MacCormack 방법의 개량에 대한 연구

하영수* · 유승재**

요 약

MacCormack 방법은 hyperbolic 편미분 방정식의 근을 구하는데 많이 쓰이는 방법으로 그 정확도가 2차 오더가 된다. 하지만 이 방법으로 편미분방정식을 풀 경우 불연속인 점에서는 엔트로피를 만족하지 않는 경우가 있어 우리는 임의의 항을 첨가하여 근을 구해야한다. 이 임의의 항을 첨가하지 않고 직접 방정식으로부터 구하는 방법을 생각하는데 있어서 기존의 MacCormack 방법에서 central scheme의 개념을 이용하면 전형적인 MacCormack 방법의 정확도와 장점을 보존할 수 있다. 이 새로운 방법을 이용하여 1D Burgers' 방정식과 1D Euler gas dynamic 방정식에 활용하여 그 결과를 살펴본다.

Some Modifications of MacCormark's Methods

Youngsoo Ha* · Seung Jae Yoo**

ABSTRACT

MacCormack's method is an explicit, second order finite difference scheme that is widely used in the solution of hyperbolic partial differential equations. Apparently, however, it has shown entropy violations under small discontinuity. This non-physical shock grows fast and eventually all the meaningful information of the solution disappears. Some modifications of MacCormack's methods follow ideas of central schemes with an advantage of second order accuracy for space and conserve the high order accuracy for time step also. Numerical results are shown to perform well for the one-dimensional Burgers' equation and Euler equations gas dynamic.

Key words : MacCormack's Method, Burgers' Equation, Hyperbolic Partial Differential Equations, Euler Equations Gas Dynamic

* Division of Applied Mathematics, KAIST
** Dept. of Information Security, Joongbu Univ.

1. Introduction

MacCormack's method for hyperbolic partial differential equations was introduced in [8, 5] and has been widely used in aerodynamic applications. We consider the general form of systems of conservation laws. Let Ω be an open subset of $\mathbb{R}^m$, and let F be a smooth function from Ω into $\mathbb{R}^m$. Then the general form of a system of conservation laws in one dimensional space is

$$\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0, \qquad (1.1)$$

for $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$, $t > 0$ where $u = [u_1, u_2, \cdots, u_m]^T$ and $F(u) = [f_1, f_2, \cdots, f_m]^T$

Here u is the vector of conserved variables, and $F = F(u)$ is the vector of fluxes and each of its components f_i is a function of u.

An equation of the form (1.1) is written in conservation form and is called a set of conservation laws. We shall be concerned with numerical solutions of hyperbolic conservation laws (1.1).

MacCormack's method, originally introduced in [8] :

$$\begin{aligned} u_i^* &= u_i^n - \frac{\triangle t}{\triangle x}[f(u_{i+1}^n) - f(u_i^n)] \\ u_i^{**} &= u_i^* - \frac{\triangle t}{\triangle x}[f(u_i^*) - f(u_{i-1}^*)] \qquad (1.2) \\ u_i^{n+1} &= \frac{1}{2}(u_i^n + u_i^{**}) \end{aligned}$$

This scheme is second-order accurate for space and time without using Jacobian matrices or Riemann solvers but it produces spurious oscillations unless artificial viscosity is explicitly added.

2. Modifications of MacCormack's method

Before deriving the modifications of MacCormack's method we overview the predictor-corrector form of the fully discrete methods of central schemes [2, 3, 6];

$$u_i^{n+\frac{1}{2}} = u_i^n - \frac{1}{2}\frac{\triangle t}{\triangle x} f_i' \qquad (2.1)$$

$$\begin{aligned} u_{i+\frac{1}{2}}^{n+1} &= \frac{1}{2}[u_i^n + u_{i+1}^n] + \frac{1}{8}[u_i' - u_{i+1}'] \\ &- \frac{\triangle t}{\triangle x}[f(u_{i+1}^{n+\frac{1}{2}}) - f(u_i^{n+\frac{1}{2}})] \end{aligned} \qquad (2.2)$$

where the numerical flux derivatives

$$\frac{1}{\Delta x} f_i' = f(u)_x|_{u=u(x_i)} + O(\Delta x)$$

and

$$\frac{1}{\Delta x} u_i' = u_x|_{u=u(x_i)} + O(\Delta x).$$

The appropriate choice of approximate derivatives guarantees that equation (2.2) is TVD in the scalar case. For more detail, see [6].

For instance, one way to ensure TVD stability for equation (2.2) is

$$\begin{aligned} (u_x)_i^n &= minmod \\ &(\alpha \Delta u_{i-\frac{1}{2}}, \frac{1}{2}(u_{i+1} - u_i), \alpha \Delta u_{i+\frac{1}{2}}) \end{aligned} \qquad (2.3)$$

and

$$\begin{aligned} (f)_i' &= minmod \\ &(\alpha \Delta f_{i-\frac{1}{2}}, \frac{1}{2}(f_{i+1} - f_i), \alpha \Delta f_{i+\frac{1}{2}}) \end{aligned} \qquad (2.4)$$

where $\alpha \in [1, 2]$ and $\Delta u_{i+1/2} = u_{i+1} - u_i$ and

$$\min{}_{-}B(x_1, x_2, \cdots) = \begin{cases} \min_j\{x_j\} & \text{if } x_j > 0 \text{ for } all\ j \\ \max_j\{x_j\} & \text{if } x_j < 0 \text{ for } all\ j \\ 0 & otherwise \end{cases} \tag{2.5}$$

We apply this scheme to MacCormack's method (1.2) then we have the following formula:

$$\begin{aligned} u_i^* &= \frac{1}{2}(u_i^n + u_{i+1}^n) + \frac{1}{8}(u_i' - u_{i+1}') \\ &\quad - \frac{\triangle t}{\triangle x}[f(u_{i+1}^n) - f(u_i^n)] \\ u_i^{**} &= \frac{1}{2}(u_{i-1}^* + u_i^*) + \frac{1}{8}(u_{i-1}^{*\prime} - u_i^{*\prime}) \\ &\quad - \frac{\triangle t}{\triangle x}[f(u_i^*) - f(u_{i-1}^*)] \\ u_i^{n+1} &= \frac{1}{2}(u_i^n + u_i^{**}) \end{aligned} \tag{2.6}$$

with (2.3) and equation (2.4).

3. Numerical results

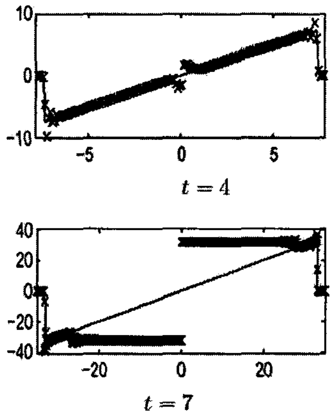
In this section we present the results of our numerical experiments for the 1D Burgers' equation with uniform grids. Also we compute the Euler equations and compare the results with exact solutions.

Example 1.(Burgers' equation) We consider the entropy of a scalar conservation law with a linear source term [1].

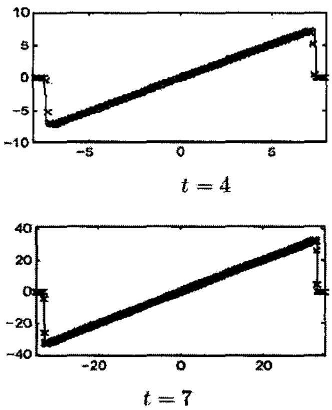
$$u_t + f(u)_x = u, \tag{3.1}$$

for $u(x,0) = u_0(x) \in L^1(\mathbb{R})$, and $x, u \in \mathbb{R}$, $t > 0$, where the flux is given by the convex power law

$$f(u) = \frac{1}{2}u^2. \tag{3.2}$$



(Figure 1) MacCormack's Scheme (1.2), * : Numerical, - : exact, : A non-physical shock emerges from the sign-changing point and finally destroys the computed solution completely



(Figure 2) Modification of MacCormack's Scheme (2.5), * : Numerical, - : exact

Now we examine the properties of MacCormack's method from numerical experiments. In figure 1 exact and computed solutions to the reaction-convection equation (1.2) are given. In the figure at time $t=4$ one can observe a small discontinuity that violates the entropy condition. However when we apply the modified MacCormack's method (2.5) to this equation we can fix entropy violations and oscillations at the shock (Figure 2).

Example 2.(1-D Euler equations of gas dynamics) We solve the 1-D Euler system

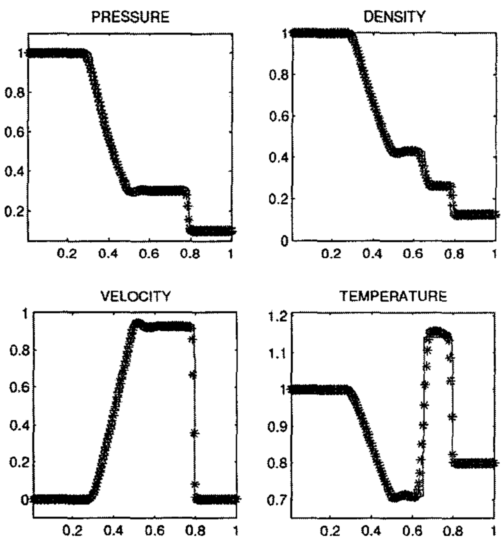
$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}(F(U)) = 0 \qquad (3.3)$$

with

$$U = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}, \quad F(U) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{pmatrix},$$

$$p = (\gamma - 1)(E - \frac{\rho u^2}{2}). \qquad (3.4)$$

with initial data $(\rho_l, u_l, p_l) = (1.0, 0.0, 1.0)$ if $x \le 0.5$, $(\rho_r, u_r, p_r) = (0.125, 0.0, 1.0)$ if $x > 0.5$ which are two sets of Riemann data proposed by Sod in [4].



(Figure3) Modification of MacCormack's Scheme (2.5), $*$: Numerical. - : exact. at $t = 0.1644$

Even though the original MacCormark's method is second-order accurate it produces spurious oscillations and blows up without adding artificial viscosity explicitly. In figure 3 we shows results of the modified MacCormark's scheme for Sod's problem, in comparison to the corresponding exact solutions at $t = 0.1644$.

4. Conclusions

In this paper, we have presented a modified MacCormack's method to solve the 1-D hyperbolic partial differential equations. To generalize this method to the nonlinear 2-D hyperbolic partial differential equations, more work is needed and this is left for future research.

References

[1] Y. Ha and Y.-J. Kim, "Explicit solu- tions of a convection-reaction equation and defects of difference schemes", preprint.

[2] G.-S. Jiang, D. Levy, C.-T. Lin, S. Osher and E. Tadmor, "High-resolution non-oscillatory central schemes with non-staggered grids for hyperbolic conservation laws", SIAM J. Numer. Anal., Vol. 35, pp. 2147-2168, 1998.

[3] A. Kurganov and E. Tadmor, "New high-resolution central schemes for nonlinear conservation laws and convection-diffusion equations", J. Comput. Phys., Vol. 160, pp. 408-463, 2000.

[4] G. Sod, "A surey of several finite difference methods for systems of nonlinear hyper-

bolic conservation laws", J. Comput. Phys., Vol. 22, pp. 1-31, 1978.
[5] R. J. LeVeque, 'Numerical methods for conservation laws', Lectures in Mathematics ETH Zürich', Birkhäuser Verlag, Basel, 1990.
[6] H. Nessyahu and E. Tadmor, "Non- oscillatory central differencing for hyperbolic conservation laws", J. Comput. Phys., Vol. 87, pp. 408-463, 1990.
[7] C.-W. Shu and S. Osher, "Efficient implementation of essentially non-oscillatory shock capturing schemes", J. Comput. Phys. Vol. 77, pp. 439-471, 1988.
[8] R. W. MacCormack, "The effects of viscosity in hypervelocity impact creating", AIAA Paper, pp. 69-354, 1969.

하 영 수
동국대학교 수학과(이학석사)
애리조나 주립대학 수학과(이학박사)
현재 KIAST 응용수학전공

유 승 재
동국대학교 수학과(이학박사)
현재 중부대학교 정보보호학과 부교수