots, n\}$. We denote BC the normed vector space of bounded functions $\phi : \mathbb{R} \to \mathbb{R}^n$ with the norm $\|\phi\| = \sum_{j=1}^n \sup_{t\in\mathbb{R}} |\phi_j(t)|$ where $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^T$. For each $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, the norm of x is defined as $|x|_0 = \sum_{j=1}^n |x_j|$, where we say that x is "positive" whenever $x \in \mathbb{R}^n_+$.

In this paper we not only carry the work of [17] to the continuous case, but we generalize it to systems. In this research, the set up of the mapping is the same as in [10] in which $\lambda = 1$. In arriving at our results, we make use of Krasnosel'skii fixed point theorem ([11]). The existence of positive periodic solutions of nonlinear functional differential equations have been studied extensively, in recent years. For some appropriate

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references we refer the reader to [1], [2], [3], [4], [5], [6], [7], [8], [9], [12], [13], [14], [15], [16], [19] and the references therein. This work is mainly motivated by the work of [9], [17], and [18].
In section 2, we state Krasnosel'skii fixed point theorem ([11]), prove two Lemmas that are essential to this research and construct the cone of interest . In section 3, we present four theorems and a corollary. In each of the theorems and the corollary an open interval of eigenvalues is determined, which in returns, implies the existence of a positive periodic solution of (1.1), by appealing to Krasnosel'skii fixed point theorem.

2. Preliminaries

THEOREM 2.1. (Krasnosel'skii) *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P}$ be a cone in $\mathcal{B}$. Suppose Ω_1 and Ω_2 are bounded open subsets of $\mathcal{B}$ such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ and suppose that*

$$T : \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{P}$$

is a completely continuous operator such that

(i) $\|Tu\| \leq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_1,$ *and* $\|Tu\| \geq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_2;$ *or*
(ii) $\|Tu\| \geq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_1,$ *and* $\|Tu\| \leq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_2.$

Then T has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

We denote $f = (f_1, f_2, \dots, f_n)^T$ and assume

$$(H1) \qquad \int_0^\omega a_j(s)ds < 0 \quad \text{for } j = 1, 2, \dots, n.$$

DEFINITION 2.2. Let X be a Banach space and K be a closed, nonempty subset of X. K is a cone if

(i) $\alpha u + \beta v \in K$ for all $u, v \in K$ and all $\alpha, \beta \geq 0$
(ii) $u, -u \in K$ imply $u = 0$.

Define the set C_ω by

$$C_\omega = \{x \in C(\mathbb{R}, \mathbb{R}^n) :\ x(t + \omega) = x(t),\ t \in \mathbb{R}\}.$$

Then it is clear that $C_\omega \subset BC$ when it is endowed with supremum norm $||x|| = \sum_{j=1}^n ||x_j||_0$, where $||x_j||_0 = \sup_{t\in[0,\omega]} |x_j(t)|$.
Next, we consider the scalar differential equation

$$(2.1) \qquad x'(t) = a(t)x(t) + \lambda f(t, x(t - \tau(t))),$$

where λ is constant, $a \in C(\mathbb{R}, \mathbb{R}), \tau : \mathbb{R} \to \mathbb{R}$, are continuous and ω-periodic with $\omega > 0$. The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and

ω-periodic in t. The proof of the next Lemma is trivial, and hence we omit it.

LEMMA 2.3. *$x(t) \in C_\omega$ is a solution of (2.1) if and only if*

$$x(t) = \lambda \int_t^{t+\omega} \frac{\exp(\int_s^t a(u)du)}{\exp(-\int_0^\omega a(u)du) - 1} f(s, x(s - \tau(s))ds. \tag{2.2}$$

Now, we define the cone K and the Green's function $G(t,s)$ for equation (1.1). For $(t,s) \in \mathbb{R}^2$, $j = 1, 2, \ldots, n$, we define

$$\sigma := \min\Big\{ \exp(-2\int_0^\omega |a_j(s)|ds), j = 1, 2, \ldots, n\Big\}, \tag{2.3}$$

$$G_j(t,s) = \frac{\exp(\int_s^t a_j(\nu)d\nu)}{\exp(-\int_0^\omega a_j(\nu)d\nu) - 1}. \tag{2.4}$$

We also define

$$G(t,s) = \text{diag}[G_1(t,s), G_2(t,s), \ldots, G_n(t,s)].$$

It is clear that $G(t,s) = G(t+\omega, s+\omega)$ for all $(t,s) \in \mathbb{R}^2$ and by (H1) and the assumption on f we have,

$$G_j(t,s) > 0,\ f_j(u, \phi(u - \tau(u)) > 0$$

for $(t,s) \in \mathbb{R}^2$ and $(u, \phi) \in \mathbb{R} \times BC(\mathbb{R}, \mathbb{R}^n_+)$. Let K be the set defined by

$$K = \{x \in C_\omega : x_j(t) \geq \sigma\|x_j\|,\ t \in [0,\omega],\ x = (x_1, x_2, \ldots, x_n)^T\}.$$

It is straight forward to verify that K is a cone.

Now, we are in a position to define an operator $\psi : K \to K$ as

$$(\psi x)(t) = \int_t^{t+\omega} G(t,s) f(s, x(s - \tau(s)))ds \tag{2.5}$$

for $x \in K$, $t \in \mathbb{R}$, where $G(t,s)$ is defined following (2.4). We denote

$$(\psi x) = \Big(\psi_1 x, \psi_2 x, \ldots, \psi_n x\Big)^T.$$

Before we proceed any further we state the followings:

$$A_j = \frac{e^{-\int_0^\omega |a_j(u)|du}}{e^{-\int_0^\omega a_j(u)du} - 1} \tag{2.6}$$

and

$$B_j = \frac{e^{\int_0^\omega |a_j(u)|du}}{e^{-\int_0^\omega a_j(u)du} - 1}, \tag{2.7}$$

for $j = 1, 2, \ldots, n$. It is easy to see that for for $j = 1, 2, \ldots, n$,

$$A_j \leq G_j(t, s) \leq B_j$$

for all $s \in [t, t + \omega]$.

If we set $A = \min_{1 \leq j \leq n} A_j$ and $B = \max_{1 \leq j \leq n} B_j$, then

$$A \leq G_j(t, s) \leq B \text{ for } j = 1, 2, \ldots, n.$$

LEMMA 2.4. *If $(\psi x)(t)$ is given by (2.5), then $\psi : K \to K$ is completely continuous.*

Proof. For each $x \in K$, since $f(t, x(t - \tau(t)))$ is a continuous function of t, we have $(\psi x)(t)$ is continuous in t and

$$\begin{aligned}
(\psi x)(t + \omega) &= \int_{t+\omega}^{t+2\omega} G(t + \omega, s) f(s, x(s - \tau(s))) ds \\
&= \int_{t}^{t+\omega} G(t + \omega, s + \omega) f(s + \omega, x(s + \omega - \tau(s + \omega))) ds \\
&= \int_{t}^{t+\omega} G(t, s) f(s, x(s - \tau(s))) ds = (\psi x)(t).
\end{aligned}$$

Thus, $(\psi x) \in C_\omega$. Next we show that (ψx) is continuous. For θ, $\vartheta \in C_\omega$, $\|\theta - \vartheta\| < \delta$ imply

$$\sup_{0 \leq s \leq \omega} |f_j(s, \theta(s - \tau(s))) - f_j(s, \vartheta(s - \tau(s)))| < \frac{\varepsilon}{\lambda n B_j \omega}.$$

If x, $y \in K$ with $\|x - y\| < \delta$, then

$$\begin{aligned}
|(\psi_j x)(t) &- (\psi_j y)(t)| \\
&\leq \lambda \int_{t}^{t+\omega} |G_j(t, s)| |f_j(s, x(s - \tau(s))) - f_j(s, y(s - \tau(s)))| ds \\
&\leq \lambda B_j \omega \sup_{0 \leq t \leq \omega} |f_j(t, \theta(t - \tau(t))) - f_j(t, \vartheta(t - \tau(t)))| \\
&< \frac{\varepsilon}{n}
\end{aligned}$$

for all $t \in [0, \omega]$. This yields to

$$||(\psi_j x)(t) - (\psi_j y)(t)||_0 < \frac{\varepsilon}{n}.$$

Thus,

$$||(\psi x) - (\psi y)|| < \varepsilon.$$

Hence, ψ is continuous. For $x \in K$, let

$$(\psi_j x)(t) = \lambda \int_0^\omega G_j(t,s) f_j(s, x(s-\tau(s)))ds.$$

Then,

$$(\psi_j x)(t) \leq \lambda B_j \int_0^\omega |f_j(s, x(s-\tau(s)))|ds$$

and

$$\begin{aligned}(\psi_j x)(t) &\geq \lambda A_j \int_0^\omega |f_j(s, x(s-\tau(s)))|ds \\ &\geq \frac{A_j}{B_j}\|\psi_j x\|_0 = \sigma\|\psi_j x\|_0, \;\; j = 1, 2, \ldots, n.\end{aligned}$$

Therefore, $(\psi x) \in K$. The proof of ψ being completely continuous is similar to the proof of [10], and hence we omit it. This completes the prove. □

3. Main results

Now we are ready to state and proof our results. But before we proceed we state the following.

(L1) $\lim\limits_{x_j \to 0^+} \frac{f_j(s,x(s-\tau(s)))}{x_j} = \infty$,

(L2) $\lim\limits_{x_j \to \infty} \frac{f(s,x(s-\tau(s)))}{x_j} = \infty$,

(L3) $\lim\limits_{x_j \to 0^+} \frac{f_j(s,x(s-\tau(s)))}{x_j} = 0$,

(L4) $\lim\limits_{x_j \to \infty} \frac{f_j(s,x(s-\tau(s)))}{x_j} = 0$,

(L5) $\lim\limits_{x_j \to 0^+} \frac{f_j(s,x(s-\tau(s)))}{x_j} = l_j$ uniformly in s with $0 < l_j < \infty$,
and

(L6) $\lim\limits_{x_j \to \infty} \frac{f_j(s,x(s-\tau(s)))}{x_j} = L_j$ uniformly in s with $0 < L_j < \infty$,

for $x \in \mathbb{R}^n$. For notational convenience, we let

$$L_M = \max_{1\leq j\leq n} L_j, \; L_m = \min_{1\leq j\leq n} L_j, \; l_M = \max_{1\leq j\leq n} l_j,$$
$$l_m = \min_{1\leq j\leq n} l_j \text{ and } |G(t,s)| = \max_{1\leq j\leq n} |G_j(t,s)|.$$

THEOREM 3.2. *Assume that* $(H1)$, $(L5)$, *and* $(L6)$ *hold. Then, for each* λ *satisfying*

$$\frac{1}{\omega\sigma A L_m} < \lambda < \frac{1}{\omega B l_M} \tag{3.1}$$

(1.1) *has at least one positive periodic solution.*

Proof. We construct the sets Ω_1 and Ω_2 in order to apply Theorem 2.1. Let λ be defined by (3.1), and choose $\epsilon > 0$ such that

$$\frac{1}{\omega\sigma A(L_m - \epsilon)} \leq \lambda \leq \frac{1}{\omega B(l_M + \epsilon)}.$$

By condition $(L5)$, there exists $H_1 > 0$ such that $f_j(t,y) \leq (l_j + \epsilon)y_j \leq (l_M + \epsilon)y_j$, for $0 < y_j \leq H_1$. Define $\Omega_1 = \{x \in K : \|x_j\|_0 < H_1,\ j = 1, \cdots, n\}$ and assume $x \in K \cap \partial\Omega_1$. Then

$$\begin{aligned}(\psi_j x)(t) &\leq \lambda B \int_0^\omega f_j(s, x(s - \tau(s)))ds \\ &\leq \lambda B\omega(l_j + \epsilon) \int_0^\omega x_j(s, x(s - \tau(s)))ds \\ &\leq \lambda B\omega(l_j + \epsilon)\|x_j\|_0 \\ &\leq \lambda B\omega(l_M + \epsilon)\|x_j\|_0 \\ &\leq \|x_j\|_0.\end{aligned}$$

In particular,

$$||\psi_j x||_0 \leq ||x_j||_0$$

and

$$||\psi x|| = \sum_{j=1}^{n} ||\psi_j x||_0 \leq \sum_{j=1}^{n} ||x_j||_0 = ||x|| \text{ for all } x \in K \cap \partial\Omega_1. \tag{3.2}$$

Next we construct the set Ω_2. Considering $(L6)$ there exists $\overline{H_2}$ such that $f_j(t,y) \geq (L_j - \epsilon)y_j \geq (L_m - \epsilon)y_j$, for all $y_j \geq \overline{H_2}$. Let $H_2 = \max\{2H_1, \frac{\overline{H_2}}{\sigma}\}$ and set

$$\Omega_2 = \{x \in K : \|x_j\|_0 < H_2,\ j = 1, \cdots, n\}.$$

If $x \in K$ with $||x|| \geq H_2$, then

$$x_j \geq \sigma||x_j|| \geq \overline{H_2}.$$

Thus

$$(\psi_j x)(t) \ \geq \ \lambda A \int_0^\omega f_j(s, x(s - \tau(s)))ds \geq \lambda A\omega\sigma(L_m - \epsilon)||x_j||_0.$$

Hence

$$||\psi_j x||_0 \geq ||x_j||_0$$

and

$$(3.3) \qquad ||\psi x|| = \sum_{j=1}^{n} ||\psi_j x||_0 \geq \sum_{j=1}^{n} ||x_j||_0 = ||x|| \text{ for all } x \in K \cap \partial\Omega_2.$$

Applying (i) of Theorem 2.1 to (3.2) and (3.3) yields that ψ has a fixed point $x \in K \cap (\overline{\Omega}_2 \backslash \Omega_1)$. The proof is complete. □

THEOREM 3.3. *Assume that* $(H1)$, $(L5)$, *and* $(L6)$ *hold. Then, for each* λ *satisfying*

$$(3.4) \qquad \frac{1}{\omega\sigma A l_m} < \lambda < \frac{1}{\omega B L_M}$$

(1.1) *has at least one positive periodic solution.*

Proof. We construct the sets Ω_1 and Ω_2 in order to apply Theorem 2.1. Let λ be given as in (3.4), and choose $\epsilon > 0$ such that

$$\frac{1}{\sigma A(l_m - \epsilon)} \leq \lambda \leq \frac{1}{B(L_M + \epsilon)}.$$

By condition $(L5)$, there exists $H_1 > 0$ such that $f_j(t,y) \geq (l_j - \epsilon)y_j \geq (l_m - \epsilon)y_j$, for $0 < y_j \leq H_1$. Define $\Omega_1 = \{x \in K : \|x_j\|_0 < H_1, j = 1, 2, \cdots, \cdot, \cdot, n\}$, and assume $x \in K \cap \partial\Omega_1$. Then

$$\begin{aligned}
(\psi_j x)(t) &\geq \lambda A \int_0^{\omega} f_j(s, x(s - \tau(s)))ds \\
&\geq \lambda A\omega(l_m - \epsilon)x_j(t - \tau(t)) \\
&\geq \lambda A\sigma\omega(l_m - \epsilon)\|x_j\|_0 \\
&\geq \|x_j\|_0.
\end{aligned}$$

In particular,

$$||\psi_j x||_0 \geq ||x_j||_0 \text{ for all } x \in K \cap \partial\Omega_1$$

and

$$(3.5) \qquad ||\psi x|| = \sum_{j=1}^{n} ||\psi_j x||_0 \geq \sum_{j=1}^{n} ||x_j||_0 = ||x||, \text{ for all } x \in K \cap \partial\Omega_1.$$

Next we construct the set Ω_2. Considering $(L6)$ there exists $\overline{H_2}$ such that $f_j(t,y) \leq (L_j + \epsilon)y_j \leq (L_M + \epsilon)y_j$, for $y_j \geq \overline{H_2}$.
We consider two cases; $f_j(t,y)$ is bounded and $f_j(t,y)$ is unbounded.

The case where $f_j(t,y)$ is bounded is straight forward. If $f_j(t,y)$ is bounded by $Q>0$, set

$$H_2 = \max\{2H_1, \omega\lambda QB\}.$$

Then if $x \in K$ and $||x||_0 = H_2$, we have

$$\begin{aligned}(\psi_j x)(t) &\le \lambda B \int_0^{\omega} f_j(s, x(s-\tau(s)))ds \\ &\le \omega\lambda BQ \le \|x_j\|_0.\end{aligned}$$

Consequently, $||\psi_j x||_0 \le ||x_j||_0$, and hence $||\psi x|| \le ||x||$.
So, if we set

$$\Omega_2 = \{y \in K : \|y_j\| < H_2, j = 1, 2, \cdot, \cdot, \cdot, n\},$$

then

$$||\psi x|| \le ||x||, \text{ for } x \in K \cap \partial\Omega_2. \tag{3.6}$$

When f is unbounded, we let $H_2 > \max\{2H_1, \overline{H_2}\}$ be such that $f_j(t,y) \le f_j(t, H_2)$, for $0 < y_j \le H_2$. For $x \in K$ with $||x_j||_0 = H_2$,

$$\begin{aligned}(\psi_j x)(t) &\le \lambda B \int_0^{\omega} f_j(s, x(s-\tau(s)))ds \\ &\le \lambda B \int_0^{\omega} f_j(s, H_2)ds \\ &\le \lambda B \int_0^{\omega} (L_j + \epsilon)H_2 ds \\ &\le \lambda B\omega(L_M + \epsilon)\|x_j\|_0 \\ &\le \|x_j\|_0.\end{aligned}$$

Consequently, $||\psi_j x|| \le ||x_j||_0$, which implies that

$$||\psi x|| = \sum_{j=1}^{n} ||\psi_j x||_0 \le \sum_{j=1}^{n} ||x_j||_0 = ||x||.$$

So, if we set

$$\Omega_2 = \{x \in \mathcal{K} : \|x_j\|_0 < H_2, j = 1, 2, \cdot, \cdot, \cdot, n\},$$

then

$$||\psi x|| \le ||x||, \text{ for } x \in K \cap \partial\Omega_2. \tag{3.7}$$

Applying (ii) of Theorem 2.1 to (3.5) and (3.6) yields that T has a fixed point $x \in K \cap (\overline{\Omega}_2 \backslash \Omega_1)$. Also, applying (ii) of Theorem 2.1 to (3.5) and (3.7) yields that ψ has a fixed point $x \in K \cap (\overline{\Omega}_2 \backslash \Omega_1)$. The proof is complete. □

THEOREM 3.4. *Assume that* $(H1)$, $(L1)$, *and* $(L6)$ *hold. Then, for each* λ *satisfying*

$$0 < \lambda < \frac{1}{\omega A L_M}, \tag{3.8}$$

(1.1)–(1.2) *has at least one positive solution.*

Proof. Apply $(L1)$ and choose $H_1 > 0$ such that if $0 < x_j < H_1$, then

$$f_j(t, x) \geq \frac{x_j}{\lambda \gamma A}.$$

Define

$$\Omega_1 = \{x \in K : \|x_j\|_0 < H_1\}.$$

If $x \in K \cap \partial\Omega_1$, then

$$\begin{aligned}
(\psi_j x)(t) &\geq \lambda A \int_0^\omega f_j(s, x(s - \tau(s)))ds \\
&\geq \lambda A \int_0^\omega \frac{x_j(s, s - \tau(s))}{\lambda \sigma A} ds \\
&\geq \lambda A \int_0^\omega \frac{\sigma \|x_j\|_0}{\lambda \sigma A} ds \\
&= \|x_j\|_0.
\end{aligned}$$

In particular, $\|\psi x\| \geq \|x\|$, for all $x \in K \cap \partial\Omega_1$. In order to construct Ω_2, we let λ be given as in (3.8), and choose $\epsilon > 0$ such that

$$0 \leq \lambda \leq \frac{1}{B\omega(L_M + \epsilon)}.$$

The construction of Ω_2 follows along the lines of the construction of Ω_2 in Theorem 3.3, and hence we omit it. Thus, by (ii) of Theorem 2.1, equation (1.1) has at least one positive solution. □

THEOREM 3.5. *Assume that* $(H1)$, $(L2)$, *and* $(L5)$ *hold. Then, for each* λ *satisfying*

$$0 < \lambda < \frac{1}{B\omega l_M}, \tag{3.9}$$

(1.1)–(1.2) *has at least one positive solution.*

Proof. Assume $(L5)$ holds. Then, we may take the set Ω_1 to be the one obtained for Theorem 3.2. That is,

$$\Omega_1 = \{x \in K : \|x_j\|_0 < H_1,\ j = 1, 2, ...n\}.$$

Hence, we have

$$||\psi x|| \le ||x||, \text{ for } x \in K \cap \partial\Omega_1.$$

Next, we assume $(L2)$. Choose $\overline{H_2} > 0$ such that $f_j(t,x) \ge \frac{x_j}{\lambda\sigma A}$, for $x_j \ge \overline{H_2}$. Let $H_2 = \max\{2H_1, \frac{\overline{H_2}}{\sigma}\}$ and set

$$\Omega_2 = \{x \in K : \|x_j\|_0 < H_2\}.$$

If $x \in K$ with $||x||_0 = H_2$,

$$\begin{aligned}(\psi_j x)(t) &\ge \lambda A \int_0^\omega f_j(s, x(s-\tau(s)))ds \\ &\ge \lambda A \int_0^\omega \frac{x_j(s, s-\tau(s))}{\lambda\sigma A} ds \\ &\ge \lambda A \int_0^\omega \frac{\sigma||x_j||_0}{\lambda\sigma A} ds \\ &= \|x_j\|_0.\end{aligned}$$

In particular, $||\psi x|| \ge ||x||$, for all $x \in K \cap \partial\Omega_2$. Consequently,

$$||\psi x|| \ge ||x||, \text{ for } x \in K \cap \partial\Omega_2.$$

Applying (i) of Theorem 2.1 yields that ψ has a fixed point $x \in K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

We state the next results as corollary, because by now, its proof can be easily obtained from the proofs of the previous results.

COROLLARY 3.6. *Assume that $(H1)$ hold. Also, if either $(L3)$ and $(L6)$ hold, or, $(L4)$, and $(L5)$ hold, then* (1.1)–(1.2) *has at least one positive solution if λ satisfies either $1/(\sigma A L_m) < \lambda$, or, $1/(\sigma A l_m) < \lambda$.*

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Department of Mathematics
University of Dayton
Dayton, OH 45469-2316, U.S.A
E-mail: youssef.raffoul@notes.udayton.edu