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A Sequence of Improvement over the Lindley Type Estimator with the Cases of Unknown Covariance Matrices[1]

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Abstract.

In this paper, the problem of estimating a p-variate ($p \geq 4$) normal mean vector is considered in decision-theoretic set up. Using a simple property of the noncentral chi-square distribution, a sequence of estimators dominating the Lindley type estimator with the cases of unknown covariance matrices has been produced and each improved estimator is better than previous one.

Keywords : normal mean vector, noncentral chi-square, Lindley type estimator

1. Introduction

Let $X=(X_1,\cdots,X_p)$ and S be independent observation with $X \sim N_p(\theta, \sigma^2 I_p)$ and $S \sim \sigma^2 \chi^2_k$, $\theta \in R^p$, where σ^2 is unknown. For any estimator $\delta(X)$ of θ , the loss in estimating θ by $\delta(X)$ is

$$L(\delta, \theta) = \frac{\|\delta - \theta\|^2}{\sigma^2} \tag{1.1}$$

The best location invariant estimator of θ is

$$\delta^0 = X$$

which is admissible for p=1 or 2. James and Stein (1961) showed that δ^0 is inadmissible for $p \geq 3$ and it is dominated by classes of

$$\delta^{JS} = \left(1 - \frac{(p-2)S}{(k+2)\|X\|^2}\right)X, \quad p \geq 3 \ .$$

Since this pioneering work, many shrinkage estimators which dominate δ^0 have been proposed. Guo and Pal(1992) considered a sequence of improved estimators providing

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successive improvement over δ^{JS} . Kubokawa(1991) and Guo and Pal(1992) constructed a sequence of estimators (dominating δ^{JS}) where each estimator is better than the previous one and the sequence converges to an admissible estimator. Subsequently a number of authors provided classes of stein-type estimators dominating X (Efron and Morris(1976), Ghosh, Hwang, and Tsui(1983) where other references are cited). One common feature of these estimators dominating X is that they are all spherically symmetric shrinking X toward some particular point, not necessarily the origin. The Lindley(1962) type estimator is

$$\delta^1 = \overline{X}1 + \left(1 - \frac{(p-3)S}{(k+2)\|X-\overline{X}\mathbf{1}\|}\right)(X-\overline{X}1), \qquad p\geq 4 .$$

where $\overline{X} = (x_1 + \cdots + X_p)/p$ and $\mathbf{1} = (1, \cdots, 1)$. The Lindley type estimator possesses better risk properties than the ordinary James-Stein estimator over a large region of the parameter space, suggesting that from a sampling theoretic viewpoint the shrinkage should be taken toward $\overline{X}1$ as opposed to the origin.

In this paper, a sequence of improved estimators providing successive improvements over δ^1 is constructed. Each of these improved estimators can be dominated by using a technique of Kubokawa(1991). In Section 2, such improved estimators are derived. In Section 2, these results are generalized when $X \sim N_p(\theta, \Sigma)$, where the covariance matrix Σ is completely unknown.

2. Improved Estimators Dominating δ^1

Consider a sequence of estimators of the form

$$\delta^n = \overline{X}1 + K_n(X - \overline{X}1), \quad n = 1, 2, 3, \cdots \tag{2.1}$$

where $K_n = K_n(X)$ is suitable function of X . We choose

$K_1 = \left(1 - \frac{(p-3)S}{(k+2)\|X-\overline{X}\mathbf{1}\|^2}\right)$ to make the first element δ^1 of the sequence $\{\delta^n\}$.

Our goal is to construct δ^n $(n\geq 2)$ such that for any integer $n\geq 1$ and $p\geq 4$,

$$R(\delta^{n+1}, \theta) \leq R(\delta^n, \theta), \quad \forall\ \theta \in R^p.$$

To dominate the estimator δ^n for any $n \geq 1$, define δ^{n+1} as

$$\delta^{n+1} = \delta^n + r_n^*(X - \overline{X}1), \text{ i. e,}$$

$K_{n+1} = K_n + r_n^*$, where $r_n^* = r_n^*(X)$ is a suitable real valued function.

Let $r_n = r_n^*(X - \overline{X}\mathbf{1})$. Define the ask difference (RD) between $R(\delta^{n+1}, \theta)$ and $R(\delta^n, \theta)$ as

$$R(n+1, n) = R(\delta^{n+1}, \theta) - R(\delta^n, \theta)$$

$$= \frac{1}{\sigma^2} E[\|\delta^n + r_n - \theta\|^2 - \|\delta^n - \theta\|^2] \tag{2.2}$$

$$= \frac{1}{\sigma^2} E[\sum_{i=1}^{p} r_{ni}^2 + 2\sum_{i=1}^{n} r_{ni}(\delta_i^n - \theta_i)]$$

where δ_i^n, θ_i and r_{ni} denote the i^{th} elements of δ^n, θ and r_n, respectively. The second term of (2.2) can be simplified as

$$E[\sum_{i=1}^{p} r_{ni}(\delta_i^n - \theta_i)] = \sum_{i=1}^{p} E[(K_n-1)r_{ni}(X_i - \overline{X}) + r_n(X_i - \theta_i)]$$

$$= \sum_{i=1}^{p} E\left[(K_n-1)r_{ni}(X_i - \overline{X}) + \frac{\partial}{\partial X_i} r_{ni}\right] \tag{2.3}$$

The expression (2.3) is obtained by using Stein's normal identity assuming that $r_{ni}'s$ $(i=1,2,\cdots,p)$ satisfy all the regularity conditions of the identity. Combining (2.2) and (2.3), we get

$$RD(n+1,n) = E\left[\sum_{i=1}^{p}\left\{r_{ni}^2 + 2(K_n-1)\, r_{ni}(X_i - \overline{X}) + 2\sigma^2 \frac{\partial}{\partial X_i} r_{ni}\right\}\right].$$

We now look for suitable $r_n = r_n^*(X - \overline{X}\mathbf{1})$ such that

$$RD(n+1,\ n) \le 0, \quad \forall\ n \ge 1.$$

Before we derive the general result, let us look at some special cases.

2.1 The case of $n=1$

To dominate δ^1 (Lindley), take $r_1^* = c_1 S^{1+\frac{\alpha_1}{2}} \|X - \overline{X}\mathbf{1}\|^{-(2+\alpha_1)}$,where $\alpha_1 > 0$ an c_1 is a suitable constant. Then,

$$2\sigma^2 \sum_{i=1}^{p} \frac{\partial}{\partial X_i} r_{1i} = \frac{2\sigma^2 c_1\{p-(3+\alpha_1)\} S^{1+\frac{\alpha_1}{2}}}{\|X - \overline{X}\mathbf{1}\|^{2+\alpha_1}},$$

$$\sum_{i=1}^{p} r_{1i}^2 = \frac{c_1^2 S^{2+\alpha_1}}{\|X - \overline{X}\mathbf{1}\|^{2+2\alpha}}, \text{ and } 2(K_1-1)\sum_{i=1}^{p} r_{1i} = -\frac{2c_1(p-3) S^{2+\frac{2}{\alpha_1}}}{(k+2)\|X - \overline{X}\mathbf{1}\|^{2+\alpha_1}}.$$

Therefore, from (2.4) one can get

$$RD(2,1) = E_{T,S}\left[c_1^2 \frac{1}{T^{1+\alpha_1}}\left(\frac{S}{\sigma^2}\right)^{2+\alpha_1} + 2c_1(p-(3+\alpha_1))\frac{1}{T^{1+\frac{\alpha_1}{2}}}\left(\frac{S}{\sigma^2}\right)^{1+\frac{\alpha_1}{2}}\right.$$

$$\left. - 2c_1 \frac{p-3}{k+2}\frac{1}{T^{1+\frac{\alpha_1}{2}}}\left(\frac{S}{\sigma^2}\right)^{2+\frac{\alpha_1}{2}}\right] \tag{2.5}$$

where $T = \frac{\|X - \overline{X}\mathbf{1}\|^2}{\sigma^2}$ ~non central $\chi_{p-1}^2(\lambda)$ with $\lambda = \frac{\|\theta - \overline{\theta}\mathbf{1}\|^2}{\sigma^2}$ and

$\overline{\theta} = (\theta_1 + \cdots + \theta_p)/p$, $\frac{S}{\sigma^2} \sim \chi^2_k$, and they are independent. It is well known that T can be treated as a mixture of central χ^2_{p-2+2U} and $U \sim Poisson(\frac{\lambda}{2})$. Let $\beta_U = U + \frac{p-1}{2}$, then

$$RD(2,1) = E_U \Bigg[c_1^2 \cdot 2 \frac{\Gamma(\beta_U - (1+\alpha_1))}{\Gamma(\beta_U)} \frac{\Gamma(\frac{k}{2}+2+\alpha_1)}{\Gamma(\frac{k}{2})} + 2c_1(p-(3+\alpha_1)) \frac{\Gamma(\beta_U - (1+\frac{\alpha_1}{2}))}{\Gamma(\beta_U)} \frac{\Gamma(\frac{k}{2}+1+\frac{\alpha_1}{2})}{\Gamma(\frac{k}{2})} - 4c_1 \frac{p-3}{k+2} \frac{\Gamma(\beta_U - (1+\frac{\alpha_1}{2}))}{\Gamma(\beta_U)} \frac{\Gamma(\frac{k}{2}+1+\frac{\alpha_1}{2})}{\Gamma(\frac{k}{2})} \Bigg] \tag{2.6}$$

To make $RD(2,1) \le 0$, it is enough to have the expression inside [] in (2.6) ≤ 0 for all U, $U = 0, 1, 2, \cdots$. Hence, the condition on c_1 is

$$0 < c_1 < \alpha_1 \frac{k+p-1}{k+1} \frac{\Gamma(\beta_U - (1+\alpha_1/2))\, \Gamma(\frac{k}{2}+1+\frac{\alpha_1}{2})}{\Gamma(\beta_U - (1+\alpha_1))\, \Gamma(\frac{k}{2}+2+\alpha_1)}$$

for $U = 0, 1, 2, \cdots$. Let

$$\varepsilon_1(p, \alpha_1) = \min_U \frac{\Gamma(\beta_U - (1+\alpha_1/2))\, \Gamma(\frac{k}{2}+1+\frac{\alpha_1}{2})}{\Gamma(\beta_U - (1+\alpha_1))\, \Gamma(\frac{k}{2}+2+\alpha_1)} \tag{2.7}$$

Then a sufficient condition on c_1 is

$$0 < c_1 < \alpha_1 \frac{k+p-1}{k+1} \varepsilon_1 \ (p, \alpha_1) \tag{2.8}$$

provided $p-1 > 2(1+\alpha_1)$. In fact, the optimal value of c_1 which minimizes the quadratic expression inside [] in (2.6) for all U is

$$c_1^0 = \alpha_1 \frac{k+p-1}{k+1} \varepsilon_1 \ (p, \alpha_1). \tag{2.9}$$

It can be proved that the minimum in (2.7) is attained at $U = 0$, $i.e.$, $\beta_U = \frac{p-1}{2}$. (See Guo and Pal(1992).) The condition that $p-1 > 2(1+\alpha_1)$ is necessary to ensure that all the expectations exist. The following result is immediate from the above derivation.

Proposition 2.1. The estimator $\delta^2 = \delta^1 + (c_1^0 S^{1+\frac{\alpha_1}{2}} / \|X - \overline{X}\mathbf{1}\|^{2+\alpha_1})(X - \overline{X}\mathbf{1})$ with

$a_1 > 0$ dominate δ^1 (Lindley) uniformly under the quadratic loss(1.1) provided $p-1 > 2(1+a_1)$

2.2 The case of $n=2$

To dominate $\delta^2 = \overline{X}\mathbf{1} + K_2(X-\overline{X}\mathbf{1})$ with $K_2 = 1 - \frac{(p-3)S}{(k+1)\|X-\overline{X}\mathbf{1}\|^2} + \frac{c_1 S^{1+\frac{a_1}{2}}}{\|X-\overline{X}\mathbf{1}\|^{2+a_1}}$, choose $r_2^* = \frac{c_2 S^{1+\frac{a_2}{2}}}{\|X-\overline{X}\mathbf{1}\|^{2+a_2}}$, where $a_2 > a_1 > 0$ and c_2 is suitable constant. Similar to the case $n=1$, $RD(3,2)$ can be derived from (2.4) as

$$RD(3,2) = E_{T,S}\Bigg[c_2^2 \frac{1}{T^{1+a_2}}\Big(\frac{S}{\sigma^2}\Big)^{1+\frac{a_2}{2}} + 2c_2(p-(3+a_2))\frac{1}{T^{1+\frac{a_2}{2}}}\Big(\frac{S}{\sigma^2}\Big)^{1+\frac{a_2}{2}}$$

$$-2c_2 \frac{p-3}{(k+2)T^{1+\frac{a_2}{2}}}\Big(\frac{S}{\sigma^2}\Big)^{2+\frac{a_2}{2}} + 2c_1c_2\frac{1}{T^{1+\frac{a_1+a_2}{2}}}\Big(\frac{S}{\sigma^2}\Big)\Big)^{2+\frac{a_1+a_2}{2}}\Bigg]$$

Following the earlier approach, a sufficient condition for $RD(3,2) \le 0$ is $0 < c_2 < a_2 \frac{k+p-1}{k+2} \varepsilon_2\ (p, a_1, a_2)$ provided that $p-1 < 2(1+a_2)$, where

$$\varepsilon_2(p, a_1, a_2) = \min_U \frac{\Gamma(\beta_U-(1+\frac{a_2}{2}))\Gamma(\frac{k}{2}+1+\frac{a_2}{2})}{\Gamma(\beta_U-(1+a_2))\Gamma(\frac{k}{2}+2+a_2)}\Bigg\{1-\frac{2(k+1)}{a_2(k+p-1)}$$

$$\times c_1 \frac{\Gamma(\frac{k}{2}+2+\frac{a_1+a_2}{2})\Gamma(\beta_U-(1+\frac{a_1+a_2}{2}))}{\Gamma(\frac{k}{2}+1+\frac{a_2}{2})\Gamma(\beta_U-(1+\frac{a_2}{2}))}\Bigg\}$$

Again, the optimal value of c_2 is $c_2^0 = a_2 \frac{k+p-1}{2(k+2)} \varepsilon_2(p, a_1, a_2)$.

2.3 General case

Consider the estimator δ^n in (2.1) with

$$K_n = 1 - \frac{(p-3)S}{(k+2)\|X-\overline{X}\mathbf{1}\|^2} + \sum_{j=1}^{n-1} \frac{c_j S^{1+\frac{a_j}{2}}}{\|X-\overline{X}\mathbf{1}\|^{2+a_j}}, \tag{2.10}$$

where $a_{n-1} > a_{n-2} > \cdots > a_1 > 0$ and $0 < c_j < a_j \frac{k+p-1}{k+2} \varepsilon_j\ (p, a_1, \cdots, a_j)$, $j=1,2,$

$\cdots, n-1$. Take $r_n^* = \dfrac{c_n S^{1+\frac{a_n}{2}}}{\|X-\overline{X}1\|^{2+a_n}}$, where $a_n > a_{n-1}$ and c_n is a suitable constant. Similar to the special cases

$$
\begin{aligned}
RD(n+1, n) = E_U\Bigg[& c_n^2 \cdot 2 \frac{\Gamma(\beta_U-(1+a_n))}{\Gamma(\beta_U)} \frac{\Gamma(\frac{k}{2}+2+a_n)}{\Gamma(\frac{k}{2})} \\
& + 2c_n(p-(3+a_n)) \frac{\Gamma(\beta_U-(1+\frac{a_n}{2}))}{\Gamma(\beta_U)} \frac{\Gamma(\frac{k}{2}+1+\frac{a_n}{2})}{\Gamma(\frac{k}{2})} \\
& - 4c_n \frac{p-3}{k+2} \frac{\Gamma(\beta_U-(1+\frac{a_n}{2}))}{\Gamma(\beta_U)} \frac{\Gamma(\frac{k}{2}+2+\frac{a_n}{2})}{\Gamma(\frac{k}{2})} \\
& + 4c_n \sum_{j=1}^{n-1} c_j \frac{\Gamma(\beta_U-(1+\frac{a_j+a_n}{2}))}{\Gamma(\beta_U)} \frac{\Gamma(\frac{k}{2}+2+\frac{a_j+a_n}{2})}{\Gamma(\frac{k}{2})} \Bigg]
\end{aligned} \tag{2.11}
$$

The expectation in (2.11) exist provided $p-1 > 2(1+a_n)$.

Define $\varepsilon_n(p, a_1, \cdots, a_n)$ as,

$$
\begin{aligned}
\varepsilon(p, a_1, \cdots, a_n) = \min_U & \Bigg\{ \frac{\Gamma(\beta_U-(1+\frac{a_n}{2}))\Gamma(\frac{k}{2}+1+\frac{a_n}{2})}{\Gamma(\beta_U-(1+a_n))\Gamma(\frac{k}{2}+2+a_n)} \\
& \Bigg(1-\frac{2(k+2)}{a_n(k+p-1)} \sum_{j=1}^{n-1} c_j \frac{\Gamma(\frac{k}{2}+2+\frac{a_j+a_n}{2})\Gamma(\beta_U-(1+(a_j+a_n)/2))}{\Gamma(\frac{k}{2}+1+\frac{a_n}{2})\Gamma(\beta_U-(1+\frac{a_n}{2}))}\Bigg)\Bigg\}
\end{aligned} \tag{2.12}
$$

The optimal value c_n is $c_n^0 = a_n \dfrac{k+p-1}{2(k+2)} \varepsilon_n(p, a_1, \cdots, a_n)$. Then a sufficient condition for δ^{n+1} dominating δ^n is

$$0 < c_n < a_n \frac{k+p-1}{k+2} \varepsilon_n(p, a_1, \cdots, a_n),$$

(2.13) and the optimal value c_n is

$$c_n^0 = a_n \frac{k+p-1}{2(k+2)} \varepsilon_n(p, a_1, \cdots, a_n).$$

The minimum in (2.12) in attained at $U=0$ (See Guo and Pal(1992)). We now state the main theorem of this section.

Theorem 2.1. An estimator δ^n of the form (2.10) is uniformly dominated by

$$\delta^{n+1} = \delta^n + \left(\frac{c_n S^{1+\frac{\alpha_n}{2}}}{\|X-\overline{X}1\|^{2+\alpha_n}} \right)(X-\overline{X}1)$$ provided $p-1 > 2(1+\alpha_n)$ and c_n satisfies the condition (2.13).

Remark 2.1. Note that the function r_{ni}, $i = 1, \cdots, p$ satisfies the regularity conditions of Stein's normal identity which enables us to derive (2.11).

3. The Case of Completely Unknown Covariance Matrix $\sum$

Let X and S be independent where $X \sim N_p(\theta, \sum)$ and $S_{pxp} \sim$ Wishart $(\sum | k)$. Here we estimate θ under the loss function

$$L(\delta, \theta) = (\delta-\theta)' \Sigma^{-1}(\delta-\theta). \tag{3.1}$$

The Lindley type estimator dominating the usual estimator $\delta^0 = X$ is

$$\delta^1 = \overline{X}1 + \left(1 - \frac{p-3}{(k-p+4)(X-\overline{X}1)'S^{-1}(X-\overline{X}1)}\right)(X-\overline{X}1).$$

Again, the difference in risks of δ^n and $\delta^{n+1} = \delta^n + r_n$ is

$$\begin{aligned} RD(n+1, n) &= E[r_n'\Sigma^{-1}r_n + 2r_n'\Sigma^{-1}(\delta^n-\theta)] \\ &= E[r_n'\Sigma^{-1}r_n + 2r_n'\Sigma^{-1}(\delta^n-X) + 2r_n'\Sigma^{-1}(X-\theta)]. \end{aligned} \tag{3.2}$$

Let $\delta^n = \overline{X}1 + K_n(X-\overline{X}1)$, where

$$K_n = \left(1 - \frac{p-3}{(k-p+4)(X-\overline{X}1)'S^{-1}(X-\overline{X}1)} + \sum_{j=1}^{n-1} \frac{c_j}{\{(X-\overline{X}1)'S^{-1}(X-\overline{X}1)\}^{1+\alpha_j}}\right)$$

and

$$r_n = r_n^*(X-\overline{X}1) = \frac{c_n}{\{(X-\overline{X}1)'S^{-1}(X-\overline{X}1)\}^{1+\alpha_n}}(X-\overline{X}1), \tag{3.3}$$

$$\alpha_{n-1} > \alpha_{n-2} > \cdots > \alpha_1 > 0.$$

Also, define $Y = \Sigma^{-\frac{1}{2}}X$, $\theta_* = \Sigma^{-\frac{1}{2}}\theta$ and $S_* = \Sigma^{-\frac{1}{2}}S\Sigma^{-\frac{1}{2}}$. Then,

$$\begin{aligned} E[r_n'\Sigma^{-1}(X-\theta)] &= E\left[\frac{c_n}{\{(X-\overline{X}1)'S^{-1}(X-\overline{X}1)\}^{1+\alpha_n}}(X-\overline{X}1)'\Sigma^{-1}(X-\theta)\right] \\ &= E\left[\frac{c_n}{\{Y-\overline{Y}1)'S_*^{-1}(Y-\overline{Y}1)\}^{1+\alpha_n}}(Y-\overline{Y}1)'(Y-\theta_*)\right] \\ &= \sum_{i=1}^{p} E\left[\frac{c_n(Y_i-\overline{Y})}{\{(Y-\overline{Y}1)'S_*^{-1}(Y-\overline{Y}1)\}^{1+\alpha_n}}(Y_i-\theta_{*i})\right], \end{aligned} \tag{3.4}$$

where $\overline{Y}=\frac{1}{p}\sum_{i=1}^{p}Y_i$, $\overline{\theta}=\frac{1}{p}\sum_{i=1}^{p}\theta_i$ and $Y_i(\sim N(\theta_{*i},1))$ is the i^{th} element of Y.

Applying Stein's normal identity in (3.4), we get

$$
\begin{aligned}
&E[r_n'\Sigma^{-1}(X-\theta)]\\
&=E\left[\sum_{i=1}^{p}\frac{\partial}{\partial Y_i}\left(\frac{c_n(Y_i-\overline{Y})}{\{(Y-\overline{Y}1)'S_*^{-1}(Y-\overline{Y}1)\}^{1+\alpha_n}}\right)\right]\\
&=E\left[\sum_{i=1}^{p}c_n\left\{\frac{(1-\frac{1}{p})\{(Y-\overline{Y}1)'S_*^{-1}(Y-\overline{Y}1)\}^{1+\alpha_n}}{\{(Y-\overline{Y}1)'S_*^{-1}(Y-\overline{Y}1)\}^{2+2\alpha_n}}\right.\right.\\
&\qquad\left.\left.-(Y_i-\overline{Y})(1+\alpha_n)\{(Y-\overline{Y}1)'S_*^{-1}(Y-\overline{Y}1)\}^{\alpha_n}\}\frac{\frac{\partial}{\partial Y_i}(Y-\overline{Y}1)'S_*^{-1}(Y-\overline{Y}1)}{\{(Y-\overline{Y}1)'S_*^{-1}(Y-\overline{Y}1)\}^{2+2\alpha}}\right]\qquad(3.5)\\
&=E\left[c_n\frac{(p-1)\{(Y-\overline{Y}1)'S_*^{-1}(Y-\overline{Y}1)\}^{1+\alpha_n}}{\{(Y-\overline{Y}1)'S_*^{-1}(Y-\overline{Y}1)\}^{2+2\alpha_n}}-c_n\frac{2(1+\alpha_n)\{Y-\overline{Y}1)'S_*^{-1}(Y-\overline{Y}1)\}^{1+\alpha_n}}{\{(Y-\overline{Y}1)'S_*^{-1}(Y-\overline{Y}1)\}^{2+2\alpha_n}}\right]\\
&=E\left[c_n\frac{p-1-2(1+\alpha_n)}{\{(Y-\overline{Y}1)'S_*^{-1}(Y-\overline{Y}1)\}^{1+\alpha}}\right].
\end{aligned}
$$

From (3.2) and (3.5),

$$
\begin{aligned}
&RD(n+1,n)\\
&=E\left[r_n'\Sigma^{-1}r_n+2r_n'\Sigma^{-1}(\delta^n-X)+2c_n\frac{(p-1-2(1+\alpha_n))}{\{(Y-\overline{Y}1)'S_*^{-1}(Y-\overline{Y}1)\}^{1+\alpha_n}}\right]\\
&=E\left[\frac{2c_n}{\{(X-\overline{X}1)S^{-1}(X-\overline{X}1)\}^{1+\alpha_n}}\left\{-\frac{(p-3)/(k-p+4)}{\{(X-\overline{X}1)S^{-1}(X-\overline{X}1)\}}\right.\right.\\
&\qquad\left.+\sum_{j=1}^{n-1}\frac{c_j}{\{(X-\overline{X}1)'S^{-1}(X-\overline{X}1)\}^{1+\alpha_j}}\right\}(X-\overline{X}1)'\Sigma^{-1}(X-\overline{X}1)\\
&\qquad+\frac{c_n^2}{\{(X-\overline{X}1)'S^{-1}(X-\overline{X}1)\}^{2(1+\alpha_n)}}(X-\overline{X}1)'\Sigma^{-1}(X-\overline{X}1)\\
&\qquad\left.+2c_n\frac{(p-1)-2(1+\alpha_n)}{\{(X-\overline{X}1)'S^{-1}(X-\overline{X}1)\}^{1+\alpha_n}}\right]\\
&=E\left[-\frac{2c_n(p-3)/(k-p+4)}{\{(X-\overline{X}1)'S^{-1}(X-\overline{X}1)\}^{2+\alpha_n}}(X-\overline{X}1)'\Sigma^{-1}(X-\overline{X}1)\right.\\
&\qquad+2c_n\left(\sum_{j=1}^{n-1}\frac{c_j}{\{(X-\overline{X}1)'S^{-1}(X-\overline{X}1)^{a+\alpha_j+\alpha_n}\}}\right)(X-\overline{X}1)'\Sigma^{-1}(X-\overline{X}1)\\
&\qquad\left.+c_n^2\frac{(X-\overline{X}1)'\Sigma^{-1}(X-\overline{X}1)}{\{(X-\overline{X}1)'S^{-1}(X-\overline{X}1)\}^{2+2\alpha_n}}+2c_n\frac{(p-1-2(1+\alpha_n))}{\{(X-\overline{X}1)'S^{-1}(X-\overline{X}1)\}^{1+\alpha_n}}\right].
\end{aligned}
$$

Note that given X, the conditional distribution of $\frac{(X-\overline{X}1)'\Sigma^{-1}(X-\overline{X}1)}{(X-\overline{X}1)'S^{-1}(X-\overline{X}1)}=$

$\dfrac{X'(I-\frac{1}{p}11')\Sigma^{-1}(I-\frac{1}{p}11')X}{X'(I-\frac{1}{p}11')S^{-1}(I-\frac{1}{p}11')X}$ is χ^2_{k-p+2} which is free from X (Graybill(1976) and Rao(1983)). So unconditionally $\dfrac{(X-\overline{X}1)'\Sigma^{-1}(X-\overline{X}1)}{(X-\overline{X}1)'S^{-1}(X-\overline{X}1)}$ is χ^2_{k-p+2} and this is independent of $(X-\overline{X}1)'\Sigma^{-1}(X-\overline{X}1)$. Therefore,

$$
\begin{aligned}
RD(n+1,n)=E\Big[&-2c_n\frac{p-3}{(k-p+4)}\left(\frac{(X-\overline{X}1)'\Sigma^{-1}(X-\overline{X}1)}{(X-\overline{X}1)'S^{-1}(X-\overline{X}1)}\right)^{2+\alpha_n}\cdot\{(X-\overline{X}1)'\Sigma^{-1}(X-\overline{X}1)\}^{-(1+\alpha_n)}\\
&+2c_n\sum_{j=1}^{n-1}c_j\left\{\frac{(X-\overline{X}1)\Sigma^{-1}(X-\overline{X}1)}{(X-\overline{X}1)S^{-1}(X-\overline{X}1)}\right\}^{2+\alpha_j+\alpha_n}\cdot\{(X-\overline{X}1)'\Sigma^{-1}(X-\overline{X}1)\}^{-(a+\alpha_j+\alpha_n)}\\
&+c_n^2\left\{\frac{(X-\overline{X}1)'\Sigma^{-1}(X-\overline{X}1)}{(X-\overline{X}1)'S^{-1}(X-\overline{X}1)}\right\}^{2+2\alpha_n}\cdot\{(X-\overline{X}1)'\Sigma^{-1}(X-\overline{X}1)\}^{-(1+2\alpha_n)}\\
&+2c_n(p-1-2(1+\alpha_n))\left\{\frac{(X-\overline{X}1)'\Sigma^{-1}(X-\overline{X}1)}{(X-\overline{X}1)'S^{-1}(X-\overline{X}1)}\right\}^{1+\alpha_n}
\end{aligned}
$$

Let

$$
\begin{aligned}
A_1&=2\frac{p-3}{k-p+4}2^{2+\alpha_n}\frac{\Gamma((k-p+6+2\alpha_n)/2)}{\Gamma((k-p+2)/2)},\\
A_2&=2^{2+2\alpha_n}\frac{\Gamma((k-p+6+4\alpha_n)/2)}{\Gamma((k-p+2)/2)},\\
A_3&=2(p-3-2\alpha_n)2^{1+\alpha_n}\frac{\Gamma((k-p+4+2\alpha_n)/2)}{\Gamma((k-p+2)/2)}
\end{aligned}
\tag{3.6}
$$

and

$$B_j=2c_j2^{2+\alpha_j+\alpha_n}\frac{\Gamma((k-p+6+2\alpha_j+2\alpha_n)/2)}{\Gamma((k-p+2)/2)}.$$

Then

$$
\begin{aligned}
RD(n+1,n)=E\,[&c_n^2A_2\{(X-\overline{X}1)'\Sigma^{-1}(X-\overline{X}1)\}^{-(1+2\alpha_n)}\\
&-c_n(A_1-A_3)\{(X-\overline{X}1)'\Sigma^{-1}(X-\overline{X}1)\}^{-(1+\alpha_n)}\\
&+c_n\sum_{j=1}^{n-1}B_j\{(X-\overline{X}1)'\Sigma^{-1}(X-\overline{X}1)\}^{-(1+\alpha_j+\alpha_n)}\Big].
\end{aligned}
$$

Since $(X-\overline{X}1)'\Sigma^{-1}(X-\overline{X}1)\sim$ *noncentral* $\chi^2_{p-1}(\lambda)$ with $\lambda=(\theta-\overline{\theta}1)'\Sigma^{-1}(\theta-\overline{\theta}1)$ we choose c_n by using earlier technique as

$$0<c_n<\min_U\left\{2^{\alpha_n}\left(\frac{A_1-A_3}{A_2}\right)\frac{\Gamma(\beta_U-(1+\alpha_n))}{\Gamma(\beta_U-(1+2\alpha_n))}-\sum_{j=1}^{n-1}2^{\alpha_n-\alpha_j}\frac{B_j\Gamma(\beta_U-(1_{\alpha_j}+\alpha_n))}{A_2\Gamma(\beta_U-(1+2\alpha_n))}\right\},\tag{3.7}$$

where A_1, A_2, A_3 and B_j 's are given in (3.6). The following result summarizes the above derivation.

Theorem 3.2. Assume $p-1>2(1+2\alpha_n)$. The estimator $\delta^{n+1}=\delta^n+r_n$ (3.3) dominates δ^n uniformly under the loss (3.1). As a result, this gives a sequence $\{\delta^n\}$ of improved estimators dominating δ^1.

Remark 3.1. Similar to the previous section, note that the function r_{ni}, $i=1,\cdots,p$ satisfies the regularity conditions of Stein's normal identity which enables us to derive (3.2).

Remark 3.2. The optimal limiting value of c_n is hard to find analytically due to the complicated structure of c_n(see (3.6) and (3.7)). Hence the problem of finding the close form of the limiting estimator of the sequence $\{\delta^n\}$ still remain open.

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