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Estimation of Parameters in a Generalized Exponential Semi-Markov Reliability Models

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Abstract. This paper deals with the stochastic analysis of a three-states semi-Markov reliability model. Using both the maximum likelihood and Bayes procedures, the parameters included in this model are estimated. Next, assuming that the lifetime and repair time are generalized exponential random variables, the reliability function of this system is obtained. Then, the distribution of the first passage time of this system is discussed. Finally, some of the obtained results are compared with those available in the literature.

Key Words : *Maximum likelihood estimators, Semi-Markov model, System reliability, Active unit, Repair facility.*

1. INTRODUCTION

A Markov chain analysis can be used to describe patterns of deposition and conditional probability of occurrence of different rock types through transition probability matrices; see, for example, Anderson and Goodman (1957), Dacay and Krumbein (1970), Harbaugh (1985), and Krumbein (1965). Markov chain models have also been used for subsurface modeling. The occurrence of lithologies is viewed as a stochastic process. The stochastic analysis of a semi-Markov reliability model is rarely investigated during the last two decades. For a more extensive overview of the reliability theory of repairable systems, see Igor (1994), Medhi (1982), Sarhan and El-Gohary (2003), and Grabski (1999).

In this section, we will display some important definitions and properties of a semi-Markov process and its kernel. The evolution of many systems naturally ends as the first failure occurs, because external intervention is not practicable. These systems are non-repairable systems. For other systems, generally of high complexity,

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renewal possibilities exist, and their effectiveness therefore depends not only on their intrinsic reliability but also on the characteristics of maintenance and repair actions.

Consider the time interval $(0, t)$. The number of renewals N_t occurring in this interval is a discrete stochastic process, called a renewal process. Once the characteristics of this process are known the reliability model, as predictions of the evolution of the system, can be made. Preventive maintenance is scheduled downtime, usually periodically, in which defined set of tasks, such as inspection and repair, replacement, cleaning, lubrication, adjustment and alignment, are preformed.

A semi Markov process $\{X(t) : t \geq 0\}$ is a stochastic process in which changes of state occur according to a Markov chain and in which the time interval between two successive transitions is a random variable whose distribution depends on the state from which the transition takes place as well as the state to which the next transition takes place Medhi (1982). Generally a semi-Markov process with discrete state space can be defined as a Markov renewal process Krumbein and Graybill (1965), Korolyuk and Swishchuk (1994) and Grabski (1999). Assuming that the state space S is finite, we can define the renewal kernel as follows:

Definition 1.1 *The stochastic matrix* $Q(t) = \Big[Q_{ij}(t); i, j \in S\Big]$, $t \geq 0$ *is said to be a renewal kernel if and only if the following conditions are satisfied:*

1. *The functions* $Q_{ij}(t)$ *are nondecreasing functions in* t.
2. $\sum_{j \in S} Q_{ij} = G_i(t)$ *are distribution functions in* t.
3. $[Q_{ij}(+\infty) = P_{ij}, i, j \in S] = P$ *is a stochastic matrix.*

Definition 1.2 *A two-dimensional Markov process* $\Big\{\xi_n, \vartheta_n, n \in N\Big\}$ *with values in* $S \times [0, \infty)$ *is called a Markov renewal process if and only if [Korolyuk and Swishchuk (1994)]*

1. $Q_{ij} = P\Big\{\xi_{n+1} = j, \vartheta_{n+1} \leq t | \xi_n = i, \vartheta_n = t_n, \ldots, \xi_0 = i_0, \vartheta_0 = t_0\Big\}$
   $= P\Big\{\xi_{n+1} = j, \vartheta_{n+1} \leq t | \xi_n = i\Big\}$,
2. $P\Big\{\xi_0 = i, \vartheta_0 = 0\Big\} = p_{i0}$

Obviously, the transition probabilities do not depend on the discrete component (do not depend on the second components). In the Markov renewal process, the non-negative random variables $\vartheta_n, n \geq 1$, define the interval between Markov renewal times:

$$\tau_n = \sum_{k=1}^{n} \vartheta_k, \ \ n \geq 1, \tau_0 = 0. \tag{1.1}$$

Now, let

$$\nu(t) = \sum_{n=1}^{\infty} I_{[0,t]}(\tau).$$

The process $\nu(t)$ is called a counting process. It determines the number of renewal times on the segment $[0, t]$.

Definition 1.3 *Grabski (1999); A stochastic process $\{X(t) : t \geq 0\}$ where $X(t) = \xi_{\nu(t)}$ is called a semi-Markov process that generated by the Markov renewal process with initial distribution $P_i^0 = p(\xi_0 = i)$ and the kernel $Q(t), t \geq 0$.*

Since the counting process $\nu(t)$ keeps constant values on the half-interval $[t_n, t_{n+1})$ and is continuous from the right, then the semi-Markov process keeps also constant values on the half intervals $[\tau_n, \tau_{n+1})$: $X_n(t) = \xi_n \text{ for } t \in [\tau_n, \tau_{n+1})$. Moreover the sequence $\{X(\tau_n) : n \in N\}$ is a Markov chain with transition probability matrix $P = \{p_{ij} = Q_{ij}(\infty), i, j \in S\}$ that is called an embedded Markov chain. The concept of a Markov renewal process is a natural generalization of the concept of the ordinary renewal process given by a sequence of independent identically non-negative random variables $\theta_n, n \geq 1$. The random variables θ_n can be interpreted as lifetimes.

Now, using the definition (1.3), the following lemma can be formulated.

Lemma 1.4 *Grabski (1999); If $\{X(t) : t \geq 0\}$ is a semi-Markov process with renewal kernel*

$$Q(t) = Q_{ij}(t), i, j \in S, \ t \in [0, \infty)$$

then

$$P\{\xi_0 = i_0, \vartheta_0 = 0, \xi_1 = i_1, \vartheta_1 \leq u_1, \ldots, \xi_n = i_n, \vartheta_n \leq u_n\} = p_{i_0} \prod_{k=1}^{n} Q_{i_{k-1}i_k}(u_k). \tag{1.2}$$

This lemma can be used to construct the likelihood function of some semi-Markov reliability models.

2. BAYESIAN ESTIMATION

Assuming that the semi-Markov renewal kernel of the reliability model depends upon a vector of unknown parameters $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, that is

$$Q(t|\underline{\alpha}) = \{Q_{ij}(t|\underline{\alpha}) : i, j \in S\}. \tag{2.1}$$

Let us assume that there is a sequence of random observations $z = \{(i_0, t_0), (i_1, t_1), \ldots, (i_n, t_n)\}$ of the random vector $(\xi_0, \vartheta_0), (\xi_1, \vartheta_1), \ldots, (\xi_n, \vartheta_n)$. We assume that there exist functions denoted by $q_{ij}(t|\alpha), i, j \in S$ such that

$$Q_{ij}(t|\underline{\alpha}) = \int_0^t q_{ij}(u|\underline{\alpha})du. \tag{2.2}$$

Using lemma 1.4, the likelihood function for the given random observations of the semi-Markov process becomes

$$L(z;\underline{\alpha}) = p_{i_0} \prod_{k=1}^{n} q_{i_{k-1}i_k}(t_k|\underline{\alpha}). \tag{2.3}$$

In the Bayesian procedure, it is assumed that $\underline{\alpha}$ is a vector of random variables. Then these random variables have a joint probability density function, say $g(\underline{\alpha})$, called a joint prior probability distribution function of $\underline{\alpha}$. If the loss incurred when the vector $\underline{\alpha}$ of the unknown parameters estimated by $\hat{\alpha}$ is quadratic, then the value of the Bayes estimator for α_i becomes the posterior expectation, given by:

$$\hat{\alpha_i} = E(\alpha_i|z) = \int \alpha_i g(\alpha_i|z)d\alpha_i, \; i = 1, 2, \ldots. \tag{2.4}$$

It is assumed that the lifetime of the system has a generalized exponential distribution with two parameters. Under the assumptions, that the life and repair times are generalized exponential, the reliability of the system is derived. The distribution of the first passage time of the system is obtained.

3. SEMI MARKOV STANDBY MODEL

The semi-Markov process is a convenient tool to describe many reliability models. In order to describe a reliability model of a standby system with a repair facility, the following assumptions are adopted:

1. The system consists of one active unit, a standby unit, a switch and a repair facility.
2. The failed units can be repaired by the repair facility and the repairs fully restore the units. This means that the element repair means its renewal.
3. The system fails when the active unit fails and repair has not been finished yet or when the active unit fails and the switch fails .
4. The lifetimes of the active units can be represented by independent and identical non-negative random variables ξ_1 with distribution function $F(t) = P\{\xi_1 \leq t\}, t \geq 0$.
5. The lengths of repair periods of the units can represented by independent and identical non-negative random variable ξ_2 with the distribution function $G(t) = P\{\xi_2 \leq t\}$.

6. The event E denotes the switch-over as the active unit fails. Then the probability that the switch performs when required is represented by $P(E) = \alpha_1, (0 \leq \alpha_1 \leq 1)$.

7. The whole system can also be repaired, and the failed system is replaced by a new identical one.

8. The replacing time is represented by a non-negative random variable ξ_3 with distribution function $H(t) = P\{\xi_3 \leq t\}$.

9. Finally, we assume that the random variables $\xi_i (i = 1, 2, 3)$ and E are independent.

Under the above assumptions, the states of the prescribed system can be considered as follows:

1. both active and standby units are "Up"

2. the failed unit is repaired and the standby unit is operating.

3. the system failure.

Let $\tau_0^*, \tau_1^*, \tau_2^*, \ldots$ denote the instants when of the state of the system changes where $\tau_0^* = 0$ and let $\{Y(t) : t \geq 0\}$ be a stochastic process with state space $S = \{1, 2, 3\}$. This process keeps constant values on the half intervals $[\tau_n^*, \tau_{n+1}^*)$ and is continuous from the right. There fore, it is not a semi-Markov process.

Let us define a new stochastic process as follows:

Assuming that $\tau_0 = 0$ and $\tau_n, n = 1, 2, \ldots$ represent the instants when the components of the system failed or the whole system renewal. The stochastic process $\{X(t) : t \geq 0\}$ defined by

$$X(0) = 0, X(t) = Y(\tau_n) \, \text{for} \, t \in [\tau_n, \tau_{n+1}) \tag{3.1}$$

is a semi-Markov process and its kernel is given by the following matrix

$$\begin{bmatrix} 0 & Q_{12} & Q_{13} \\ 0 & Q_{22} & Q_{23} \\ Q_{31} & 0 & 0 \end{bmatrix} \tag{3.2}$$

The semi-Markov process $\{X(t), t \geq 0\}$ is completely specified by its semi-Markov kernel. Let us deduce the elements of the semi-Markov kernel as follows:

$$\begin{aligned}
Q_{12}(t) &= P\{X(\tau_{n+1}) = 2, \vartheta_{n+1} \le t | X(\tau_n) = 1\} = P\{E, \xi_3 \le t\} = \alpha_1 F(t) \\
Q_{13}(t) &= P\{X(\tau_{n+1}) = 3, \vartheta_{n+1} \le t | X(\tau_n) = 1\} \\
&= P\{\bar{E}, \xi_1 \le t\} = (1-\alpha_1)F(t) \\
Q_{22}(t) &= P\{X(\tau_{n+1}) = 2, \vartheta_{n+1} \le t | X(\tau_n) = 2\} \\
&= P\{E, \xi_1 \le t, \xi_2 > \xi_1\} = \alpha_1 \textstyle\int_0^t (1 - G(t))dF(t) \\
Q_{23}(t) &= P\{X(\tau_{n+1}) = 3, \vartheta_{n+1} \le t | X(\tau_n) = 2\} \\
&= P\{\xi_1 \le t, \xi_2 > \xi_1\} + P\{\bar{E}, \xi_1 \le t, \xi_2 < \xi_1\} \\
&= \textstyle\int_0^t [1 - G(t)]dF(t) + (1-\alpha_1) \int_0^t G(t)dF(t) \\
&= F(t) - \alpha_1 \textstyle\int_0^t G(t)dF(t), \\
Q_{31}(t) &= P\{X(\tau_{n+1}) = 1, \vartheta_{n+1} \le t | Y(\tau_n) = 3\} \\
&= P\{\xi_3 \le t\} = H(t).
\end{aligned} \tag{3.3}$$

Using the relations between the elements of the semi-Markov kernel and their corresponding densities $q_{ij}, i, j \in S$ we get:

$$\begin{aligned}
&q_{12}(t) = \alpha_1 f(t), \quad q_{13}(t) = (1-\alpha_1)f(t), \quad q_{31}(t) = h(t) \\
&q_{13}(t) = \alpha_1 G(t) f(t), \quad q_{23}(t) = [1 - \alpha_1 G(t)] f(t).
\end{aligned} \tag{3.4}$$

It is observed in Gupta and Kundu (2001) that the two-parameter generalized exponential distribution can be used quite effectively in to analyze many lifetime data, particularly in place of two-gamma and two parameter Weibull distributions. The two-parameter generalized exponential distribution has increasing or decreasing failure rate depending on the shape parameter.

Now, we assume that the lifetime of the active units have identically generalized exponential distribution with parameters α_2 and α_3. That is,

$$f(t) = \alpha_2\, \alpha_3\, (1 - e^{-\alpha_3 t})^{\alpha_2 - 1}\, e^{-\alpha_3 t}, \; \alpha_2, \alpha_3 > 0, \; t \ge 0. \tag{3.5}$$

Substituting from (3.5) into the densities (3.4), we get

$$\begin{aligned}
q_{12}(t|\underline{\alpha}) &= \alpha_1\alpha_2\alpha_3(1 - e^{-\alpha_3 t})^{\alpha_2 - 1}\, e^{-\alpha_3 t}, \\
q_{13}(t|\underline{\alpha}) &= (1-\alpha_1)\alpha_2\alpha_3(1 - e^{-\alpha_3 t})^{\alpha_2 - 1}\, e^{-\alpha_3 t}, \\
q_{22}(t|\underline{\alpha}) &= \alpha_1\alpha_2\alpha_3 G(t)\, (1 - e^{-\alpha_3 t})^{\alpha_2 - 1}\, e^{-\alpha_3 t}, \\
q_{23}(t|\underline{\alpha}) &= \alpha_2\alpha_3[1 - \alpha_1 G(t)](1 - e^{-\alpha_3 t})^{\alpha_2 - 1}\, e^{-\alpha_3 t}, \\
q_{31}(t) &= h(t).
\end{aligned} \tag{3.6}$$

4. MAXIMUM LIKELIHOOD ESTIMATES

In this section the maximum likelihood estimators of the unknown vector $\underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ included in the generalized exponential reliability model are presented. Suppose that z denotes the observations $\{(i_0, t_0), (i_1, t_1), \ldots, (i_n, t_n)\}$ of two dimensional random vector of variables, $\{(\xi_0, \vartheta_0), (\xi_1, \vartheta_1), \ldots, (\xi_n, \vartheta_n)\}$ where $i_0, i_1, \ldots, t_n$ and $t_0, t_1, \ldots, t_n \in [0, \infty)$. Further, we assume that this observation is classified as follows:

Let

$$A_{ij} = \{k : i_{k-1} = i, i_k = j, k = 1, 2, \ldots, n\}$$

be the set of numbers of direct observed transition from the state i to the state j and n_{ij} is the cardinal number of the set A_{ij} which represents the number of direct transitions from the state i to state j . In the present case we find that

$$n_{12} + n_{13} + n_{22} + n_{23} + n_{31} = n. \tag{4.1}$$

Based on the above observation, the sample likelihood function $L(z; \underline{\alpha})$ can be obtained as follows:

Using (2.3) and (3.6) the sample likelihood function $L(z; \underline{\alpha})$ becomes

$$L(z; \underline{\alpha}) = \alpha_1^{n_{22}+n_{12}} (1-\alpha_1)^{n_{13}} \alpha_2^m \alpha_3^m e^{-\alpha_3 \tau} \prod_{k \in L} (1 - e^{-\alpha_3 t_k})^{\alpha_2 - 1} \prod_{k=1}^{n_{23}} [1 - \alpha_1 G(t_k)] \tag{4.2}$$

where

$$\tau = \sum_{k \in L} t_k, \;\; m = n_{23} + n_{13} + n_{22} + n_{12}, \;\; L = A_{23} U A_{13} U A_{22} U A_{12}.$$

Finally, the log of the sample likelihood function L can be written in the following form

$$\begin{aligned}\mathcal{L} = {} & m[\ln \alpha_2 + \ln \alpha_3] + (n_{22} + n_{12}) \ln \alpha_1 + n_{13} \ln(1 - \alpha_1) - \alpha_3 \tau \\ & + \sum_{k=1}^{n_{23}} \ln[1 - \alpha_1 G(t_k)] + (\alpha_2 - 1) \sum_{k \in L} \ln(1 - e^{-\alpha_3 t_k}).\end{aligned} \tag{4.3}$$

The maximum likelihood estimates $\hat{\alpha}_i$ are the values of $\alpha_i, i = 1, 2, 3$, that maximize the sample likelihood $\mathcal{L}$. Equivalently $\hat{\alpha}_i, i = 1, 2, 3$ maximize the log sample likelihood since it is a monotone function of $L(z, \underline{\alpha})$.

The maximum likelihood equations are given by :

$$\frac{\partial \mathcal{L}}{\partial \alpha_i} = 0, \quad i = 1, 2, 3. \tag{4.4}$$

Using (4.3) and (4.4) the maximum likelihood equations are

$$\frac{\partial \mathcal{L}}{\partial \alpha_1} = \frac{n_{22}+n_{12}}{\alpha_1} - \frac{n_{13}}{1-\alpha_1} - \sum_{k=1}^{n_{23}} \frac{G(t_k)}{1-\alpha_1 G(t_k)} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \alpha_2} = \frac{m}{\alpha_2} + \sum_{k\in L} \ln(1-e^{-\alpha_3 t_k}) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \alpha_3} = -\tau + \frac{m}{\alpha_3} + (\alpha_2 - 1)\sum_{k\in L} \frac{t_k\, e^{-\alpha_3 t_k}}{1-e^{-\alpha_3 t_k}} = 0. \tag{4.5}$$

The maximum likelihood estimates (MLEs) $\hat{\alpha}_i$, $i = 1, 2, 3$ for the unknown parameters α_i are the solution of the non-linear system (4.5). As it seems, the general solution of this system is very difficult to find in a closed form. The general solution is intractable and numerical procedures are required.

From (4.5), we obtain the MLE of α_2 as a function of α_3, namely $\hat{\alpha}_2(\alpha_3)$ where

$$\hat{\alpha}_2(\alpha_3) = \frac{-m}{\sum_{k\in L} \ln(1-e^{-\alpha_3 t_k})}. \tag{4.6}$$

Putting $\hat{\alpha}_2(\alpha_3)$ in (4.3) we obtain

$$g(\alpha_1,\alpha_3) = (n_{22}+n_{12})\ln\alpha_1 + n_{13}\ln(1-\alpha_1) + m\ln\alpha_3 - \tau\alpha_3$$
$$-m\,\ln\left(\sum_{k\in L} \text{-log}(1-e^{-\alpha_3 t_k})\right) - \sum_{k\in L}\ln(1-e^{-\alpha_3 t_k}). \tag{4.7}$$

Therefore, MLEs of both α_1 and α_3, namely $\hat{\alpha_1}$ and $\hat{\alpha_3}$ can be obtained by maximizing (4.7) with respect to α_1 and α_3 respectively. It is observed that both $\hat{\alpha_1}$ and $\hat{\alpha_3}$ can be obtained from the fixed point solution of $h_1(\alpha_1)$ and $h_2(\alpha_3)$ respectively, where

$$h_1(\alpha_1) = \left(n_{22}+n_{12}+n_{23} - \sum_{k=1}^{n_{23}} \frac{1}{1-\alpha_1 G(t_k)}\right)\left[m + \sum_{k=1}^{n_{23}} \frac{1}{1-\alpha_1 G(t_k)}\right]^{-1}, \tag{4.8}$$

$$h_2(\alpha_3) = \left[\frac{\sum_{k\in L} t_k e^{-\alpha_3 t_k}/(1-e^{-\alpha_3 t_k})}{\sum_{k\in L}\ln(1-e^{-\alpha_3 t_k})} + \frac{1}{m}\sum_{k\in L}\frac{t_k}{1-e^{-\alpha_3 t_k}}\right]^{-1}, \tag{4.9}$$

where, the function $G(t_k)$ can be considered as a known function of the observation data z.

An iterative procedure can be used to solve the Eqs (4.8) and (4.9) . The MLEs $\hat{\alpha}_1, \hat{\alpha}_3$ can be obtained from (4.8) and (4.9).

Now, we state the asymptotic normality results to obtain the asymptotic variance of the unknown parameters Gupta and Kundu (2001).

The elements of the Fisher information matrix are as follows, for $\alpha_2 > 2$

$$E\left(\frac{\partial^2 \mathcal{L}}{\partial \alpha_1^2}\right) = -\frac{n_{22}+n_{12}}{\alpha_1^2} - \frac{n_{13}}{(1-\alpha_1)^2} + \sum_{k=1}^{n_{23}} \frac{[G(t_k)]^2}{[1-\alpha_1 G(t_k)]^2}$$

$$E\left(\frac{\partial^2 \mathcal{L}}{\partial \alpha_2^2}\right) = -\frac{m}{\alpha_2^2},\ E\left(\frac{\partial^2 \mathcal{L}}{\partial \alpha_3^2}\right) = -\frac{m\alpha_2}{\alpha_3^2}\left\{\acute{\Psi}(1) - \Psi(\alpha_2) + [\Psi(\alpha_2) - \Psi(1)]^2\right\}$$

$$E\left(\frac{\partial^2 \mathcal{L}}{\partial \alpha_2 \partial \alpha_3}\right) = \frac{m}{\alpha_3}\left[\frac{\alpha_3}{\alpha_2 - 1}(\Psi(\alpha_2) - \Psi(1)) - (\Psi(\alpha_2+1) - \Psi(1))\right] \tag{4.10}$$

and for $0 < \alpha_2 \leq 2$, we have

$$E\left(\frac{\partial^2 \mathcal{L}}{\partial \alpha_2^2}\right) = -\frac{m}{\alpha_2^2},\ E\left(\frac{\partial^2 \mathcal{L}}{\partial \alpha_2 \partial \alpha_3}\right) = \frac{m\alpha_2}{\alpha_3}\int_0^\infty te^{-2t}(1-e^{-t})^{\alpha_2-2}dt < \infty,$$

$$E\left(\frac{\partial^2 \mathcal{L}}{\partial \alpha_3^2}\right) = \frac{m}{\alpha_3^2} - \frac{m\alpha_2(\alpha_2-1)}{\alpha_3^2}\int_0^\infty t^2 e^{-2t}(1-e^{-t})^{\alpha_2-3}dt < \infty. \tag{4.11}$$

In what follows we will study some important special cases:
In order to obtain the first special case, the following assumptions are needed:

1. The distribution of the time lengths of the repair periods of the units satisfy the condition: $1 - \alpha_1 G(t_k) = 1 - \alpha_1$ for every $k \in A_{23}$.
2. The lifetimes of the active units can be represented by identically generalized exponential random variables with one parameter α_2. That is, $\alpha_3 = 1$.

   In this case, the MLEs are given by:

$$\hat{\alpha}_1 = \frac{n_{22}+n_{12}}{m},\ \hat{\alpha}_2 = \frac{-m}{\sum_{k\in L}\ln(1-e^{-t_k})}. \tag{4.12}$$

The second special case can be obtained by considering the following assumptions:

1. The distribution of the time lengths of the repair periods of the units satisfy the condition: $1 - \alpha_1 G(t_k) = 1 - \alpha_1$ for every $k \in A_{23}$.
2. The lifetimes of the active units can be represented by identically exponential random variables with parameter α_3. That is, $\alpha_2 = 1$

   In this case, the MLEs are given by:

$$\hat{\alpha}_1 = \frac{n_{22}+n_{12}}{m},\ \hat{\alpha}_3 = \frac{\tau}{m}. \tag{4.13}$$

We assume that the value of the parameter α_2 is known and only the values of the parameters α_1 and α_3 are known. Thus $\underline{\alpha} = (\alpha_1, \alpha_3)$ is the vector of unknown parameters.

5. BAYESIAN PROCEDURE

As we have seen in the previous section, the maximum likelihood estimates of the unknown parameters have no closed forms. Therefore, we look for another approach that may enable us to derive estimations of these parameters in closed forms. As we will see, the Bayes approach gives such advantage. In order to obtain the Bayes estimate of the unknown parameters the following assumptions are needed.

1. The parameters $\alpha_i (i = 1, 2)$ behave as independent random variables.

2. The prior distribution of the parameter $\alpha_i (i = 1, 2)$ is the symmetrical triangular distribution on the interval $[a_i, b_i]$. That is, the pdf's of α_i is given by

$$g_i(\alpha_i) = \begin{cases} \frac{1}{\epsilon_i^2}(\epsilon_i - |\alpha_i - \nu_i|) & \text{for } \alpha_i \in [a_i, b_i] \quad (i = 1, 2) \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

3. The loss incurred when α_1 and α_2 are estimated respectively by $\hat{\alpha}_1$ and $\hat{\alpha}_2$ is quadratic. That is,

$$l(\underline{\alpha}, \underline{\hat{\alpha}}) = k_1 \ (\alpha_1 - \hat{\alpha}_1)^2 + k_2 \ (\alpha_2 - \hat{\alpha}_2)^2, \ k_1, k_2 > 0. \tag{5.2}$$

Using the assumptions 1 and 2, the joint prior pdf of $\underline{\alpha}$, namely $g(\underline{\alpha})$ takes the form

$$g(\underline{\alpha}) = \begin{cases} \prod_i^2 \frac{1}{\epsilon_i^2}(\epsilon_i - |\alpha_i - \nu_i|) & \text{for } \alpha_i \in [a_i, b_i], \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

To establish a theorem about the joint posterior pdf of $\underline{\alpha}$, the following lemmas are needed.

Lemma 5.1 *Let $I_{a,b}(n, U)$ be defined by the following integral*

$$I_{a,b}(n, U) = \int_a^b z^n U^z dz, \ n = 0, 1, 2, \ldots, \tag{5.4}$$

Then

$$I_{a,b}(n, U) = \sum_{k=0}^{n} \frac{(-1)^k \ n!}{(n-k)!(\ln U)^{k+1}} \left[b^{n-k} U^b - a^{n-k} U^a \right]. \tag{5.5}$$

Lemma 5.2 *Let $I_{a,b}(n,m)$ be defined by the following integral*

$$I_{a,b}(n,m)=\int_a^b z^n(1-z)^m dz,\quad m,n=0,1,2,\dots, \tag{5.6}$$

Then

$$I_{a,b}(n,m)=\sum_{k=0}^{n}\frac{n!}{(m)_k}\left[a^{n-k}(1-a)^{m+k+1}-b^{n-k}(1-b)^{m+k+1}\right] \tag{5.7}$$

where $(m)_k=m(m+1)(m+2)\dots(m+k-1)$.

Now, we proceed to present the theorem that gives the joint posterior pdf of $\underline{\alpha}$ for the given observation data z.

Theorem 5.1 *Using the given observation data z and the assumptions 1 and 2 the joint posterior pdf of $\underline{\alpha}$ is given by*

$$g(\underline{\alpha}|z)=\frac{1}{\Phi(\underline{0})}\left\{\prod_{i=1}^{2}(\epsilon_i-|\alpha_i-\nu_i|)\alpha_i^{(n_{22}+n_{12})\delta_{1i}+m\delta_{2i}}(1-\alpha_i)^{n_{13}}U^{(\alpha_2-1)\delta_{2i}}\right\}$$
$$\times\prod_{k=1}^{n_{23}}[1-\alpha_1 G(t_k)] \tag{5.8}$$

where $\Phi(\underline{0})$ is given by

$$\Phi(\underline{0})=\sum_{l=0}^{n_{23}}(-1)^l\tau_l I_{1l}(\underline{0})I_{2l}(\underline{0}) \tag{5.9}$$

and

$$I_{1l}(\underline{0})=I_{a_1,\nu_1}(n_{22}+n_{12}+l+1,n_{13})-a_1I_{a_1,\nu_1}(n_{22}+n_{12}+l,n_{13})$$
$$+b_1I_{\nu_1,b_1}(n_{22}+n_{12}+l,n_{13})-I_{\nu_1,b_1}(n_{22}+n_{12}+l+1,n_{13})$$

$$I_2(\underline{0})=I_{a_2,\nu_2}(m+1,U)-a_2I_{a_2,\nu_2}(m,U)+b_2I_{\nu_2,b_2}(m,U)-I_{\nu_2,b_2}(m+1,U). \tag{5.10}$$

Proof. The joint posterior of $\underline{\alpha}=(\alpha_1,\alpha_2)$ is related to the joint prior pdf of $\underline{\alpha}$ and the sample likelihood function according to the following relation Martz and Waller (1982).

$$g(\underline{\alpha}|z)=\frac{g(\underline{\alpha})L(z;\underline{\alpha})}{\Phi(\underline{0})} \tag{5.11}$$

where:

$$\Phi(\underline{0})=\int_{-\infty}^{+\infty}g(\underline{\alpha})L(z,\underline{\alpha})d\underline{\alpha}.$$

Since α_1 and α_2 are independent, then the joint prior of $\underline{\alpha}$ becomes $g_1(\alpha_1)g_2(\alpha_2)$. Then substituting from (4.2) and (5.1) into (5.11) we can obtain the numerator of (5.11) and the denominator $\Phi(\underline{0})$ can be obtained as follows

$$\Phi(\underline{0}) = \sum_{l=0}^{n_{23}}(-1)^l \tau_l \prod_{i=1}^{2} \int_{a_i}^{b_i} \Big\{(\epsilon_i - |\alpha_i - \nu_i|)\alpha_i^{(n_{22}+n_{12}+l)\delta_{1i}+m\delta_{2i}} \times (1-\alpha_i)^{n_{13}\delta_{1i}} U^{(\alpha_i-1)\delta_{2i}} d\alpha_i. \quad (5.12)$$

If

$$I_{1l}(\underline{0}) = \int_{a_1}^{b_1} (\epsilon_1 - |\alpha_1 - \nu_1|)\alpha_1^{(n_{22}+n_{12}+l)}(1-\alpha_1)^{n_{13}} d\alpha_1 \quad (5.13)$$

and

$$I_2(\underline{0}) = \int_{a_2}^{b_2} (\epsilon_2 - |\alpha_2 - \nu_2|)\alpha_2^m U^{\alpha_1 - 1} d\alpha_2, \quad (5.14)$$

then

$$I_{1l}(\underline{0}) = \int_{a_1}^{\nu_1} (\alpha_1 - a_1)\alpha_1^{(n_{22}+n_{12}+l)}(1-\alpha_1)^{n_{13}} d\alpha_1 + \int_{\nu_1}^{b_1} (b_1 - \alpha_1)\alpha_1^{(n_{22}+n_{12}+l)}(1-\alpha_1)^{n_{13}} d\alpha_1. \quad (5.15)$$

Using (5.5), we get

$$I_{1l}(\underline{0}) = I_{a_1,\nu_1}(n_{22}+n_{12}+l+1, n_{13}) - a_1 I_{a_1,\nu_1}(n_{22}+n_{12}+l, n_{13}) + b_1 I_{\nu_1,b_1}(n_{22}+n_{12}+l, n_{13}) - I_{\nu_1,b_1}(n_{22}+n_{12}+l+1, n_{13})$$

and using (5.7), we get

$$I_2(\underline{0}) = I_{a_2,\nu_2}(m+1, U) - a_2 I_{a_2,\nu_2}(m, U) + b_2 I_{\nu_2,b_2}(m, U) - I_{\nu_2,b_2}(m+1, U). \quad (5.16)$$

Using (5.12), (5.15) and (5.16), we can write $\Phi(\underline{0})$ as given in (5.9) which completes the proof.

Corollary 5.3 *The marginal posterior pdf of $\alpha_l (l = 1, 2)$ can be obtained from the joint posterior pdf of $\underline{\alpha} = (\alpha_1, \alpha_2)$ using the following relations*

$$g_1(\alpha_1|z) = \int_{a_2}^{b_2} g(\underline{\alpha}|z) d\alpha_2, \textit{ and } \quad g_1(\alpha_2|z) = \int_{a_1}^{b_1} g(\underline{\alpha}|z) d\alpha_1. \quad (5.17)$$

The proof of this corollary can be reached by substituting from (5.8) into (5.17) and making some arrangements.

Theorem 5.2 *Under the assumptions $1 - 3$, we have*

1. *The Bayes estimators for* α_1 *and* α_2 *are*

$$\hat{\alpha}_l = \frac{\Phi(\delta_{l1}, \delta_{l2})}{\Phi(\underline{0})}, \quad l = 1, 2 \tag{5.18}$$

2. *The minimum posterior risks associated with* $\hat{\alpha}_1$ *and* $\hat{\alpha}_2$ *are*

$$Var(\alpha_l|z) = \frac{\Phi(2\delta_{1l}, 2\delta_{2l})}{\Phi(\underline{0})} - \left\{\frac{\Phi(\delta_{1l}, \delta_{2l})}{\Phi(\underline{0})}, \quad l = 1, 2\right\}^2 \tag{5.19}$$

where

$$\Phi(p_1, p_2) = \sum_{l=0}^{n_{23}} (-1)^l \tau_l I_{1l}(p) I_2(p) \tag{5.20}$$

and $I_{1l}(p), I_2(p)$ *are given by*

$$I_{1l}(p) = I_{a_1,\nu_1}(n_{22} + n_{12} + l + p + 1, n_{13}) - a_1 I_{a_1,\nu_1}(n_{22} + n_{12} + l + p, n_{13})$$
$$+ b_1 I_{\nu_1,b_1}(n_{22} + n_{12} + l + p, n_{13}) - I_{\nu_1,b_1}(n_{22} + n_{12} + l + +p1, n_{13})$$

and

$$I_2(\underline{0}) = I_{a_2,\nu_2}(m+p+1, U) - a_2 I_{a_2,\nu_2}(m+p, U) + b_2 I_{\nu_2,b_2}(m+p, U) - I_{\nu_2,b_2}(m+p+1, U). \tag{5.21}$$

Proof. If the squared error loss function is used, then the Bayes estimator of the unknown parameter $\alpha_i (i = 1, 2)$ and associated minimum posterior risk are defined as the posterior expectation and posterior variance of that parameter respectively Igor (1994). That is, the Bayes estimator for α_i is

$$\hat{\alpha}_i = \int_{a_i}^{b_i} \alpha_i g_i(\alpha_i|z) d\alpha_i, \ i = 1, 2 \tag{5.22}$$

and the minimum posterior risk associated with $\hat{\alpha}_i$ is

$$\text{Var}(\alpha_i|z) = \int_{a_i}^{b_i} \alpha_i^2 g_i(\alpha_i|z) d\alpha_i - \left\{\int_{a_i}^{b_i} \alpha_i g_i(\alpha_i|z) d\alpha_i\right\}^2, \ i = 1, 2 \tag{5.23}$$

Substituting from (5.17) into (5.22) and (5.23) and making some arrangements, we can reach the formula (5.18) and (5.19), which completes the proof.

6. SYSTEM RELIABILITY

In what follows, we will obtain the system reliability of the semi-Markov reliability model. Now, we will define the first passage time. To define the first passage time, we must answer the question "how many transitions will the process take to reach state j for the first time if the system is in state i at time zero". The first passage time of the continuous-time semi-Markov process can be measured in time or in terms of the number of transitions. We will obtain the distribution $\Theta_{iA}(t)$ of the first passage time from the state i to a state in a subset $A \subset S$ given that state i was entered at time zero and zeroth transition.

Assuming that $A \subset S = \{1,2,3\}$ and $\bar{A} = S - A$, we introduce the following notations

$$\Delta_A = \inf\{n \in: X(\tau_n) \in A\}, \tag{6.1}$$

and

$$f_{iA}(n) = P\{\Delta_A = n|X(0) = i\}, T_A = \tau_{\Delta A}. \tag{6.2}$$

Thus, the function $\Theta_{iA}(t)$ is given by

$$\Theta_{iA}(t) = P\{T_A \leq t|X(0) = i\},\ i \in \bar{A} \tag{6.3}$$

represents the distribution of the first passage time of the semi-Markov process $\{X(t) : t \geq 0\}$, from the state $i \in \bar{A}$ to state in the subset A.

Now, we will define, a mean and the second moment of the first passage time distribution as follows

$$\bar{\Theta}_{iA} = \int_0^\infty t d\Theta_{iA}(t),\ and,\quad \bar{\Theta}^2_{iA} = \int_0^\infty t^2 d\Theta_{iA}(t). \tag{6.4}$$

If A denotes the subset of the failed states of the model and $i \in \bar{A}$ is an initial operating state such that $P\{X(0) = i\} = 1$, then the random variable T_A represents the lifetime or the time to failure of the considered system. That is, the reliability of the system is

$$R(t) = 1 - \Theta_{iA}(t),\ t \geq 0. \tag{6.5}$$

Using Korolyuk and Swishchuk (1994) and Grabski (1999), some of the reliability characteristics of the system can be defined as follows:

$$\bar{q}_{ik} = \int_0^\infty t q_{ik}(t)dt,\ \bar{q}^2_{ik} = \int_0^\infty t^2 q_{ik}(t)dt. \tag{6.6}$$

To derive the reliability of the system, we will establish the following theorem.

Theorem 6.1 *Consider the following systems:*

1.

$$\Theta_{iA}(t) = \sum_{j\in A} Q_{ij}(t) + \sum_{k\in\bar{A}} \int_0^t dQ_{kA}(t-u)q_{ik}(u)du,\ i \in \bar{A} \tag{6.7}$$

2.

$$\Leftrightarrow \bar{\theta}_{iA} = \bar{g}_i + \sum_{k \in \bar{A}} p_{ik}\bar{\theta}_{ik}, \quad i \in \bar{A} \tag{6.8}$$

3.

$$\bar{\theta}_{iA} = \bar{g}_i^2 + 2\sum_{k \in \bar{A}} \bar{q}_{ik}\bar{\theta}_{kA} + \sum_{k \in \bar{A}} p_{ik}\bar{\theta}^2_{ik}, \quad i \in \bar{A}, \tag{6.9}$$

which consist of a system of integral equations (6.7) and two linear algebraic systems of equations (6.8) and (6.9). These systems have the only solution $\Theta_{iA}(t), \bar{\theta}_{iA}$ *and* $\bar{\theta}^2_{iA}$ *respectively, if the following conditions are satisfied*

1.

$$f_{iA} = 1 \ \forall \ i \in \bar{A} \tag{6.10}$$

2.

$$\forall i, j \in S \ \exists \ d > 0 \ s.t. \ \bar{q}^2_{ij} < d \tag{6.11}$$

3.

$$\sum_{k=1}^{\infty} k^2 f_{iA} < \infty \ \forall, \ \ i \in \bar{A} \tag{6.12}$$

Proof. The system of integral equations (6.7) is equivalent to its Laplace-Stieltjes system

$$\tilde{\Theta}_{iA}(s) = \sum_{j \in A} \tilde{q}_{ij}(s) + \sum_{k \in \bar{A}} \tilde{q}_{ik}(s)\tilde{\Theta}_{kA}(s), \ i \in \bar{A} \tag{6.13}$$

where

$$\tilde{\Theta}_{iA}(s) = \int_0^{\infty} e^{-st}\frac{d\Theta_{iA}(t)}{dt}dt, \ \tilde{q}_{ij}(s) = \int_0^{\infty} e^{-st} q_{ij}(t)dt. \tag{6.14}$$

In the present model $A = \{3\}$ and $\bar{A} = \{1, 2\}$. From the solution of the system (6.14), we have

$$\tilde{\Theta}_{23}(s) = \frac{\tilde{q}_{23}(s)}{1 - \tilde{q}_{22}}, \quad \tilde{\Theta}_{13}(s) = \tilde{q}_{13}(s) + \frac{\tilde{q}_{12}\tilde{q}_{23}}{1 - \tilde{q}_{22}}. \tag{6.15}$$

Using the Laplace transformation, the reliability function (6.5) of the present model is given by

$$\tilde{R}(s) = \frac{1 - \tilde{\Theta}_{13}(s)}{s}. \tag{6.16}$$

From the system of equations (6.8), we can get

$$\bar{\theta}_{13} = \bar{g}_1 + \frac{p_{12}\bar{g}_2}{1 - p_{22}}. \tag{6.17}$$

For the present model we have:

$$\bar{g}_1 = \bar{g}_2 = E(\xi_1) = \int_0^\infty \alpha_2\alpha_3\, te^{-\alpha_3 t}(1 - e^{-\alpha_3 t})^{\alpha_2 - 1}dt$$
$$= \frac{1}{\alpha_3}\{\Psi(\alpha_2 + 1) - \Psi(1)\} \tag{6.18}$$

where $\Psi(.)$ denotes the digamma function.

$$P_{21} = \int_0^\infty q_{21}(t)dt = \alpha_1.$$

Substituting (6.17) into (6.16) we can obtain the mean of the lifetime of the present system as follows:

$$E(T_A|X(0) = 1) = \bar{\theta}_{13} = \frac{1}{\alpha_3}\{\Psi(\alpha_2 + 1) - \Psi(1)\}\{1 + \frac{\alpha_1}{1 - p_{22}}\} \tag{6.19}$$

where

$$p_{22} = \alpha_1\alpha_2\alpha_3 \int_0^\infty G(t)(1 - e^{-\alpha_3 t})\, e^{-\alpha_3 t}\, dt. \tag{6.20}$$

Important special case can be obtained when $\alpha_2 = 1$, the lifetimes of the active units can be represented by identically exponential random variables with parameter α_3. Therefore, the mean of the lifetime of the system is given by

$$E(T_A|X(0) = 1) = \bar{\theta}_{13} = \frac{1}{\alpha_3} + \frac{\alpha_1}{\alpha_3(1 - p_{22})}, \quad p_{22} = \alpha_1\alpha_3 \int_0^\infty G(t)e^{-\alpha_3 t}. \tag{6.21}$$

This result approves of the result obtained by Grabski (1999) and El-Gohary (2004). This shows the effectiveness of the present method.

7. CONCLUSION

Finally, we conclude that the likelihood and Bayes procedures are used to obtain estimations of the parameters included in a three-state standby with repair semi-Markov model. The distribution of the first passage time is discussed. The reliability function of this model is derived. Some special cases are discussed.

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