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LATTICE ACTION ON FINITE VOLUME HOMOGENEOUS SPACES

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ABSTRACT. We study the distribution of a dense orbit of a lattice Λ acting by the right multiplication on the space $\Gamma\backslash G$ where G is a connected simple Lie group and Γ its lattice. We show that for $G = \mathrm{SL}_n(\mathbb{R})$, every dense orbit is equidistributed with respect to the Euclidean norm.

1. Introduction

Let G be a connected non-compact simple (linear) Lie group with finite center. A discrete subgroup in G of finite co-volume is called a lattice in G. Two lattices Γ and Δ in G are called commensurable with each other if the intersection $\Gamma \cap \Delta$ is of finite index both in Γ and Δ.

Let Δ and Γ be lattices in G and consider the right translation action of Δ on the homogeneous space $\Gamma\backslash G$. If Δ is commensurable with Γ, then the orbit $\Gamma\backslash\Gamma\Delta$ consists of only finitely many points of cardinality $[\Delta : \Delta \cap \Gamma]$ and in particular is discrete in $\Gamma\backslash G$. If Δ is *not* commensurable with Γ, then the orbit $\Gamma\backslash\Gamma\Delta$ is dense in $\Gamma\backslash G$. Indeed, it is a rather simple consequence of Ratner's theorem([10]) that the orbit of $(\Gamma \times \Delta)$ on $(\Gamma \times \Delta)\backslash(G \times G)$ is dense under the diagonal action of G (cf. Lemma 3.1). Therefore the set $\{(\Gamma g, \Delta g) : g \in G\}$ is dense in $G \times G$. It follows that $\Gamma\Delta$ is dense in G, being the image of a dense subset of $G \times G$ under the continuous map $G \times G \to G$ given by $(g, h) \mapsto gh^{-1}$. This fact was first stated by Vatsal in [13] (for p-adic case) where he used this fact as a crucial ingredient in proving the main result in [13].

In this paper, we study finer properties of the distribution of the dense orbit $\Gamma\backslash\Gamma\Delta$ on $\Gamma\backslash G$. Fix a maximal compact subgroup K of G and consider a well rounded (see Definition 2.1) family $\{G_R \subset G : R > 0\}$

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of left K-invariant subsets of G. For a subset $S \subset G$, S_R denotes the intersection $S \cap G_R$ for any $R > 0$.

DEFINITION 1.1. We say that Δ_R is equidistributed on $\Gamma\backslash G$ as $R \to \infty$ if

$$\frac{\#\{\delta \in \Delta_R : \Gamma\delta^{-1} \in \Omega_1\}}{\#\{\delta \in \Delta_R : \Gamma\delta^{-1} \in \Omega_2\}} \sim \frac{\mathrm{vol}(\Omega_1)}{\mathrm{vol}(\Omega_2)} \quad \text{as } R \to \infty$$

for any two compact subsets $\Omega_1, \Omega_2 \subset \Gamma\backslash G$ with non-empty interior and piecewise smooth boundary.

Let $G = KAN$ be an Iwasawa decomposition and set $B = AN$. We first show that the equidistribution of Δ_R on $\Gamma\backslash G$ as $R \to \infty$ holds if B_R is uniformly distributed on the product space $(\Gamma \times \Delta)\backslash(G \times G)$ via the diagonal action, in the sense that, denoting by ρ a *right* invariant Haar measure on B,

$$(1.2) \quad \frac{1}{\rho(B_R)} \int_{B_R} f(\Gamma b^{-1}, \Delta b^{-1})d\rho(b) \to \int_{(\Gamma\times\Delta)\backslash(G\times G)} f\, d\mu \ \text{ as } R \to \infty$$

for any continuous function f on $(\Gamma \times \Delta)\backslash(G \times G)$ with compact support, where μ denotes the probability Haar measure on $(\Gamma \times \Delta)\backslash(G \times G)$ (see Theorem 2.6).

Secondly, we verify the uniform distribution of B_R on $(\Gamma\times\Delta)\backslash(G\times G)$, as described above, in the special case when $G = \mathrm{SL}_n(\mathbb{R})$, B is the identity component of the upper triangular subgroup and the family $\{G_R : R > 0\}$ is given by

$$(1.3) \qquad G_R = \{(g_{ij}) \in \mathrm{SL}_n(\mathbb{R}) : \sqrt{\textstyle\sum g_{ij}^2} \le R\}.$$

Therefore we obtain:

THEOREM 1.4. *Let $n \ge 2$. Let Δ and Γ be lattices in $\mathrm{SL}_n(\mathbb{R})$ which are not commensurable with each other. Then for any nice* (see 2.3) *compact subset $\Omega \subset \Gamma\backslash G$,*

$$\#\{g = (g_{ij}) \in \Delta : \Gamma g^{-1} \in \Omega, \sqrt{\textstyle\sum g_{ij}^2} \le R\} \sim \frac{\mathrm{vol}(G_R)\cdot\mathrm{vol}(\Omega)}{\mathrm{vol}(\Delta\backslash G)\cdot\mathrm{vol}(\Gamma\backslash G)} \quad \text{as } R \to \infty,$$

where G_R is given as in (1.3). *In particular, Δ_R is equi-distributed on $\Gamma\backslash G$ as $R \to \infty$.*

All volumes appearing in the right hand side of the above asymptotic are to be computed with respect to one fixed Haar measure on G.

The basic ingredient of the proof of the uniform distribution of B_R in (1.2) is Ratner's measure classification theorem invariant under unipotent flows ([9]) as well as the work of Dani and Margulis on the behavior of unipotent flows near cusps ([2]). Setting $X = (\Gamma \times \Delta)\backslash(G \times G)$, the left hand side of (1.2) defines a probability measure, say ρ_R, on the homogeneous space X. Then (1.2) is precisely saying that ρ_R weakly converges to μ as $R \to \infty$. One first shows that any weak limit of ρ_R is a probability measure on X, resorting to the work of Dani and Margulis[2] in the form refined by Shah[12]. In this step the unipotent subgroup we use is just N. However in order to apply Ratner's measure classification theorem in showing that any weak limit of ρ_R is indeed $G \times G$-invariant, we need to know that the weak limits of ρ_R are invariant under some unipotent flows. For a general well rounded family in a general simple Lie group, this seems a hard part to prove, and that is the essential reason why our theorem is proved only for $\mathrm{SL}_n(\mathbb{R})$ and the family G_R as in (1.3).

Gorodnik showed the uniform distribution of B_R on $\Gamma\backslash \mathrm{SL}_n(\mathbb{R})$ for any lattice Γ ([4]). The second part of the proof of Theorem 1.3, which is the uniform distribution of B_R on the product space $(\Gamma\times\Delta)\backslash(\mathrm{SL}_n(\mathbb{R})\times \mathrm{SL}_n(\mathbb{R}))$, closely follows his work.

Remark. Recently in [5] Gorodnik and the author were able to prove that (1.2) holds for any connected non-compact simple Lie group G with finite center, where G_R is defined as the Riemannian balls of radius R, by a completely different approach from the methods described here as well as from those of [4]. Together with our results in section 2 of this paper, this implies that Theorem 1.3 holds for any connected non-compact simple Lie group G with finite center with respect to a family of Riemannian balls in G.

On the other hand, another recent work of Gorodnik and Weiss[6] also gives a different method of proving Theorem 1.3 in a greater generality.

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2. Relation between Δ-action on $\Gamma\backslash G$ and B-action on $(\Gamma \times \Delta)\backslash(G \times G)$

Let G be a connected semisimple (real) linear Lie group with finite center. Consider an Iwasawa decomposition $G = KAN$ where K is a maximal compact subgroup of G, A is the connected component of a maximal real split torus and N the unipotent radical of a minimal parabolic subgroup normalized by A. Set $B = AN$.

Fix a right invariant Haar measure ρ on B and let dk be the probability Haar measure on K. Denote by μ_G the measure on G defined as follows: for any $f \in C_c(G)$,

$$\int_G f\, d\mu_G = \int_B \int_K f(kb)\, dk d\rho(b).$$

It is standard to check that μ_G is a Haar measure on G.

For any discrete subgroup Γ of G, there exists a unique right invariant measure (cf. [8]), which we also denote by μ_G by a slight abuse of notation, such that for any $f \in C_c(G)$,

$$\int_G f\, d\mu_G = \int_{[g]\in\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(\gamma g) d\mu_G([g]).$$

DEFINITION 2.1. For a given family $\mathcal{F} = \{G_R \subset G : R > 0\}$ of subsets of G, we say $\mathcal{F}$ is well rounded if the following conditions hold:

(A) for all sufficiently small $\epsilon > 0$, there exists a neighborhood U_ϵ of e in G and $k_\epsilon \geq 1$ such that $k_\epsilon \to 1$ as $\epsilon \to 0$ and

$$G_{k_\epsilon^{-1}R} \subset U_\epsilon G_R \subset G_{k_\epsilon R} \text{ for all sufficiently large } R > 0;$$

(B) for all sufficiently small $\epsilon > 0$, there exist $a(k_\epsilon) \geq 1$ and $b(k_\epsilon) \leq 1$ such that

$$b(k_\epsilon) \leq \liminf_{R\to\infty} \frac{\mu_G(G_{k_\epsilon^{-1}R})}{\mu_G(G_R)} \leq \limsup_{R\to\infty} \frac{\mu_G(G_{k_\epsilon R})}{\mu_G(G_R)} \leq a(k_\epsilon)$$

with $a(k_\epsilon)$ and $b(k_\epsilon)$ going to 1 as $k_\epsilon \to 1$.

EXAMPLE 2.2. Let $G \subset \mathrm{M_n}(\mathbb{R})$. Suppose that $\|\cdot\|$ is a norm on $\mathrm{M_n}(\mathbb{R})$ which is bi K-invariant, i.e., $\|k_1 X k_2\| = \|X\|$ for all $X \in \mathrm{M_n}(\mathbb{R})$ and $k_1, k_2 \in K$. Then it follows from [3, Lemma 2.2 and appendix 1] that for

$$G_R := \{g \in G : \|g\| \leq R\},$$

the family $\{G_R : R > 0\}$ is well rounded.

For each $R > 0$ and any subset S of G, we set

$$S_R = G_R \cap S.$$

DEFINITION 2.3. Let Γ be a lattice in G. A compact subset $\Omega \subset \Gamma\backslash G$ is *nice* if for any $\epsilon > 0$, there exists a neighborhood U_ϵ of e in G such that

$$(2.4) \qquad (1-\epsilon)\mu_G(\Omega_\epsilon^+) \leq \mu_G(\Omega) \leq (1+\epsilon)\mu_G(\Omega_\epsilon^-),$$

where

$$\Omega_\epsilon^+ = \cup_{g\in U_\epsilon}\Omega g \quad\text{and}\quad \Omega_\epsilon^- = \cap_{g\in U_\epsilon}\Omega g.$$

Let Δ and Γ be lattices in G. Set

$$X = (\Gamma \times \Delta)\backslash(G \times G)$$

and denote by μ the normalized Haar measure on X. In fact,

$$d\mu = \frac{1}{\mu_G(\Gamma\backslash G)\cdot \mu_G(\Delta\backslash G)}(d\mu_G \times d\mu_G).$$

For the rest of this section, we fix a *well rounded* family $\{G_R \subset G : R > 0\}$ of subsets of G. We also assume that each G_R is left K-invariant, i.e., $KG_R = G_R$. It follows that $G_R = KB_R$ for each $R > 0$.

The aim of this section is to relate the uniform distribution of B_R on X as $R \to \infty$ with the equidistribution of Δ_R on $\Gamma\backslash G$.

DEFINITION 2.5. For any $f \in C_c(X)$, set

$$\rho_R(f) = \frac{1}{\rho(B_R)}\int_{B_R} f(\Gamma b^{-1}, \Delta b^{-1})\, d\rho(b).$$

By the Riesz representation theorem, ρ_R defines a probability measure on X. We denote by $\mathcal{P}(X)$ the space of probability measures on X with the weak*-topology.

The main theorem in this section is the following:

THEOREM 2.6. *If $\rho_R \to \mu$ as $R \to \infty$ in $\mathcal{P}(X)$, then for any nice compact subset Ω in $\Gamma\backslash G$,*

$$\#\{\delta \in \Delta_R : \Gamma\delta^{-1} \in \Omega\} \sim \frac{\mathrm{vol}(G_R)\cdot \mathrm{vol}(\Omega)}{\mathrm{vol}(\Gamma\backslash G)\cdot \mathrm{vol}(\Delta\backslash G)} \text{ as } R \to \infty$$

(here all volumes are computed with respect to μ_G).

Fixing a piecewise continuous function ϕ on $\Gamma\backslash G$ with compact support, and $R > 0$, we define for any $g \in G$

$$F_R^\phi(g) := \sum_{\delta\in\Delta} \chi_{G_R^{-1}}(\delta g)\cdot \phi(\Gamma\delta g).$$

Notice that F_R^ϕ is a well defined function on $\Delta\backslash G$. If χ_Ω is the characteristic function of a set $\Omega \subset \Gamma\backslash G$, then we set, for simplicity,

$$F_R^{\chi_\Omega} = F_R^\Omega.$$

The reason we define this function is that the value of F_R^Ω at the identity e is precisely the left hand side of the asymptotic in Theorem 2.6:

$$F_R^\Omega(e) = \#\{\delta \in \Delta_R : \Gamma\delta^{-1} \in \Omega\}.$$

To show Theorem 2.6, we approximate the value $F_R^\Omega(e)$ using the inner products of functions in $L^2(\Delta\backslash G)$.

Fix a nice compact subset $\Omega \subset \Gamma\backslash G$. For each $\epsilon > 0$, let U_ϵ be a bounded symmetric neighborhood of e in G as in the definition (2.3). We may assume that $U_\epsilon \cap \Delta = \{e\}$ for each $\epsilon > 0$ without loss of generality. In what follows, all inner products $\langle \cdot, \cdot \rangle$ are taken in the space $L^2(\Delta\backslash G)$ with respect to the measure μ_G. Let k_ϵ be as in Definition 2.1(A).

LEMMA 2.7. *Let ψ_ϵ be a non-negative continuous function on $\Delta\backslash G$ supported on $\Delta\backslash\Delta U_\epsilon$ with $\int_{\Delta\backslash G} \psi_\epsilon d\mu_G = 1$. Then for any $R > 0$,*

$$\langle F_{k_\epsilon^{-1}R}^{\Omega_\epsilon^-}, \psi_\epsilon \rangle \leq F_R(e) \leq \langle F_{k_\epsilon R}^{\Omega_\epsilon^+}, \psi_\epsilon \rangle. \tag{2.8}$$

Proof. Observe that

$$F_R^\Omega(g) = \#\{\delta \in \Delta : \delta \in G_R^{-1} g^{-1}, \delta \in \Omega g^{-1}\}.$$

It then follows from (2.3) and (2.1(A)) that for any $g \in U_\epsilon$,

$$F_{k_\epsilon^{-1}R}^{\Omega_\epsilon^-}(g) \leq F_R^\Omega(e) \leq F_{k_\epsilon R}^{\Omega_\epsilon^+}(g) \quad \text{for all } g \in \Delta\backslash\Delta U_\epsilon, \tag{2.9}$$

where Ω_ϵ^+ and Ω_ϵ^- are defined as in Definition 2.3. Hence, taking integrals against ψ_ϵ in (2.9) yields the claim. □

In the following lemma and the proposition, assume that ϕ is a piecewise continuous function on $\Gamma\backslash G$ with compact support and the set of points of discontinuity has measure zero.

LEMMA 2.10. *For any $\psi \in C_c(\Delta\backslash G)$,*

$$\langle F_R^\phi, \psi \rangle = \int_{g \in G_R} (\phi \times \psi)(\Gamma g^{-1}, \Delta g^{-1}) d\mu_G(g).$$

Proof. Observe that

$$\begin{aligned}\langle F_R^\phi, \psi\rangle &= \int_{\Delta\backslash G}\left(\sum_{\delta\in\Delta}\chi_{G_R^{-1}}(\delta g)\phi(\Gamma\delta g)\right)\psi(\Delta g)\,d\mu_G(g)\\ &= \int_{\Delta\backslash G}\left(\sum_{\delta\in\Delta}\chi_{G_R^{-1}}(\delta g)\phi(\Gamma\delta g)\psi(\Delta\delta g)\right)d\mu_G(g)\\ &= \int_{g\in G}\chi_{G_R^{-1}}(g)\phi(\Gamma g)\psi(\Delta g)\,d\mu_G(g)\\ &= \int_{g\in G_R}\phi(\Gamma g^{-1})\psi(\Delta g^{-1})d\mu_G(g).\end{aligned}$$

□

PROPOSITION 2.11. *If $\rho_R \to \mu$ as $R\to\infty$ in $\mathcal{P}(X)$, then for any $\psi\in C_c(\Delta\backslash G)$*

$$\frac{1}{\mu_G(G_R)}\langle F_R^\phi,\psi\rangle \to \int_X \phi\times\psi\,d\mu \text{ as } R\to\infty.$$

Proof. Define a function f on X by

$$f(\Gamma g,\Delta h) := \int_{k\in K}\phi(\Gamma gk)\psi(\Delta hk)\,dk \quad \text{for } g,h\in G.$$

Since $G_R = KB_R$, we deduce from Lemma 2.10 that

$$\begin{aligned}\langle F_R^\phi,\psi\rangle &= \int_{g\in G_R}\phi(\Gamma g^{-1})\psi(\Delta g^{-1})d\mu_G(g)\\ &= \int_K\int_{b\in B_R}\phi(\Gamma b^{-1}k^{-1})\psi(\Delta b^{-1}k^{-1})\,d\rho(b)dk\\ &= \int_{b\in B_R}\left(\int_K\phi(\Gamma b^{-1}k)\psi(\Delta b^{-1}k)dk\right)d\rho(b)\\ &= \int_{b\in B_R} f(\Gamma b^{-1},\Delta b^{-1})\,d\rho(b).\end{aligned}$$

Since f is piecewise continuous on X with compact support and the set of points of discontinuity has measure zero, the assumption now implies that

$$\langle F_R^\phi,\psi\rangle \sim \rho(B_R)\cdot\int_X f\,d\mu \text{ as } R\to\infty.$$

It follows from the invariance of the measure μ that

$$\int_X f\,d\mu = \frac{1}{\mu_G(\Gamma\backslash G)\cdot\mu_G(\Delta\backslash G)}\int_{\Gamma\backslash G}\phi\,d\mu_G\cdot\int_{\Delta\backslash G}\psi\,d\mu_G.$$

Since $G_R = KB_R$ and $\int_K dk = 1$, we have $\mu_G(G_R) = \rho(B_R)$. Therefore the claim follows. □

Proof of Theorem 2.6. By Proposition 2.11, the left and right hand sides of (2.8) are asymptotically, as $R \to \infty$, equal to the product of $\frac{1}{\mathrm{vol}(\Gamma\backslash G)\cdot\mathrm{vol}(\Delta\backslash G)}$ with

$$\mu_G(G_{k_\epsilon^{-1}R}) \cdot \mu_G(\Omega_\epsilon^-) \quad \text{and} \quad \mu_G(G_{k_\epsilon R}) \cdot \mu_G(\Omega_\epsilon^+)$$

respectively.

Since by (2.1(B))

$$\frac{b(k_\epsilon)}{a(k_\epsilon)} \le \frac{\mu_G(G_{k_\epsilon R})}{\mu_G(G_{k_\epsilon^{-1}R})} \le \frac{a(k_\epsilon)}{b(k_\epsilon)},$$

we have

$$\frac{\mu_G(G_{k_\epsilon R})}{\mu_G(G_{k_\epsilon^{-1}R})} \to 1 \text{ as } R \to \infty \text{ and } \epsilon \to 0.$$

Also by (2.3)

$$\frac{1-\epsilon}{1+\epsilon} \le \frac{\mu_G(\Omega_\epsilon^+)}{\mu_G(\Omega_\epsilon^-)} \le \frac{1+\epsilon}{1-\epsilon},$$

we have

$$\frac{\mu_G(\Omega_\epsilon^+)}{\mu_G(\Omega_\epsilon^-)} \to 1 \quad \text{as } \epsilon \to 0.$$

Therefore we obtain as $R \to \infty$

$$F_R^\Omega(e) \sim \frac{\mu_G(G_R)\cdot\mu_G(\Omega)}{\mu_G(\Gamma\backslash G)\cdot\mu_G(\Delta\backslash G)},$$

finishing the proof. □

3. Unipotent flows on $(\Gamma \times \Delta)\backslash(G \times G)$

Let G be a connected simple non-compact linear Lie group with finite center. This guarantees that G is generated by unipotent one parameter subgroups in G. We denote by G_0 the image of G under the diagonal embedding into $G \times G$,

$$G_0 = \{(g,g) : G \times G : g \in G\}.$$

The reason we insist G being simple rather than semisimple is to ensure that G_0 is a maximal connected closed subgroup of $G \times G$.

Let Δ and Γ be lattices in G which are not commensurable with each other. The following is a well known consequence of Ratner's theorem.

THEOREM 3.1. *The orbit* $(\Gamma \times \Delta)G_0$ *is dense in* $(\Gamma \times \Delta)\backslash(G \times G)$.

Proof. Since G is generated by unipotent one parameter subgroups, by the theorem of Ratner[10], there exists a connected closed subgroup H of $G \times G$ containing G_0 such that the orbit $(\Gamma \times \Delta)H$ is closed and $H \cap (\Gamma \times \Delta)$ is a lattice in H. Since G_0 is a maximal closed subgroup in $G \times G$, it follows that either $H = G \times G$ or $H = G_0$. In the latter case, $(\Gamma \times \Delta) \cap G_0$ is a lattice in G_0, or equivalently $\Gamma \cap \Delta$ is a lattice in G. It follows that $\Gamma \cap \Delta$ has finite index both in Δ and Γ, that is, Δ and Γ are commensurable with each other, contradicting our assumption. Hence $H = G \times G$, proving our claim. □

Denoting by $\mathfrak{g}$ the Lie algebra of G, consider the vector space

$$V = \oplus_{i=1}^{\dim(G\times G)} \wedge^i (\mathfrak{g} \oplus \mathfrak{g}), \tag{3.2}$$

with a fixed norm $\|\cdot\|$. The group $G \times G$ acts on V via the adjoint representation. For any closed subgroup H in $G \times G$ with the Lie algebra $\mathfrak{h}$, we define a unit vector in V:

$$p_H \in \wedge^{\dim(H)}\mathfrak{h}.$$

DEFINITION 3.3. Set $\mathcal{H}$ to be the collection of all *proper* non-trivial closed connected subgroups $H \subset G \times G$ such that $(\Gamma \times \Delta)H$ is closed in $(\Gamma \times \Delta)\backslash(G \times G)$, $(\Gamma \times \Delta) \cap H$ is a lattice in H and the subgroup generated by all one parameter unipotent subgroups of H acts ergodically on $((\Gamma \times \Delta) \cap H)\backslash H$ with respect to the H-invariant probability measure.

If $H \in \mathcal{H}$, $\mathrm{Ad}(H \cap (\Gamma \times \Delta))$ is Zariski dense in $\mathrm{Ad}(H)$ (see [12]).

THEOREM 3.4. [1, Theorem 3.4] *For any* $H \in \mathcal{H}$, *we have* $(\Gamma \times \Delta)p_H$ *is discrete in* V *and* $[\Gamma \times \Delta]N_{G\times G}(H)^1$ *is closed in* $(\Gamma \times \Delta)\backslash(G \times G)$, *where*

$$N_{G\times G}(H)^1 := \{g \in G \times G : gp_H = p_H\}.$$

Write $V = V_0 \oplus V_1$ where V_0 is the subspace of all G_0-invariant vectors and V_1 a G_0-invariant complement. Let $\mathrm{pr}_i : V \to V_i$ be the projection for each $i = 0, 1$.

LEMMA 3.5. *Let* $H \in \mathcal{H}$ *be such that* $H \neq G \times \{e\}, \{e\} \times G$. *Then*

$$0 \notin \mathrm{pr}_1((\Gamma \times D)p_H).$$

Proof. Suppose not. Then for some $x \in \Gamma \times \Delta$, $xG_0x^{-1} \subset N_{G\times G}(H)^1$. Since $(\Gamma \times \Delta)G_0$ is dense in $G \times G$ by Theorem 3.1 and $(\Gamma \times \Delta)N_{G\times G}(H)^1$ is closed in $G \times G$ by the above theorem, we have $(\Gamma \times \Delta)N_{G\times G}(H)^1 = G \times G$. By Baire category theorem, it implies that $G \times G = N_{G\times G}(H)^1$.

Hence H is a normal subgroup of $G \times G$. The only proper connected normal subgroups of $G \times G$ are $\{e\} \times G$ and $G \times \{e\}$. This proves the claim. $\square$

Fix $m \in \mathbb{N}$. For any $d, n \in \mathbb{N}$, the notation $\mathcal{P}_{d,n}$ denotes the set of functions $q : \mathbb{R}^m \to G \times G$ such that for any $a, b \in \mathbb{R}^m$, the map

$$t \mapsto \mathrm{Ad}(q(at + b))$$

is a polynomial of degree at most d with respect to some basis of the Lie algebra of $G \times G$.

We now state two main ingredients of our proof of the uniform distribution of B_R as in (1.2). Both theorems below hold for any connected semisimple Lie group and its lattices without any change. We state this way merely for our convenience for later use.

THEOREM 3.6. [12, Theorem 2.2] *Let $m \in \mathbb{N}$ be fixed. Suppose that $(\Gamma \times \Delta)\backslash(G \times G)$ is not compact. There exists closed subgroups $U_1, \cdots, U_l$ such that each is the unipotent radical of a proper parabolic subgroup in $G \times G$, $(\Gamma \times \Delta)U_i$ is compact and for any given $d, n \in \mathbb{N}$, and $\epsilon, \delta > 0$, there exists a compact subset $C \subset (\Gamma \times \Delta)\backslash(G \times G)$ such that for any $q \in \mathcal{P}_{d,n}$ and a bounded open convex subset $D \subset \mathbb{R}^m$, one of the following holds:*

(1) *There exist $x \in (\Gamma \times \Delta)$ and $1 \le i \le l$ such that*

$$\sup_{t \in D} \|q(t)^{-1} x p_{U_i}\| \le \delta.$$

(2) *$|\{t \in D : (\Gamma \times \Delta)q(t) \notin C\}| \le \epsilon|D|$, where $|\cdot|$ denotes the usual Lebesgue measure on $\mathbb{R}^m$.*

For each $H \in \mathcal{H}$ and a subgroup $U_0 \subset G \times G$, define

$$X(H, U_0) = \{(g_1, g_2) \in G \times G : (g_1, g_2)U_0 \subset H(g_1, g_2)\}.$$

THEOREM 3.7. [11] *Let $\epsilon > 0$ and $H \in \mathcal{H}$. Let U_0 be any one parameter unipotent subgroup of $G \times G$. For every compact $C \subset (\Gamma \times \Delta)\backslash(\Gamma \times \Delta)X(H, U_0)$, there exists a compact subset $F \subset V$ such that for every neighborhood K of F in V there exists a neighborhood Ψ of C in $(\Gamma \times \Delta)\backslash G \times G$ such that for any $q \in \mathcal{P}_{d,n}$ and every bounded open convex subset $D \subset \mathbb{R}^m$ one of the following holds:*

(1) *for some $x \in (\Gamma \times \Delta)$, $\{(q(t))^{-1} x p_H : t \in D\} \subset K$.*
(2) *$|\{t \in D : (\Gamma \times \Delta)q(t) \in \Psi\}| \le \epsilon|D|$.*

4. Translates of B_R by a unipotent element in $\mathrm{SL}_n(\mathbb{R})$

For the rest of paper, we set $G = \mathrm{SL}_n(\mathbb{R})$ and consider the Iwasawa decomposition $G = KAN$ where K is given by

$$\{g \in \mathrm{SL}_n(\mathbb{R}) : {}^t gg = I_n\},$$

A is the diagonal subgroup of G consisting of positive diagonals, and N is the strictly upper triangular subgroup of G.

Then $B = AN$ is precisely the identity component of the upper triangular subgroup of G.

Consider the norm $\|\cdot\|$ on the vector space $\mathrm{M_n}(\mathbb{R})$ of $n \times n$ matrices given by

$$\|(g_{ij})\| = \sqrt{\sum g_{ij}^2}.$$

Define for each $R > 0$

$$G_R := \{g \in G : \|g\| \le R\}. \tag{4.1}$$

Then the family $\{G_R : R > 0\}$ is well rounded (see Example 2.2) and clearly each G_R is bi K-invariant.

Set

$$U = \{g \in G : (I_n - g)_{ij} = 0 \text{ for all } (i,j) \text{ expect for } (1,n)\}. \tag{4.2}$$

Note that U is the one parameter unipotent subgroup whose only non-zero entry is the $(1,n)$-entry, except for 1's on the diagonal. Even though it is not formulated this way, the following is the essential content in the proof of Lemma 18 in [4]:

THEOREM 4.3. *Let ρ be a right invariant Haar measure on B. Then for any $u \in U$,*

$$\frac{\rho(uB_R \triangle B_R)}{\rho(B_R)} \to 0 \text{ as } R \to \infty.$$

The proof of the above theorem we give below is basically taken from [4]. However we have simplified some calculations. For instance, we do not need the precise asymptotic of $\rho(B_R)$.

The Lie algebra $\mathfrak{a}$ of A can be identified with the set

$$\{s = (s_1, \cdots, s_n) \in \mathbb{R}^n : \sum_{i=1}^n s_i = 0\},$$

and the exponential map $e := \exp : \mathfrak{a} \to A$ is an isomorphism whose inverse is given by log. The Lie algebra $\mathfrak{n}$ of N can be identified with

$\{t : I + t \in N\}$ and define the map $n : \mathfrak{n} \to N$ by

$$n(t) = I + t.$$

We use the right invariant measure ρ on B defined by

$$d\rho(e^s n(t)) = e^{2\delta(s)} ds_1 \cdots ds_{n-1} dt \tag{4.4}$$

where $\delta(s)$ is half the sum of all positive roots $s_i - s_j$ with $1 \le i < j \le n$.

For $R > 0$ and $s \in \mathfrak{a}$, define the subset $N_{s,R}$ of N by

$$N_{s,R} = \{n \in N : \|e^s n\| \le R\},$$

so that $e^s N_{s,R} = e^s N \cap B_R$. For $c > 0$, set

$$A^c = \{e^s \in A : \min_{1 \le i \le n-1} s_i > c\}.$$

In the following the notation, $f(R) \ll g(R)$ means that there exists a constant $C > 0$ such that $f(R) \le C \cdot g(R)$ for all sufficiently large R.

LEMMA 4.5. *Let $c > 1$. Setting $B_R^c := A^c N \cap B_R$, we have*

$$\rho(B_R) \sim \rho(B_R^c) \text{ as } R \to \infty.$$

Proof. Write $\delta(s) = \sum_{i=1}^{n-1} r_i s_i$. Then $r_i > 0$ for each $1 \le i \le n-1$. Note that $B_R - B_R^c = \cup_{i=1}^{n-1} B_R^i$ where

$$B_R^i := \{e^s n \in B_R : s_i < c\}.$$

First by the change of variable $e^{s_i} n_{ij} \mapsto n'_{ij}$, we see

$$\int_{N_{s,R}} e^{\sum_{i=1}^{n-1} r_i s_i} \, dn \ll e^{\sum_{i=1}^{n-1} r_i s_i} \cdot R^{\dim(N)}.$$

Hence

$$\begin{aligned} \rho(B_R^i) &\le \int_{\{s \in \mathfrak{a} : \|e^s\| \le R,\, s_i \le c\}} \int_{N_{s,R}} e^{2\sum_{i=1}^{n-1} r_i s_i} \, dn \, ds \\ &\ll R^{\dim(N) + \sum_{j \ne i} r_j}. \end{aligned} \tag{4.6}$$

On the other hand, if we set

$$S_R := \{s \in \mathfrak{a} : \frac{1}{2} \log \frac{R^2}{2n} \ge s_1 \ge \cdots \ge s_{n-1} \ge 0\},$$

we have

$$R^2 - \|e^s\|^2 \ge R^2/2 \text{ for any } s \in S_R.$$

Hence

$$\rho(B_R) \geq \int_{s\in S_R}\int_{N_{s,R}} e^{2\sum_{i=1}^{n-1} r_i s_i}\, dn\, ds \tag{4.7}$$

$$\gg \int_{s\in S_R} e^{\sum_{i=1}^{n-1} r_i s_i}\left(R^2 - \|e^s\|^2\right)^{\dim(N)/2} ds \tag{4.8}$$

$$\gg R^{\dim(N)+\sum_{j=1}^{n-1} r_j}. \tag{4.9}$$

Since $r_j > 0$ for each $1 \leq j \leq n-1$, it follows from (4.6) and (4.7) that

$$\frac{\rho(B_R^i)}{\rho(B_R)} \to 0 \text{ as } R \to \infty.$$

This proves the claim. □

LEMMA 4.10. *For any $R > 0$, $s \in \mathfrak{a}$ and $u \in U$,*

$$\begin{aligned} &\{n \in N : \|e^s n\| \leq R - e^{s_n}\|u\|\} \\ &\subset \{n \in N : \|ue^s n\| \leq R\} \\ &\subset \{n \in N : \|e^s n\| \leq R + e^{s_n}\|u\|\}. \end{aligned} \tag{4.11}$$

Proof. Note that

$$\|ue^s n\| \leq \|e^s n\| + \|(u-I)e^s n\|.$$

By direct computation we see that $(u-I)e^s n = e^{s_n}(u-I)$. Since $\|u-1\| \leq \|u\|$, this proves the first inclusion. The other direction can be proven similarly. □

Set

$$\tilde{A}_R := \{e^s \in A_R : R^2 - \|e^s\|^2 > R\}.$$

LEMMA 4.12. [1, Proof of Lemma 18] *Let $r > 0$ be fixed. For any $\epsilon > 0$, there exists $0 < \delta_0 = \delta_0(\epsilon) \leq 1/2$ such that for any $R > 2$, $e^s \in \tilde{A}_R$, and $\delta < \delta_0$,*

$$((R+\delta)^2 - \|e^s\|^2)^r - ((R-\delta)^2 - \|e^s\|^2)^r \leq \epsilon((R-\delta)^2 - \|e^s\|^2)^r.$$

Proof of Theorem 4.3. Fix $\epsilon > 0$ and $R > 2$. Set $\tilde{A}_R^c = \tilde{A}_R \cap A^c$. Take a sufficiently large c so that $\|u\|e^{s_n} < \delta_0(\epsilon)$ for all $s \in \tilde{A}_R^c$. Then it is shown in the proof of Lemma 18 in [4] that

$$\lim_{R\to\infty} \frac{\rho(B_R^c - \tilde{A}_R^c N)}{\rho(B_R^c)} = 0.$$

Therefore

$$\limsup \frac{\rho(uB_R^c \triangle B_R^c)}{\rho(B_R^c)}$$
$$\leq \limsup \frac{1}{\rho(B_R^c)} \int_{e^s\in \tilde{A}_R^c} \int_{n\in N} |\chi_{N_{s,R}}(e^{-s}ue^s n) - \chi_{N_{s,R}}(n)| e^{2\delta(s)}\, dn\, ds.$$

Hence by Lemmas 4.10 and 4.12, we have

$$\limsup \frac{\rho(uB_R^c \triangle B_R^c)}{\rho(B_R^c)} \leq \epsilon \cdot \limsup \frac{1}{\rho(B_R^c)} \int_{\tilde{A}_R^c} \int_{N_{s,R}} e^{2\delta(s)}\, dn\, ds \leq \epsilon.$$

By Lemma 4.5, this proves Theorem 4.3. □

5. Uniform distribution of B_R in $\mathrm{SL}_n(\mathbb{R})$-case

We continue notation $G = \mathrm{SL}_n(\mathbb{R})$, U and G_R set up in section 4. In particular, recall that $G = \mathrm{SL}_n(\mathbb{R})$ and $G_0 = \{(g,g) : g \in \mathrm{SL}_n(\mathbb{R})\}$. Let Δ and Γ be lattices in G which are not commensurable with each other. We continue notation for $X = (\Delta\times\Gamma)\backslash(G\times G)$, μ, ρ_R, etc., from section 2.

Our goal is to show that as $R \to \infty$, ρ_R converges to μ in the space $\mathcal{P}(X)$.

Let $V = V_0 \oplus V_1$, and pr_i, $i = 0,1$ be as in section 3. For $g \in G$ and $v \in V$, we simply write gv for $(g,g)v$. The following is a special case of [4, Lemma 16] applied to the representation which is the restriction to G_0 of the $G\times G$ representation on V:

THEOREM 5.1. [5, Lemma 16] *For any relatively compact subset $K \subset V$ and $r > 0$, there exists $0 < \alpha < 1$ and $c > 0$ such that for any $e^s \in A^c$ and $x \in V$ with $\|\mathrm{pr}_1(x)\| \geq r$*

$$\{e^s n(t)x : \|t\| \leq e^{-\alpha s_1}\} \not\subset K.$$

5.1. No escaping to ∞

If X is non-compact, consider the one point compactification $X\cup\{\infty\}$ of X. The following proposition implies that every weak limit of ρ_R in $\mathcal{P}(X\cup\{\infty\})$ is supported on X:

THEOREM 5.2. *For any $\epsilon > 0$, there exists a compact subset $C \subset X$ such that*

$$\liminf_{R\to\infty} \rho_R(C) \geq 1-\epsilon.$$

Proof. Let $U_1, \cdots, U_l$ be as in Theorem 3.6. Let V be the vector space defined in 3.2. Let π be the restriction to G_0 of the adjoint representation of $G \times G$ on V. Let V_0, V_1 and pr_i be as in Theorem 5.1.

We claim that for any $\delta > 0$, there exist $c > 0$ and $0 < \alpha < 1$ such that for any $1 \le i \le l$, for any $e^s \in A^c$ and any $x \in \Gamma \times \Delta$,

$$\sup\{\|(e^s n(t))xp_{U_i}\| : \|t\| < e^{-\alpha s_1}\} \ge \delta. \tag{5.3}$$

Suppose not. Then there exists a $\delta > 0$ such that for any $c > 0$ and $0 < \alpha < 1$, there exists $e^s \in A^c$, $x \in \Gamma \times \Delta$ and $1 \le i \le l$ such that

$$\sup\{\|(e^s n(t))xp_{U_i}\| : \|t\| < e^{-\alpha s_1}\} < \delta. \tag{5.4}$$

Fixing $0 < \alpha < 1$ and considering a sequence $c_j \to \infty$, we can find $1 \le i_0 \le l$ and sequences $x_j \in \Gamma \times \Delta$, $e^{s^j} \in A_{c_j}$ such that for all j,

$$\sup\{\|(e^{s^j} n(t))x_j p_{U_{i_0}}\| : \|t\| < e^{-\alpha s_1^j}\} < \delta. \tag{5.5}$$

Since $U_{i_0} \in \mathcal{H}$, $(\Gamma \times \Delta)p_{U_{i_0}}$ is discrete. By Lemma 3.5, we have $0 \notin \mathrm{pr}_1((\Gamma \times \Delta)p_{U_{i_0}})$. Therefore by passing to a subsequence we have either $\inf_j \|\mathrm{pr}_1(x_j p_{U_{i_0}})\| > 0$ or $\|\mathrm{pr}_0(x_j p_{U_{i_0}})\| \to \infty$.

If the second case happens, then we have for any s and t,

$$\|(e^s n(t))x_j p_{U_{i_0}}\| \ge \|\mathrm{pr}_0(x_j p_{U_{i_0}})\| \to \infty.$$

This contradicts 5.5. Hence for some $r > 0$, $\|\mathrm{pr}_1(x_j p_{U_{i_0}})\| \ge r$ for all j.

Then by Theorem 5.1, for any $\delta > 0$ (by taking $K = \{x \in V : \|x\| \le \delta\}$), there exists $c > 0$ and $0 < \alpha < 1$ such that for any $e^s \in A^c$,

$$\sup\{\|(e^s n(t))x_j p_{U_i}\| : \|t\| < e^{-\alpha s_1}\} \ge \delta.$$

This contradicts 5.5 since $c_j \to \infty$. This proves our claim.

Fix any $\delta > 0$ and $\epsilon > 0$. Let c and α be as in the claim.

For each fixed $e^s \in A^c$, we apply Theorem 3.6 to

$$q(t) = ((e^s n(t))^{-1}, (e^s n(t))^{-1}) : \mathfrak{n} \to G \times G. \tag{5.6}$$

The inequality (5.3) now implies that for any fixed $e^s \in A^c$, and for any open convex subset $D \subset \mathfrak{n}$ containing $\{t \in \mathfrak{n} : \|t\| < e^{-\alpha s_1}\}$, we have

$$|\{t \in D : (\Gamma(e^s n(t))^{-1}, \Delta(e^s n(t))^{-1}) \notin C\}| < \epsilon |D|. \tag{5.7}$$

Set

$$\tilde{A}_R^c := \{e^s \in A_R^c : \|e^s\|^2 + (\max_{1\le i\le n} e^{2s_i})e^{-\alpha s_1} \le R^2\}. \tag{5.8}$$

Note that for $c > 1$, which we assume in what follows,

$$\max_{1\le i\le n} e^{s_i} = \max_{1\le i\le n-1} e^{s_i}.$$

Observe that for any $e^s \in \tilde{A}^c_R$, we have

$$\{t \in \mathfrak{n} : \|t\| \le e^{-\alpha s_1}\} \subset \{t \in \mathfrak{n} : n(t) \in N_{s,R}\}. \tag{5.9}$$

Since, as $R \to \infty$,

$$\frac{1}{\rho(B_R)} \rho(\tilde{A}^c_R N \cap B_R) \to 1$$

as shown in [5, Lemma 19], it suffices to note:

$$\limsup_{R\to\infty} \frac{1}{\rho(B_R)} \int_{\tilde{A}^c_R} \int_{n \in N_{s,R}} \chi_{X-C}((\Gamma \times \Delta)(e^s n)^{-1}) e^{2\delta(s)} \, dn \, ds$$
$$\le \limsup_{R\to\infty} \frac{1}{\rho(B_R)} \epsilon \cdot \int_{\tilde{A}^c_R} \int_{n \in N_{s,R}} e^{2\delta(s)} \, dn \, ds \quad \text{by (5.7)}$$
$$= \epsilon \cdot \limsup_{R\to\infty} \frac{\rho((\tilde{A}^c_R N) \cap B_R)}{\rho(B_R)}$$
$$\le \epsilon. \qquad \square$$

5.2. Showing $G \times G$-invariance

Recall the unipotent one parameter subgroup U defined in (4.2) and set

$$U_0 := \{(u, u) \in \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R}) : u \in U\}.$$

As a corollary of Theorem 4.3, we obtain:

PROPOSITION 5.10. *Any weak limit of ρ_R in $\mathcal{P}(X)$ is U_0-invariant.*

Proof. Let $f \in C_c(X)$. Even though ρ is not left invariant, it is still invariant under left translations by an element of N. Hence for any $u \in N$,

$$\int_{B_R} f(\Gamma b^{-1} u, \Delta b^{-1} u) d\rho(b) = \int_{b \in u^{-1} B_R} f(\Gamma b^{-1}, \Delta b^{-1}) d\rho(b).$$

Note that

$$\left| \int_{B_R} f(\Gamma b^{-1} u, \Delta b^{-1} u) d\rho(b) - \int_{B_R} f(\Gamma b^{-1}, \Delta b^{-1}) d\rho(b) \right|$$
$$\le \max_{x \in X} |f(x)| \cdot \int_{u^{-1} B_R \triangle B_R} d\rho(b)$$
$$= \max_{x \in X} |f(x)| \cdot \frac{\rho(u^{-1} B_R \triangle B_R)}{\rho(B_R)}.$$

Hence by Theorem 4.3, for any $u \in U$,

$$\limsup_{R\to\infty} ((u, u)\rho_R(f) - \rho_R(f)) = 0.$$

This proves the claim. □

Recall the notation $\mathcal{H}$ from 3.3. For each $H \in \mathcal{H}$, recall

$$X(H, U_0) = \{(g_1, g_2) \in G \times G : (g_1, g_2)U_0 \subset H(g_1, g_2)\}.$$

It is shown by Ratner[9] that $\mathcal{H}$ is countable. Moreover by [7, Theorem 2.2], for any probability measure, say η, on X which is invariant under U_0, if $\eta\left((\Gamma \times \Delta)\backslash(\Gamma \times \Delta)X(H, U_0)\right) = 0$ for each $H \in \mathcal{H}$ then η is $G \times G$-invariant.

Therefore the following shows that any weak limit of ρ_R in $\mathcal{P}(X)$ is $G \times G$-invariant.

THEOREM 5.11. *For any subgroup $H \in \mathcal{H}$ and for any compact subset $C \subset (\Gamma \times \Delta)\backslash(\Gamma \times \Delta)X(H, U_0)$, we have*

$$\lim_{R \to \infty} \rho_R(C) = 0.$$

Proof. Notice that $X(H, U_0) \neq \emptyset$ if and only if H contains a conjugate of U_0. Hence we may assume that H contains a conjugate of U_0.

Fix any $\epsilon > 0$ and a compact subset C as in the theorem. We obtain F as in Theorem 3.7. Let K be a neighborhood of F. We claim that there exists $c > 0$ and $0 < \alpha < 1$ such that for any $e^s \in A^c$ and for any $x \in \Gamma \times \Delta$

$$\{e^s n(t) x p_H : \|t\| \leq e^{-\alpha s_1}\} \not\subset K.$$

Suppose not. Then for any fixed $0 < \alpha < 1$, there exists sequences $c_j \to \infty$, $e^{s^j} \in A_{c_j}$ and $x_j \in \Gamma \times \Delta$ such that for all j,

$$\{e^{s^j} n(t) x_j p_H : \|t\| \leq e^{-\alpha s^j_1}\} \subset K. \tag{5.12}$$

Note that $(\Gamma \times \Delta)p_H$ is a discrete subset of V by 3.4, and by Lemma 3.5, we have $0 \notin \mathrm{pr}_1((\Gamma \times \Delta)p_H)$, since H contains a conjugate of U_0.

Therefore by passing to a subsequence we have either

$$\inf_j \|\,\mathrm{pr}_1(x_j p_H)\| > 0$$

or

$$\lim \|\,\mathrm{pr}_0(x_j p_H)\| = \infty.$$

If the first case does not happen and hence the second case happens, then for any s and t,

$$\|e^s n(t)(x_j p_H)\| \geq \|\,\mathrm{pr}_0(x_j p_H)\| \to \infty$$

as $j \to \infty$. This contradicts 5.12. Therefore for some $r > 0$, we have $\| \mathrm{pr}_1(x_j p_H)\| \geq r$ for all j. By Theorem 5.1, there exists $c > 0$ and $0 < \alpha < 1$ such that for any $e^s \in A^c$,

$$\{(e^s n(t))x_j p_H : \|t\| \leq e^{-\alpha s_1}\} \not\subset K.$$

This is a contradiction to 5.12 since $c_j \to \infty$ and $A^c \subset A^d$ if $c > d$.

Therefore this proves the claim. Now fix c and α as in the claim. By applying Theorem 3.7, we obtain for any $e^s \in A^c$ and for any convex open $D \subset \mathfrak{n}$ containing

$$D_s := \{t \in \mathfrak{n} : \|t\| \leq e^{-\alpha s_1}\},$$

$$|\{t \in D : (\Gamma \times \Delta)(e^s n(t), e^s n(t)) \in C\}| \leq \epsilon |D|.$$

Recall that (5.9) holds for any $e^s \in \tilde{A}_R^c$ (see (5.8)). Similarly to the proof of Theorem 5.2, it suffices to note that

$$\begin{aligned}
&\limsup_{R\to\infty} \frac{1}{\rho(B_R)} \int_{\tilde{A}_R^c} \int_{n \in N_{s,R}} \chi_C(\Gamma(e^s n)^{-1}, \Delta(e^s n)^{-1}) e^{2\delta(s)}\, dnds \\
&\leq \limsup_{R\to\infty} \frac{1}{\rho(B_R)} \epsilon \cdot \int_{\tilde{A}_R^c} \int_{n \in N_{s,R}} e^{2\delta(s)}\, dnds \\
&= \epsilon \cdot \limsup_{R\to\infty} \frac{\rho((\tilde{A}_R^c N) \cap B_R)}{\rho(B_R)} \\
&\leq \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this shows Theorem 5.11. □

Now Theorems 5.2 and 5.11 yield:

THEOREM 5.13. *For $G = \mathrm{SL}_n(\mathbb{R})$, B the identity component of the upper triangular subgroup, the family $\{G_R : R > 0\}$ with*

$$G_R := \{(g_{ij}) \in G : \sqrt{\sum g_{ij}^2} \leq R\}$$

and for lattices Γ and Δ in G not commensurable with each other,

$$\lim_{R\to\infty} \rho_R = \mu \text{ in } \mathcal{P}((\Gamma \times \Delta)\backslash(G \times G))$$

for any right-invariant Haar measure ρ on B.

By Theorem 2.6, the above theorem implies Theorem 1.4.

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