The Korean Communications
in Statistics Vol. 12 No. 2, 2005
pp. 435-442

A General Solution of the Integral Equation for Erlang Distribution[1]

Yoon Dong Lee[2], Hyemi Choi[3], and Eun-kyung Lee[4]

Abstract

The mathematical properties of the sequentially operated systems are often described by integral equations. Reservoir system of a product and sequential probability ratio test (SPRT) are typical examples of sequentially operated systems. When the underlying random quantities follow Erlang distribution, a systematic method was developed to solve the integral equations. We extend the method to the cases having accrual functions of more general types. The solutions of the integral equations are represented as a linear combination of distribution functions, and the coefficients of the linear combination are obtained by solving linear system derived from the continuity condition of the solutions.

Keywords : Erlang distribution, SPRT, Integral equation.

I. Introduction

The Erlang distribution function $F_n(x)$ with parameter $n \in N$ is denoted by

$$F_n(x) = (1 - e_{n-1}(x)\, e^{-x}) I(x \geq 0)$$

where $e_n(x) = \sum_{k=0}^{n} x^k / k!$ and the indicator function $I(\cdot)$ is 1 when the argument is true; 0 otherwise. In theories of renewal process and optimal stopping, the problems related to the integral equations with respect to the Erlang distribution are frequently used.

As an example, when the random sample $X_i,\ i = 1, 2, \cdots$ follows the Erlang distribution $F_n(x)$ with the inverse scale parameter θ, the SPRT based on the log-likelihood, to test null hypothesis H_0: $\theta = \theta_0$ versus alternative hypothesis H_1: $\theta = \theta_1$ is performed by the statistic

1) This research was supported by Konkuk University in 2004 for new faculty members
2) Dept. of Applied Statistics, Konkuk Univ., Seoul, Korea,
E-mail poisson@dreamwiz.com
3) Division of Applied Mathematics, KAIST, Daejeon, Korea
4) Department of Statistics, Seoul National University, Seoul, Korea

$T_t = \sum_{i=1}^{t}(X_i - k_0)$ with the parameter $k_0 = n(\log\theta_1 - \log\theta_0)(\theta_1 - \theta_0)^{-1}$. The test goes on while $0 \le T_t \le h$ for some $h > 0$ and stops at the time t whenever $T_t < 0$ or $T_t > h$. It is well-known that the SPRT with k_0 has the optimality in the meaning that the expected sample size is minimized when H_1 is true out of the test having the same expected sample size when H_0 is true. To view the optimality and other properties of SPRT in detail we need to consider a slightly generalized statistic $T_t(s) = s + \sum_{i=1}^{t}(X_i - k)$. One of the important properties of sequential tests is the acceptance probability that the test will end with $T_t < 0$. For any $s \in [0, h]$, the acceptance probability $P(s)$ is written in integral equation,

$$P(s) = F_n(k - s) + \int_0^h P(x)\,dF_n(x - s + k) \tag{1}$$

The first term $F_n(k-s)$ in RHS of (1) is interpreted as the pure amount of accrual of $P(s)$ due to a step of sampling. Other versions of sequential statistics of the form, $T_t^u = \max(0,\ T_{t-1}^u + (X_t - k))$ and $T_t^d = \min(0,\ T_{t-1}^d + (X_t - k))$ with the initial condition $T_0^u = s$, $T_0^d = -s$ for some $s \in [0, h]$, have also corresponding integral equations of the same type with (1) in describing their properties. For all those examples, the differences in their integral equations are the parts of the pure amount of accrual due to a step of sampling. We consider the general type of accrual function $\pi(s)$, which is represented exactly or approximately with the functions in vector space $\Xi = \{F_j(s) : j = 0, 1, \cdots\}$. Khamis (1961) suggested a method how to approximate a function with elements in Ξ

As an example of different types, we consider the reservoir with capacity h. When the amount of daily supply of a product is random and has distribution $F(\cdot)$, the cost function $\pi(s)$ due to the short of the product will be written in the same form with respect to the initial amount s in the reservoir and the daily demand k. The integral equation with the accrual function $\pi(s)$ written in the form $c + \sum_{j=0}^{\infty} a_j F_j(k - s)$, is generally defined in a framework with generic notation $G(s)$, as follows;

$$G(s) = c + \sum_{j=0}^{\infty} a_j F_j(k - s) + \int_b^{b+h} G(x)\,dF_n(x - s + k) \tag{2}$$

The notation $G_n(s)$ is used to show the explicit dependence of $G(s)$ to n. The constant b takes an arbitrary value provided (2) is meaningfully defined. In practical situations, the constant b has the value 0 or h, depending on what the function $G(s)$ is.

When $n = 1$, Vardeman and Ray (1985) found the exact solutions for the expected sample size of the statistics, T_t^u and T_t^d, and Stadje (1987) obtained the solution for T_t. With a different approach, Kohlruss (1994) showed for functions of expected sample size and acceptance probability for T_t when $n \geq 1$. In this paper, we will show the solutions of (2) will be shown as a linear combination of $F_n(ik-s)$, and the coefficients of the linear combination will be obtained by solving linear system derived from the continuity conditions of the solution.

The goal of this paper is to extend the result of Lee (2004). Lee (2004) only dealt with the simple accrual function $\pi(s)$. We consider general type of accrual function $\pi(s)$ and provide the solution of the integral equation (2). When $c=0$, and $a_n=1$ and $a_j=0$, $j \neq n$ the solutions obtained in this paper coincide with the result of Lee (2004).

Lee, et. al. (2005) suggested a better algorithm to compute the solutions of Lee (2004). The suggested algorithm is more efficient computationally for the same problem and soves the wider problem than Gan and Choi (1994). Elrang distribution is frequently used for modelling of queueing systems. Choi and Kim (1998) and Kim and Lee (2000) also considered related topics.

2. General Results

The function $F_0(s)$ used in (2) is consistently defined as $F_0(s) = I(s \geq 0)$ so that the relation $(1-D_s)F_n(-s) = F_{n-1}(-s)$ is satisfied for all $n \geq 1$, when D_s is the differential operator with respect to s. Moreover it is worth to mention that $(1-D_s)F_0(-s) = F_0(-s)$. For the simplicity of arguments, we define the followings.

$$G*F(s) = \int_b^{b+h} G(x)dF(x-s)$$

$$V_n^{q(s)} = c + \sum_{j=0}^{q} a_j F_0(s) + \sum_{j=1}^{\infty} a_{j+q} F_{j+n}(s)$$

for $q \geq 0$. Since $(1-D_s^u) V_n^q(-s) = V_{p(n-u)}^{q+p(u-n)}(-s)$, with $p(x) = \max(0, x)$, we have

$$(1-D_s)^u G_n(s) = V_0^u(k-s) + G_n * F_{\max(0, n-u)}(s-k) \tag{3}$$

for $u \geq 0$. Especially, for the case $u=n$ (equivalently for $u > n$ also) in (3),

$$G_n * F_0(s) = G_n(s) \cdot I(b \leq s \leq b+h)$$

Thus, when $(s-k) \notin [b, b+h]$, equation (3) implies $(1-D_s)^n G_n(s) = V_0^n(k-s)$ which is an ordinary linear differential equation of order n. Since the differential equation $(1-D_s)^n G(s) = 0$ for $s \leq b+k$ has the general solution of the form

$$G(s) = \sum_{j=1}^{n} b_j \overline{F}_j(b+k-s),$$

where $\overline{F}_j(s)$ defined as $1-F_j(s)$, the solution of $G_n(s)$ in the region $[b, b+k)$ is easily obtained as

$$G_n(s) = V_n^n(k-s) + \sum_{j=1}^{n} b_{0,j}\overline{F}_j(b+k-s) \tag{4}$$

for arbitrary constants $b_{0,j}$, $j=1,2,\cdots,n$. The function might be meaningfully defined in wider region, but our main interests are the value of $G(s)$ in the region $s\in[b, b+h]$. The region is divided into m sub-intervals of $\Psi_i=[b+ik, b+(i+1)k)\cap[b, b+h]$, $i=0,1,\cdots,(m-1)$, where $(m-1)k\le h< mk$. Let $G_{n,i}(s)$ be the restriction of $G_n(s)$ on Ψ_i. Then, $G_{n,0}(s)$ is given in the form (4). From (3) for $s\in\Psi_i$, $i=1,2,\cdots,m$, we have the recursive relation

$$(1-D_s)^n G_{n,i}(s) = V_0^n(k-s) + G_{n,i-1}(s-k) \tag{5}$$

For $i<0$, $G_{n,i}$ is defined to be identical 0. The relation (5) is applicable in the both of forward and backward directions; that is, in increasing order of i and decreasing order of i. Since we get the solution easily for $i=0$, forward direction is favorable. The general solution of (5) is given in the following theorem.

Theorem 1: For $n\ge 0$ the integral equation (5) has the solution

$$G_{n,i}(s) = \sum_{q=1}^{i+1} V_{qn}^n(qk-s) + \sum_{q=0}^{i}\sum_{j=1}^{n} b_{q,j}\overline{F}_{j+(i-q)n}(b+(i+1)k-s) \tag{6}$$

for $s\in\Psi_i$, $i=0,1,\cdots,(m-1)$.

(Proof) Direct from the facts described in Lee (2004), and abbreviated.

The function $G_n(s)$ is completely determined by the nm constants $\{b_{q,j}\}$, for $q=0,\cdots,(m-1)$ and $j=1,\cdots,n$. To determine the constants $b_{q,j}$ we need to set the constraints from the continuity conditions at the points $s_i=b+ik$, $i=1,2,\cdots,(m-1)$.

Lemma 2: For the function $G_n^*(s)$ defined as $G_n(s)-\sum_{j=0}^{n} a_jF_j(k-s)$, the u-th derivative $(1-D_s)^u G_n^*(s)$ is continuous for $u=0,\cdots,(n-1)$.

(Proof) In (3), the term $G_n * F_{\max(0,n-u)}(s)$ is continuous regardless of the continuity of $G_n(s)$ provided $u<n$, and the term $V_0^u(k-s)-\sum_{j=0}^{u} a_jF_j(k-s)$ is also continuous. The result directly follows.

Since $G^{*}_{n,i}(s)-G^{*}_{n,i-1}(s)=G_{n,i}(s)-G_{n,i-1}(s)$, when $G^{*}_{n,i}(s)$ is the restriction of $G^{*}_{n}(s)$ on Ψ_i, we derive the following condition directly from the continuity of $(1-D^u_{s)}G^{*}_{n}(s)$.

$$(1-D_s)^u[G_{n,i}(s)-G_{n,i-1}(s)]_{s=s_i}=0 \tag{7}$$

for $u=0,1,\cdots,(n-1)$ and $i=1,2,\cdots,(m-1)$. So the constraints are converted to the linear relation with respect to $b_{q,j}$,

$$\sum_{q=0}^{m-1}\sum_{j=1}^{n} b_{q,j}[A^u_{q,j}(i-1,0)-A^u_{q,j}(i,k)]=V^n_{(i+1)n-u}(k-b) \tag{8}$$

where $A^u_{q,j}(i,s)=I(q\leq i)\ \overline{F}_{\max(0,(i-q)n+j-u)}(s)$.

The remaining n constraints are derived from (3) and (6) for the point $s=0$. Here, if $b=0$ or $b=-h$, one of two boundary points $s=b$ or $s=b+h$ will be 0. That is,

$$[(1-D^u_{s)}G_{n(s)}]_{s=0}=V^u_0(k)+G_n*F_{n-u}(-k) \tag{9}$$

for $u=0,\cdots,(n-1)$. $G_n(s)$ in LHS of (9) is $G_{n,0}(s)$ if $b=0$, and $G_{n,(m-1)}(s)$, if $b=-h$. Let $\phi\in\{0,\cdots,(m-1)\}$ be the case $0\in\Psi_\phi$. The function G_n in RHS of (9) is substituted with the solution obtained from (6). The linear constraints are set to be

$$\sum_{q=0}^{m-1}\sum_{j=1}^{n} b_{q,j}[B^u_{q,j}(k)-A^u_{q,j}(\phi,b+(\phi+1)k)]=C(\phi,u) \tag{10}$$

where

$$B^u_{q,j}(y)=\sum_{i=q}^{m-1}\int_{\Psi_i}\overline{F}_{j+(i-q)n}(b+(i+1)k-x)\ dF_{n-u}(x+y)$$

$$C(\phi,u)=\sum_{q=1}^{\phi+1}V^{n(qk)}_{qn-u}-V^{n(k)}_0-\sum_{i=0}^{m-1}\sum_{q=1}^{i+1}\int_{\Psi_i}V^{n(qk-x)}_{qn}dF_{n-u}(x+k).$$

The key in calculation of $B^u_{q,j}$ and $C(\phi,u)$ is the integral of the type $\int F_n(-x)dF_m(x)$. From the followings we get the results easily.

$$F_n(-x)D_xF_m(x)=\int F_n(-x)\,dF_{m-1}(x)-\int F_{n-1}(-x)\,dF_m(x)$$

The term $D_xF_m(x)$ is given by $F_m(x)-F_{m-1}(x)$.

Now the solution of the integral equation (2) exists for any constants a_j, $j\geq 0$. As a special case, some of a_j might be given to be a value of $G(s)$, for example, $G(0)$. In those cases we also find the solution by adding more constraints derived from the meaning of the values. As long as we have the linearly independent constraints as many as the number of unknown constants, the solution will be completely solved. Later we will consider the case with specific examples.

In the next section we will consider the specific examples. The examples are completely described by specifying the accrual function $\pi(s)$. We select $\pi(s)$ which is popular in process control problems.

3. Applications

From the theorem and lemma stated in the previous section we obtain the solution of the integral equation $G(s)=\mathrm{pi}(s)+G*F_n(s-k)$ for a class of accrual function $\pi(s)$. Now we consider the cases $\pi_P(s)=F_n(k-s)$, $\pi_N(s)=1$, $\pi_H(s)=1+G(0)F_n(k-s)$, and $\pi_L(s)=1+G(0)(1-F_n(k-s))$ for $\pi(s)$. For the cases of $\pi_P(s)$, $\pi_N(s)$, and $\pi_H(s)$, we assume $b=0$, and for $\pi_L(s)$ we assume $b=-h$ for simplicity, without loss of applicability. We will call the solution $G(s)$ of (2) as $P(s)$, $N(s)$, $H(s)$ and $L(s)$ when $\pi(s)=\pi_P(s)$, $\pi_N(s)$, $\pi_H(s)$ and $\pi_L(s)$ respectively. The function $N(s)$ is defined as $E(t)$ when $t(\geq 1)$ is the first time that $T_t \notin [b, b+h]$ as defined in the previous section. $H(s)$ and $L(s)$ corresponds to the cases of T_t^u and T_t^d. For $P(s)$ and $N(s)$, the solutions are directly obtained from the system of linear equations (8) and (10). A special care need to be taken for $H(s)$ and $L(s)$ because of the term $G(0)$ in $\pi(s)$.

From the characteristics of $\pi_H(s)$ and $\pi_L(s)$, we have another constraint for each of $H(s)$ and $L(s)$. By the definitions of $H(0)$ and $L(0)$, we get $H(0)=[1-F_n(k)]^{-1}[1+H*F_n(-k)]$ and $L(0)=F_n(k)^{-1}[1+L*F_n(-k)]$ and $L(0)$ are constants, and represented by $\{b_{q,j}\}$ which appear in $G*F_n(-k)$. Due to the fact the linear constraints (8) and (10) are re-written w.r.t. $\{b_{q,j}\}$ when we substitute $H(0)$ and $L(0)$ to the original integral equations.

Fig. 1 shows the numerical results for the functions $P(s)$, $N(s)$, $H(s)$ and $L(s)$. $P(s)$ and $N(s)$ are obtained in the case $n=2$ and $h=6$. When k is small (k=1.6), the effects of the initial value s to the functions $P(s)$ and $N(s)$ appear to be more sensitive than the case when k is large (k=2.4). Three cases k=0.75, 1.0, 1.25 are plotted for $H(s)$. For $L(s)$, we searched the value k giving the similar value of $L(0)$ with the case of $H(0)$. When $H(0)=L(0)$, the function $L(s)$ more rapidly decreases as s increases.

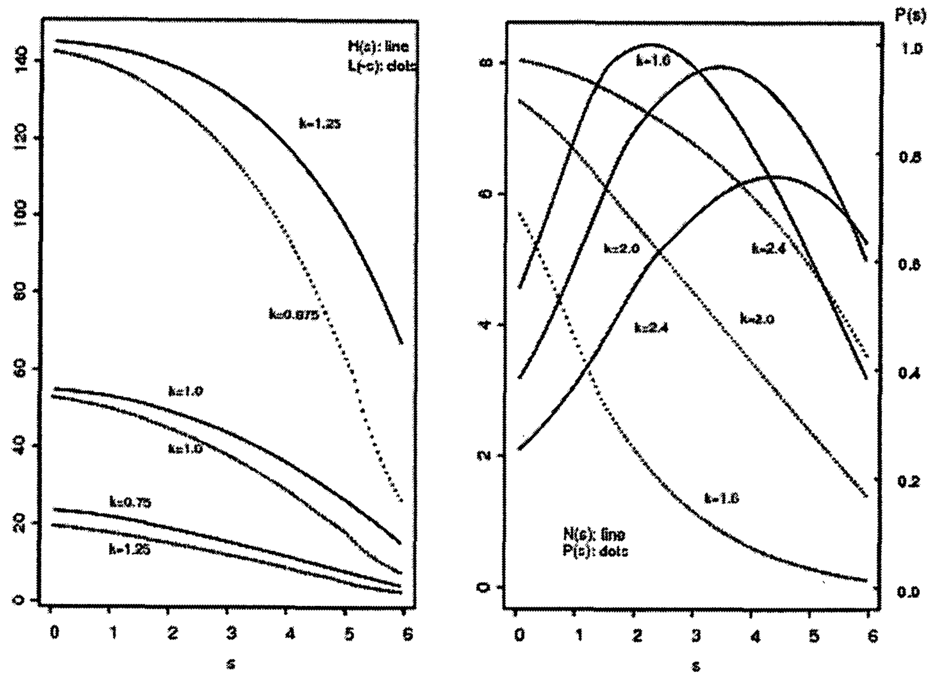

Fig. 1. Examples of $H(s)$ and $L(-s)$ for exponential distribution (left) and $P(s)$ and $N(s)$, for Erlang distribution with $n=2$ (right), when $h=6$.

4. Final remarks

In previous sections, we assumed implicitly that $\theta=1$ in $X_i \sim F_n(\theta x)$, and the results $G_n(s)$ was obtained. Here, the function $G_n(s)$ is the function of not only s, but also k, h and even θ. To show this relation explicitly, we use $G_n^\theta(s, h, k)$ instead of simple notation $G_n(s)$. Thus, $G_n(s) = G_n^1(s, h, k)$. For the reason of $\theta X_i \sim F_n(x)$, we get $G_n^\theta(s, h, k) = G_n^1(\theta s, \theta h, \theta k)$.

Although we obtained the results for Erlang distributions and their mixture type extension, the results have quite important applications in many areas. As mentioned in previous researches, monitoring system of Poisson process is a main application. In communication systems, many of random phenomena are explained by Erlang distribution, which is the gamma distribution with integer shape parameter. An important application of this paper is the area of monitoring process variance, when the process has the normally distributed characteristics. Chang and Gan (1995) compared the methods to monitor the process variance. They considered sequential version with the statistic of sample variance S^2 which follows χ^2

distribution. Log-normal approximation to χ^2 distribution and simulation study were adopted in the research. Instead, the method we suggest provides exact solution simply.

For testing the properties of the variants of SPRT our results are useful. Also, The result of this paper has application in finding the optimal parameters h and k of a reservoir system with a specific example of $\pi(s)$.

References

[1] Chang, T. and Gan, F. (1995). A cumulative sum control chart for monitoring process variance, *Journal of Quality Technology*, **25**, 109-119.

[2] Choi, K and Kim, H. (1998). Bayesian computation for superposition of MUSA-OKUMOTO and Erlang (2) processes, *Journal of Korean Applied Statistics*, **11**, 377-387.

[3] Gan, F. and Choi, K. (1994). Computing Average Run Lengths for Exponential CUSUM Schemes, *Journal of Quality Technology*, **26**, 134-139.

[4] Khamis, S. H. (1961). Incomplete Gamma Functions Expansions of Statistical Distribution Functions, *Statistique Mathematique*, 385-396.

[5] Kim, H. and Lee, S. (2000). Bayesian inference for mixture failure model of Rayleigh and Erlang pattern, *Journal of Korean Applied Statistics,* **13**-2, 505-514.

[6] Kohlruss, D. (1994). Exact formulas for the OC and the ASN functions of the SPRT for Erlang distributions, *Sequential Analysis*, **13**, 53-62.

[7] Lee, E., Na, M.H. and Lee, Y.D. (2005). Solutions of integral equations related to SPRT for Erlang distribution, *Journal of Korean Applied Statistics*, **18**, 57-66.

[8] Lee, Y. D. (2004). Unified solutions of integral equations of SPRT for exponential random variables, *Communications in Statistics, Series A, Theory and Method* , **33**, 65-74.

[9] Stadje, W. (1987). On the SPRT for the mean of an exponential distribution, *Statistics & Probability Letters*, **5**, 389-395.

[10] Vardeman, S. and Ray, D. (1985). Average Run Lengths for CUSUM Schemes When Observations Are Exponentially Distributed, *Technometrics*, **27**, 145-150.

[Received March 2005, Accepted June 2005]