International Journal of Reliability and Applications
Vol. 6, No. 1, pp. 31-39, 2005

Design of a Life Test Sampling Plan Based on the Cost Model

Young Il Kwon
Department of Industrial Engineering, Cheongju University, Naedok-dong 36 Cheongju, Chungbuk, 360-764, Korea

Abstract. An economic life test sampling plan for products with exponential lifetime distribution is developed. To reduce test time, a test plan with curtailed Type II censoring is considered. A cost model is constructed which involves three cost components; test cost, accept cost, and reject cost. Determination of optimal plan minimizing the expected average cost per lot is discussed with a constraint related to consumer's risk. Some numerical examples are provided.

Key Words : *Life test sampling plan, Type II censoring, Warranty, Economic design, Cost model.*

1. INTRODUCTION

A sampling plan that determines the acceptability of a product with respect to the product lifetime is called a life test sampling plan. In life testing, a fixed number of items are often tested simultaneously and the test continues for some fixed period of time (Type I censoring) or until some fixed number of items on test fail (Type II censoring).

Expenses for conducting life tests are usually high because the tests are time consuming and destructive in that some or all of the items fail during the tests. Therefore it is desirable to design economic life test procedure that minimizes expected cost for conducting the test to achieve a specified precision. A life test sampling plan based on cost model for the Weibull distribution with known shape parameter is considered by Soland (1968) and one for exponential distribution with Type II censoring by Thyregod (1975). Dunsmore and Wright (1985) proposed a Bayesian sequential sampling plan based on cost model for exponential distribution. Blight (1972) and Ebrahimi (1988) considered economic choice of a life test procedure that minimizes expected test cost consists of costs proportional to the number of items on test and the test duration to achieve a given precision. Kwon (1996) proposed a Bayesian sampling plan for products with Weibull lifetime distribution that are sold under a warranty policy.

Corresponding Author.
E-mail address: yikwon@cju.ac.kr

In this article, an economic life test sampling plan for exponential population with curtailed Type II censoring is considered. One of the major shortcomings of the sampling plan with Type II censoring only is that the test duration becomes longer as the reliability of a product becomes better. Curtailed Type II censoring allows early acceptance of products with high reliability and therefore the effectiveness of a sampling plan will be increased by adopting curtailed Type II censoring. Costs associated with life test sampling plans are test cost, accept cost, and reject cost. Test cost consists of the costs proportional to the length of test time and number of sampled items used on test. Accept cost is an external failure cost due the items in the accepted lot that fail before some fixed length of time (i.e., mission time or warranty period) has been elapsed. Reject cost is scrap or reprocess cost for items in the rejected lot.

The lifetime of a product is assumed to be an exponential distribution with *p.d.f.*

$$f(t) = \frac{1}{\theta} e^{-\frac{t}{\theta}}, \quad t > 0. \tag{1.1}$$

The test procedure of the proposed sampling plan is based on curtailed Type II censoring. In the test with Type II censoring, n items are placed on test simultaneously and test continues until the r-th failure time, t_r, is observed. It is known that the best acceptance region for testing the hypothesis

$$\begin{aligned} H_0 &: \theta = \theta_0 \\ H_1 &: \theta = \theta_1 < \theta_0 \end{aligned} \tag{1.2}$$

is of the form $V(t_r) > S$ where $V(t_r)$ is the total time on test (TTT) at t_r and S is a positive constant. In the case of acceptance, the test may be terminated at $t^*(S)$ before the *r-th* failure occurs where $t^*(S)$ is the time at which TTT reaches S. Test procedure with curtailed Type II censoring can be described as follows:

n items are drawn at random from a lot and placed on test simultaneously. Items that fail during the test are not replaced by new ones. The test continues until either the r-th failure occurs or a fixed TTT S is reached. If the r-th failure occurs first, the lot is rejected and otherwise, the lot is accepted.

2. MODEL FORMULATION

Consider the (n, r, S) life test sampling plans proposed in the previous section. It is assumed that items used on test are scrapped since they, although not failed, might be damaged or deteriorated during the test. The following notations will be used for model formulation.

N	lot size
$t^*(S)$	the time at which total time on test reaches S
$D(S)$	number of failures up to $t^*(S)$
t_r	the *r-th* failure time in a sample of size n
$V(t_r)$	total time on test at t_r
T	$Min\{t_r, t^*(S)\}$, test duration
t_m	warranty period or mission time of the product
A_s	cost of putting an item on test
A_t	cost per unit of test time
A_r	cost of rejecting an item
A_f	cost associated with an external failure occurring before t_m

The sampling cost A_s will be larger than the scrap cost A_r since an item used on test shall be scraped after the test is terminated.

Let $L(\theta)$ denote the probability of accepting a lot when the mean life of a products in the lot is θ. Then, for given r and S, $L(\theta)$ is given by

$$L(\theta) = \Pr\{D(S) < r\} = \sum_{k=0}^{r-1} p(k : S/\theta) \tag{2.1}$$

where $p(k:m)$ denotes the Poisson probability mass function with parameter m, i.e., $p(k:m) = \dfrac{m^k e^{-m}}{k!}$.

For given θ, the expected number of external failures up to t_m for an accepted lot is $(N-n)t_m/\theta$. Therefore the expected average cost per lot is given by

$$k(n,r,S) = A_s n + A_t E[T] + (N-n)[A_f t_m L(\theta)/\theta + A_r\{1 - L(\theta)\}] \tag{2.2}$$

where

$$E[T] = \theta[\sum_{k=1}^{r-1} a(n,k)p(k:S/\theta) + a(n,r)\{1 - \sum_{k=0}^{r-1} p(k:S/\theta)\}] \tag{2.3}$$

and

$$a(n,k)=\sum_{i=1}^{k}\frac{1}{n-i+1}. \tag{2.4}$$

Let K_a and K_r be the costs per lot for special cases of acceptance and rejection without test respectively. Then

$$\begin{aligned}K_a &= NA_f t_m/\theta,\\ K_r &= NA_r.\end{aligned} \tag{2.5}$$

If the value of θ is known precisely, the optimal decision is to accept the lot if $\theta \geq \theta^*$, and reject it otherwise, where

$$\theta^* = A_f t_m / A_r. \tag{2.6}$$

3. THE OPTIMAL SAMPLING PLAN

In this section, sampling plan minimizing expected average cost subject to a constraint related to consumer's risk will be considered. It is assumed that the condition

$$\Pr\{accept\ \ lot \mid \theta=\theta_1\} \leq \beta \tag{3.1}$$

should be satisfied from the consumer's point of view where θ_1 is the mean life of a product which is not acceptable. Since

$$\begin{aligned}\Pr\{accept\ \ lot \mid \theta=\theta_1\} &= \Pr\{D(S)<r \mid \theta=\theta_1\}\\ &= \Pr\{V(t_r)>S \mid \theta=\theta_1\}\end{aligned} \tag{3.2}$$

and $2V(t_r)/\theta$ has a chi-square distribution with degree of freedom $2r$ when θ is the true value, constraint (3.1) can be expressed as

$$S \geq \theta_1 \chi^2_\beta(2r)/2 \tag{3.3}$$

where $\chi^2_\beta(k)$ denotes $100(1-\beta)\,th$ percentile of chi-square distribution with k degrees of freedom.

Let θ_m be the value of θ at which we want to minimize the expected average cost per lot. θ_m may be the "target value" of the production process and we may want a sampling plan which would be most economical when the process is working satisfactorily. Therefore our objective is to decide the sampling plan (n,r,S) that minimizes the expected average cost $K(n,r,S)$ at $\theta = \theta_m$ subject the constraint (3.1).

For given n, r, and S, let $T(n,r,S)$ and $Q(r,S)$ denote $E(T)$ and $L(\theta)$ respectively. Then we obtain the following results.

Theorem 3.1 $T(n,r,S)$ and $Q(r,S)$ have the following properties:

(i) $T(n,r,S)$ is decreasing in n and increasing in r and S.
(ii) $Q(r,S)$ is increasing in r and decreasing in S.

Proof. (i) From (4), we obtain

$$\begin{aligned}\Delta_n T(n,r,S) &= T(n,r,S) - T(n-1,r,S) \\ &= \theta[-\sum_{k=1}^{r-1} kh(k)/\{(n-1)(n-k-1)\}] < 0,\end{aligned} \tag{3.4}$$

$$\begin{aligned}\Delta_r T(n,r,S) &= T(n,r,S) - T(n,r-1,S) \\ &= \frac{h(k)}{n-r+1} > 0,\end{aligned} \tag{3.5}$$

and

$$\frac{\partial T(n,r,S)}{\partial S} = \sum_{k=0}^{r-1} p(k:S/\theta)/(n-k) > 0 \tag{3.6}$$

where

$$h(k) = \begin{cases} p(k:S/\theta) \ \ for \ \ k \le r-1 \\ 1 - \sum_{k=0}^{r-1} p(k:S/\theta) \ \ for \ \ k = r\ . \end{cases} \tag{3.7}$$

Therefore the desired results are proved.

(ii) Since

$$\begin{aligned}\Delta_r Q(r,S) &= Q(r,S) - Q(r-1,S) \\ &= p(r-1:S/\theta) > 0,\end{aligned} \tag{3.8}$$

and

$$\frac{\partial Q(r,S)}{\partial S} = -p(r-1:S/\theta) < 0, \tag{3.9}$$

$Q(R,S)$ is increasing in n and decreasing in S.

Let $n(r)$ and $S(r)$ be the optimal values of n and S for given r. Then we have the following results.

Theorem 3.2 When $\theta_m \geq \theta^*$, for a given r, we obtain

(i) $S(r) = \theta_1 \chi^2_\beta(2r)/2$ (3.10)

and

$$\text{(ii) } n(r) = \begin{cases} r, & \text{if } \Delta_n K(r+1, r, S(r)) \geq 0, \\ \\ \underset{n>r}{Max}\ \{n \mid \Delta_n K(n,r,S(r)) \leq 0\}, & \text{otherwise} \end{cases} \tag{3.11}$$

where

$$\Delta_n K(n,r,S(r)) = K(n,r,S(r)) - K(n-1,r,S(r)). \tag{3.12}$$

Proof. (i) Let $A_1 = A_f t_m / \theta$. Then

$$K(n,r,S) = A_s n + A_t T(n,r,S) + (N-n)\{(A_1 - A_r)Q(r,S) + A_r\}. \tag{3.13}$$

Since $A_1 - A_r \leq 0$ for $\theta_m \geq \theta^*$ and $T(n,r,S)$ and $-Q(r,S)$ are increasing in S, $K(n,r,S)$ is increasing in S. Therefore

$$\begin{aligned} S(r) &= Min\{S \mid S \geq \theta_1 \chi^2_\beta(2r)/2\} \\ &= \theta_1 \chi^2_\beta(2r)/2. \end{aligned} \tag{3.14}$$

(ii) Once $S(r)$ is decided for a given r, we have

$$K(n,r,S(r)) = \{A_s - K_{ar}(r)\}n + A_t T(n,r,S(r))\} + NK_{ar}(r) \tag{3.15}$$

where $K_{ar}(r) = A_1 Q(r,S(r)) + A_r\{1 - Q(r,S(r))\}$. The difference of $\Delta_n K(n,r,S(r))$ with respect to n is

$$\begin{aligned}&\Delta_n K(n+1,r,S(r)) - \Delta_n K(n,r,S(r))\\ &= K(n+1,r,S(r)) - 2K(n,r,S(r)) + K(n-1,r,S(r)) \\ &= A_t \theta_m \sum_{k=1}^{r} kh(k)\{\frac{1}{(n-1)(n-k-1)} - \frac{1}{n(n-k)}\} > 0.\end{aligned} \tag{3.16}$$

Therefore the results are proved.

The following theorem concerns the optimal decision for $\theta_m < \theta^*$.

Theorem 3.3 When $\theta_m < \theta^*$, optimal decision is to reject the lot without test.

Proof. Since $A_1 - A_r > 0$ when $\theta_m < \theta^*$, we obtain

$$\begin{aligned}K(n,r,S) &= A_s n + A_t T(n,r,S) + (N-n)\{(A_1 - A_r)Q(r,S) + A_r\}\\ &\geq A_s n + A_t T(n,r,S) + (N-n)A_r \\ &= NA_r + A_t T(n,r,S) + (A_s - A_r)n \\ &\geq NA_r\end{aligned} \tag{3.17}$$

and this implies that the optimal decision for $\theta_m < \theta^*$ is to reject the lot without test.

4. NUMERICAL EXAMPLES

Life test sampling plan is being used with lots of size N=2,000 and the cost constants are A_s =2.5\$, A_r =2.0\$, A_f =70.0\$, and A_t =0.2\$. Warranty period of the product is t_m =100 and a lot with items whose mean lifetime is less than θ_1 =1,950 are to be accepted no more than 10% of the time (β =0.1). Then we have θ^* =3,500. Optimal sampling plans that minimize the expected average cost per lot for some selected values of θ_m are given in Table 4.1.

Table 4.1 Optimal sampling plans for selected values of θ_m.

θ_m	n	r	S	$K(n,r,S)$	$L(\theta_m)$	$E(T)$
10,000	49	7	20,529	1,587	0.995	128
7,000	63	10	27,694	2,200	0.992	454
5,000	86	14	36,964	3,006	0.981	448
4,000	111	16	41,488	3,690	0.937	389
3,500	0	-	-	4,000	0.0	0

Let θ' be the true value of θ, that is, the mean lifetime of products being produced in current production process. In applying the sampling plans, θ' may differ from the target value θ_m. Consider a situation where θ_m = 10,000 and all cost constants and constraint are the same as those in the previous example. When θ' differ from θ_m as in Table 4.2, the corresponding expected average cost $K(n,r,S)$ is computed for $\delta = \theta' / \theta_m$ where (n,r,S) is the optimal sampling plan for θ_m =10,000. Here, (n',r',S') and $K(n',r',S')$ are optimal sampling plan and the corresponding expected average cost for $\theta = \theta'$, respectively and η denotes

$$\eta = \{K(n,r,S)/K(n',r',S') - 1.0\} \times 100 \ (\%). \tag{4.1}$$

Table 4.2 Expected average cost when θ' differ from the target value θ_m.

δ	$K(n,r,S)$	$K(n',r',S')$	η (%)
1.3	1,261.8	1,255.6	0.49
1.2	1,351.2	1,347.2	0.30
1.1	1,457.8	1,456.7	0.08
1.0	1,587.0	1,587.0	0.00
0.9	1,747.1	1,746.9	0.01
0.8	1,950.9	1,945.3	0.29
0.7	2,218.4	2,200.0	0.84

From the results in Table 4.2, we can see that the proposed sampling plans are robust to small changes in the parameter θ.

5. CONCLUDING REMARKS

A design of economic life test sampling plan for products with exponential lifetime distribution is considered. A test plan with a curtailed Type II censoring is used to reduce the test time. A cost model is constructed which involve three cost components; test cost, accept cost, and reject cost. Determination of the optimal plan that minimizes the expected average cost per lot under a constraint related to a consumer's risk is discussed and some numerical examples are provided. The results of numerical examples suggest that the proposed sampling plan seems to be robust to small changes in the mean lifetime of products.

REFERENCES

Blight, B.J.N. (1972). On the Most Economical Choice of a Life Testing Procedure for Exponentially Distributed Data. *Technometrics*, **14**, 613-618.

Dunsmore, I.R., and Wright, D.E. (1985). A Decisive Predictive Approach to the Construction of Sequential Acceptance Sampling Plans for Life Times. *Applied Statistics*, **34**, 1-13.

Ebrahimi, N. (1988). Determining the Sample Size for a Hybrid Life Test Based on the Cost Function. *Naval Research Logistics*, **35**, 63-72.

Kwon, Y.I. (1996). A Bayesian Life Test Sampling Plan for Product with Weibull Lifetime Distribution Sold under Warranty. *Reliability Engineering and System Safety*, **53**, 61-66.

Soland, R.M. (1968). Bayesial Analysis of the Weibull Process with Unknown Scale Parameter and Its Application to Acceptance Sampling. *IEEE Trans. on Reliability*, **17**, 84-90.

Thyregod, P. (1975). Bayesian Single Sampling Plans for Life Testing with Truncation of the number of failures. *Scandinavian Journal of Statistics*, **2**, 61-70.