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# A Case Study on Guiding the Mathematically Gifted Students to Investigating on the 4-Dimensional Figures

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## Abstract

Some properties on the mathematical hyper-dimensional figures by 'the principle of the permanence of equivalent forms' was investigated. It was supposed that there are 2 conjectures on the making n-dimensional figures : simplex (a pyramid type) and a hypercube(prism type). The figures which were made by the 2 conjectures all satisfied the sufficient condition to show the general Euler's Theorem(the Euler's Characteristics). Especially, the patterns on the numbers of the components of the simplex and hypercube are fitted to Binomial Theorem and Pascal's Triangle. It was also found that the prism type is a good shape to expand the Hasse's Diagram.

5 mathematically gifted high school students were mentored on the investigation of the hyper-dimensional figure by 'the principle of the permanence of equivalent forms'. Research products and ideas students have produced are shown and the 'guided re-invention method' used for mentoring are explained.

**key words** : mathematics, gifted, mentorship, guided reinvention method, dimension

# Ⅰ. Introduction

## 1. The Background of Research

Many Korean students are very interested in 4-dimensional(4-D)world. Although they would like to imagine the 4-D world, they could not experience it physically. Then, how can we guide them to experience it in mathematical world? 3-dimensional figures could be drawn on the 2-dimensional plane(especially, on the paper) by sketch, development figure, and planar graph. As the similar way, we can represent 4-dimensional figures on 2 or 3-D space by using some mathematical methods.

## 2. The Purpose of Research

This study is focused on how to guide the science school (mathematically gifted) students toward to self-oriented study. They are pre-professional researchers in the field of mathematics. So, it is important to accomplish not only the ready-made product but also the mathematical thinking, attitude and interest. The role of the teacher is to encourage and promote his students to study, on themselves, the geometrical representation of 4-dimensional figures and it's applications This study is to show a case how to guide the mathematically gifted students to investigate on the 4-dimensional figures.

## 3. Objects

The objects were all 1st grade 5 students in Gyeong-Gi Science High School. Except one of them were not attended accelerated course. Only one student was studying advanced calculus on himself.

## 4. Research Contents

The research contents are as follows.

1) To find different representations for the 4 dimensional figures and their relations.
2) To find the cases to extend and generalize the mathematical theorems and rules on the 3 dimensional figures toward to the 4 dimensional figures.
3) To suggest and to give an examples of the role of mentor.

# Ⅱ. Theories and Research Method

## 1. Theories

### 1) The principle of the permanence of equivalent forms

The principle of the permanence of equivalent forms, proposed by G. Peacock(England, 1791 1858), is a powerful guideline to generalize or expand the definitions in old number systems to new number system with maintaining the old general mathematical properties. For example, in the law of exponents, it is trivial that $a^{m+n} = a^m a^n$ for all $a \in \mathbb{Q}^+$, $m, n \in \mathbb{Z}^+$. Then, by the principle of the permanence of equivalent forms, we will get follows by defining $a^0 = 1$: $a^{x+y} = a^x a^y$ for all $a \in \mathbb{R} - \{0\}$, $x, y \in \mathbb{Q}$.

### 2) The method of guided re-invention

The method of guided re invention is proposed by H. Freudenthal(1905 1990, Netherland). The learning process has to include phases of directed invention, that is, of invention not in objective but in the subjective sense, seen from the perspective of the student. It is believed that knowledge and ability acquired by re-invention are better understood and more easily preserved than if it was acquired in a less active way. So, we have to offer into acted-out mathematics rather than as already-made product. The opposite of ready-made mathematics is mathematics in *statu nascendi*.(Freudenthal, 1973 : 115-118).

Guiding means striking a delicate balance between the force of teaching and the freedom of learning. But the learner's free choice is already restricted by the "re" of "reinvention". The learner shall invent something that is new to him but well-known to others(for example, his guide or mathematicians)(Freudenthal, 1991 : 48). The term of "guided" implies the roles of teacher.

## 2. Method

### 1) Using Tool

Zonodome System which was made in USA(Zometool, Inc.)

### 2) Process

① Face to face regular meetings(4 hours per a month for 8 months)
② Discuss and present the result on homepage : http://club.nate.com/sixpig
③ Invitation lecture meeting about planar graph
④ Tour the Exhibition of Teaching/Learning Material Tool

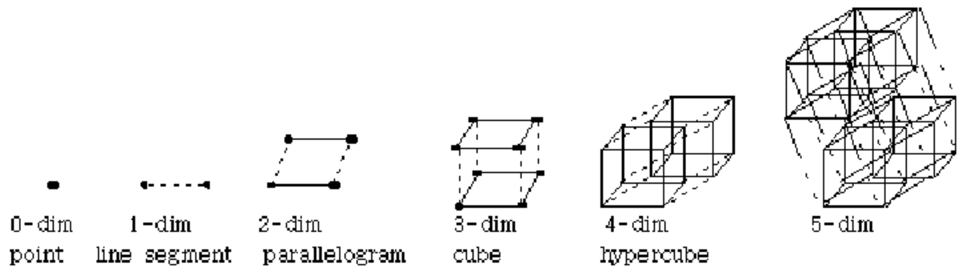
### 3) The Roles of the Mentor

Mentor is a man/woman who is guide his/her students(mentees) to investigate professional works on themselves. He/She has to enhance his/her students' inherent curiosities and encourage their desires. I guided my students to expand and generalize the given samples what I supposed for examples. They have learned how to use 'the principle of the permanence of equivalent forms'. And they have guided by the 'method of guided re-invention'.

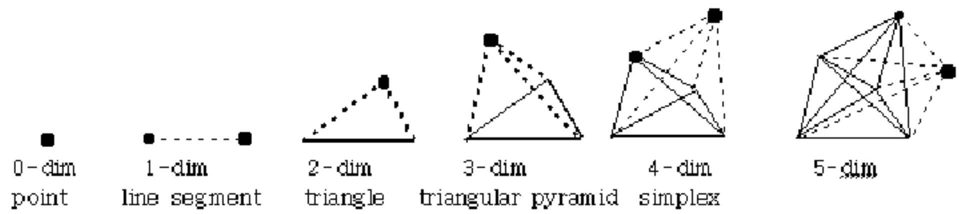
## 3. Geometrical hypotheses on 4-D figures and their verification

**Conjecture [1]:** The trajectory space which is made by translation an n-dimensional figure to another dimensional direction is a n+1 dimensional figure.(prism type)

**Conjecture [2]:** The space which is made by connecting with lines between all vertexes of an n-dimensional figure and a point out of the n-dimensional space(which is containing the given n-dimensional figure) is a n+1 dimensional figure.(pyramid type)




[Fig. 1] prism type based on conjecture [1]




[Fig. 2] pyramid type based on conjecture [2]

Now we can verify that it is reasonable to apply the two conjectures to the original mathematical theorems by the principle of the permanence of equivalent forms.

**Verification 1. Proof with the Euler's Characteristics**

We already know the Euler's theorem V-E+F=2 is satisfied for every polyhedrons, where each V, E, F is the numbers of vertexes, edges, and faces of the given polyhedrons. But if we consider the number of 3-dim figure in polyhedrons(we call this S, S=1), then the formula will be changed to V-E+F-S=1. And, now, we can expand the Euler's theorem to the beyond of the 3-dimension

**[Euler's Characteristics]** Let ${}_n\alpha_k$ is the number of $k$-dim figures in $n$-dim convex figure. Then the following formula is satisfied for all natural number $k(<=n)$

$$ {}_n\alpha_0 - {}_n\alpha_1 + {}_n\alpha_2 - {}_n\alpha_3 + \cdots\cdots + (-1)^n\, {}_n\alpha_n = \sum_{k=0}^{n} (-1)^k\, {}_n\alpha_k = 1 $$

For example, in 4-D space, we can get ${}_4\alpha_k$s from ${}_3\alpha_k$s by two conjectures, [1] and [2]. In prism type based by conjecture [1], we can get the following.

$$ {}_4\alpha_0 = {}_3\alpha_0 \times 2, \quad {}_4\alpha_1 = {}_3\alpha_1 \times 2 + {}_3\alpha_0, \quad {}_4\alpha_2 = {}_3\alpha_2 \times 2 + {}_3\alpha_1, \quad {}_4\alpha_3 = {}_3\alpha_3 \times 2 + {}_3\alpha_2. $$

Since, ${}_3\alpha_0 - {}_3\alpha_1 + {}_3\alpha_2 - {}_3\alpha_3 = 1$ by the Euler's theorem in 3-D space, it is easy to get the following.

$$ \begin{aligned} & {}_4\alpha_0 - {}_4\alpha_1 + {}_4\alpha_2 - {}_4\alpha_3 + {}_4\alpha_4 \\ = & {}_3\alpha_0 \times 2 - ({}_3\alpha_1 \times 2 + {}_3\alpha_0) + ({}_3\alpha_2 \times 2 + {}_3\alpha_1) - ({}_3\alpha_3 \times 2 + {}_3\alpha_2) + 1 \\ = & {}_3\alpha_0 - {}_3\alpha_1 + {}_3\alpha_2 - {}_3\alpha_3 - {}_3\alpha_3 + 1 = 1 - 1 + 1 = 1 \end{aligned} $$

Similarly, in pyramid type based by conjecture [2], we can get the following.

$$ {}_4\alpha_0 = {}_3\alpha_0 + 1, \quad {}_4\alpha_1 = {}_3\alpha_1 + {}_3\alpha_0 \quad {}_4\alpha_2 = {}_3\alpha_2 + {}_3\alpha_1, \quad {}_4\alpha_3 = {}_3\alpha_3 + {}_3\alpha_2. $$

Since, ${}_3\alpha_0 - {}_3\alpha_1 + {}_3\alpha_2 - {}_3\alpha_3 = 1$, it is easy to get following.

$$ {}_4\alpha_0 - {}_4\alpha_1 + {}_4\alpha_2 - {}_4\alpha_3 + {}_4\alpha_4 = 1. $$

We can get several examples which are shown as follows.

<Table 1> Euler's Characteristics on several convex 4-D Figures

| Kinds of Figures \ the number of the components | | 0-dim Vertexes ($a_0$) | 1-dim Edges ($a_1$) | 2-dim Faces ($a_2$) | 3-dim Solids ($a_3$) | 4-dim ($a_4$) | Euler's Char. |
|---|---|---|---|---|---|---|---|
| regular polytopes | 5-cell(regular simplex) | 5 | 10 | 10 | 5 | 1 | 1 |
| | 8-cell(hypercube) | 16 | 32 | 24 | 8 | 1 | 1 |
| | 16-cell | 8 | 24 | 32 | 16 | 1 | 1 |
| | 24-cell | 24 | 96 | 96 | 24 | 1 | 1 |
| | 120-cell | 600 | 120 | 720 | 120 | 1 | 1 |
| | 600-cell | 120 | 720 | 1200 | 600 | 1 | 1 |
| deformations | hexagonal prism | 7 | 12 | 7 | 1 | | 1 |
| | deformation by Conj[1] | 14 | 31 | 26 | 9 | 1 | 1 |
| | deformation by Conj[2] | 17 | 30 | 34 | 11 | 1 | 1 |
| | octagonal prism | 16 | 24 | 10 | 1 | | 1 |
| | deformation by Conj[1] | 32 | 64 | 44 | 12 | 1 | 1 |
| | deformation by Conj[2] | 17 | 30 | 34 | 11 | 1 | 1 |
| | truncated icosahedron | 60 | 90 | 32 | 1 | | 1 |
| | deformation by Conj[1] | 120 | 240 | 154 | 34 | 1 | 1 |
| | deformation by Conj[2] | 17 | 30 | 34 | 11 | 1 | 1 |

**Verification 2. Proof with the Binomial Theorem**

As you see in <Table 2>, in the expansion of $(x+2)^n$, the coefficients of $x^k$ are ${}_nC_k \cdot 2^{n-k}$, so we get the formula (1).

$$(x+2)^n = {}_nC_0x^0 \cdot 2^n + {}_nC_1x^1 \cdot 2^{n-1} + {}_nC_2x^2 \cdot 2^{n-2} + \cdots\cdots + {}_nC_{n-1}x^{n-1} \cdot 2^1 + {}_nC_nx^n \cdot 2^0 \ \cdots\cdots (1)$$

If we substitute $x=-1$ in (1), we get (2) (where, ${}_na_k$ is the number of k-dim figures in n-dim figures).

$$1 = {}_nC_0 \cdot 2^n - {}_nC_1 \cdot 2^{n-1} + {}_nC_2 \cdot 2^{n-2} - \cdots\cdots + (-1)^n\, {}_nC \cdot 2^0 = {}_na_0 - {}_na_1 + {}_na_2 - \cdots\cdots + (-1)^n\, {}_na_n$$
$$= \sum_{k=0}^{n} (-1)^k\, {}_na_k \quad \cdots\cdots (2)$$

<Table 2> the numbers of $k$-dim figures in $n$-dim figure(prism type)

| the number of the components / Kinds of Figures | 0-dim Vertexes ($\alpha_0$) | 1-dim Edges ($\alpha_1$) | 2-dim Faces ($\alpha_2$) | 3-dim Solids ($\alpha_3$) | 4-dim Figures ($\alpha_4$) | … | $n-1$dim Figures ($\alpha_{n-1}$) | $n$-dim Figures ($\alpha_n$) |
|---|---|---|---|---|---|---|---|---|
| 0-dim(point) | 1 | | | | | | | |
| 1-dim (line segment) | 2 | 1 | | | | | | |
| 2-dim (parallelogram | 4 | 4 | 1 | | | | | |
| 3-dim(quadrangular prism) | 8 | 12 | 6 | 1 | | | | |
| 4-dim(hypercube) | 16 | 32 | 24 | 8 | 1 | | | |
| ⋮ | | | | | | | | |
| $n$-dim | $2^n$ | $n \cdot 2^{n-1}$ | | | | | $2n$ | 1 |

**Verification 3. Proof with the Pascal's Triangle**

<Table 3> the numbers of $k$-dim figures in $n$-dim figure(pyramid type)

| the number of the components / Kinds of Figures | 0-dim Vertexes ($\alpha_0$) | 1-dim Edges ($\alpha_1$) | 2-dim Faces ($\alpha_2$) | 3-dim pyramid ($\alpha_3$) | 4-dim Figures ($\alpha_4$) | … | $n-1$dim Figures ($\alpha_{n-1}$) | $n$-dim Figures ($\alpha_n$) |
|---|---|---|---|---|---|---|---|---|
| 0-dim(point) | 1 | | | | | | | |
| 1-dim(line segment) | 2 | 1 | | | | | | |
| 2-dim (triangle) | 3 | 3 | 1 | | | | | |
| 3-dim(triangular pyramid) | 4 | 6 | 4 | 1 | | | | |
| 4-dim(simplex) | 5 | 10 | 10 | 5 | 1 | | | |
| ⋮ | | | | | | | | |
| $n$-dim | ${}_{n+1}C_1$ | ${}_{n+1}C_2$ | ${}_{n+1}C_3$ | | | | ${}_{n+1}C_n$ | ${}_{n+1}C_{n+1}$ |

As you see in <Table 3>, in the expansion of $(x+1)^{n+1}$, the coefficients of $x^{k+1}$ are ${}_{n+1}C_{k+1}$, so as follows:

$$(x+1)^{n+1} = {}_{n+1}C_0 + {}_{n+1}C_1 \cdot x + {}_{n+1}C_2 \cdot x^2 + \cdots\cdots + {}_{n+1}C_{n+1} \cdot x^{n+1} \quad \cdots\cdots (3)$$

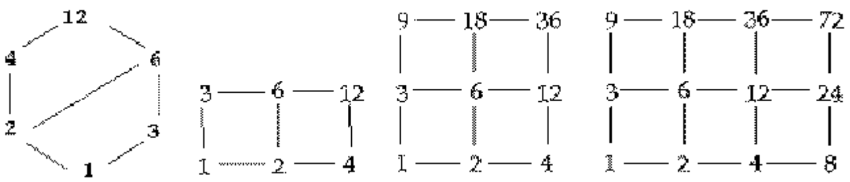
Substitute $x=-1$ in (3), we get (4) (where, ${}_n\alpha_k$ is the number of k-dim figures in n-dim fig).

$${}_{n+1}C_1 - {}_{n+1}C_2 + \cdots\cdots + (-1)^n\,{}_{n+1}C_{n+1} = {}_n a_0 - {}_n a_1 + {}_n a_2 - \cdots\cdots + (-1)^n\,{}_n a_n = 1 \quad \cdots\cdots \text{(4)}$$

Especially, <Table 3> is similar with Pascal's Triangle.

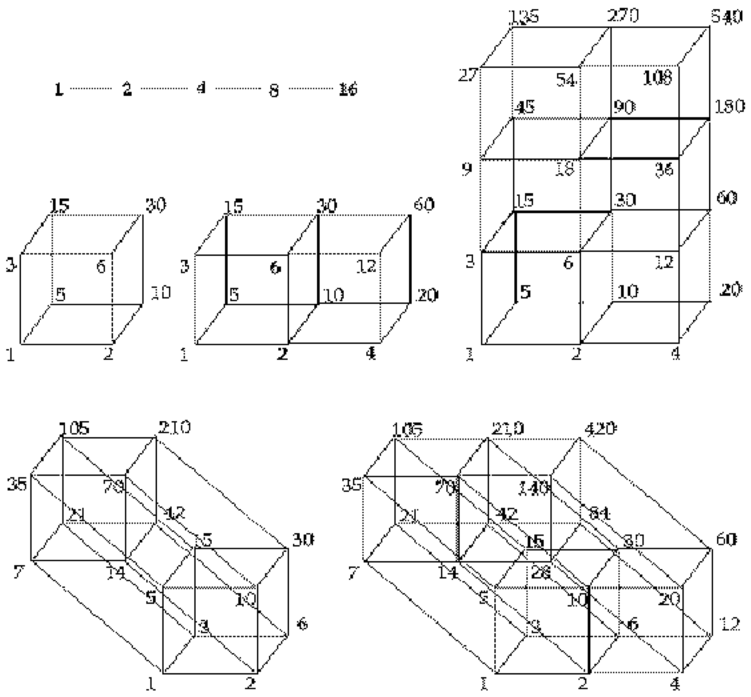
**Verification 4. Proof with the Hasse's Diagram**

Let's try to make lines for all divisors of any given natural number if two divisors are related to divisor and multiple, each other. Then as we see in [Fig. 3], as running to the right the numbers are doubled, and as running to the upwards the numbers are tripled. And it is very easy to find greatest common divisor(G.C.D.) and least common multiple(L.C.M.) of any two divisors, for example, G.C.D(18, 24)=6, L.C.M(18, 24)=72.




[Fig. 3] Examples of Hasse's Diagram(1)

If a natural number has three prime numbers in it's factorization, the Hasse's Diagram will be cubic form. As the same way, we can expand the Hasse's Diagram according to the factorization of any natural numbers $A = p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots\cdots p_k^{n_k}$. We can see some examples( $16 = 2^4$, $540 = 2^2 \times 3^3 \times 5$ and $420 = 2^2 \times 3 \times 5 \times 7$ ) in [Fig. 4].



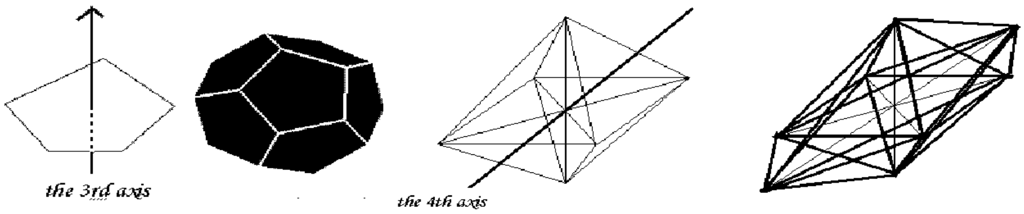

[Fig. 4] Examples of Hasse's Diagram(2)

## Ⅲ. The Results

My students(mentees) were guided to investigate on 4-D figures. And they were encouraged to re-find mathematically meaningful properties by using 'the principle of the permanence of equivalent forms'. 'The principle of the permanence of equivalent forms' was very useful to guide students in a stranger mathematical situation or a new context. Some examples of the products are as follow.

### 1. Imaging several 4-D structures and Representing them on paper or 3-D space

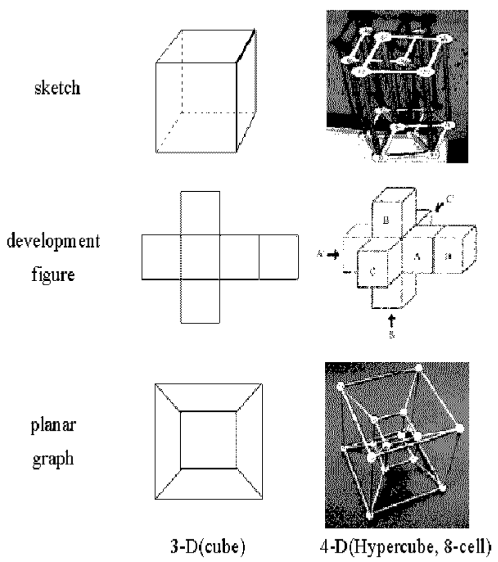
A student explained that it is possible to construct 4-D structures if he could build 3-D structures along the 4th axis as he could attach 2-D structures(polygons) along the 3rd axis.



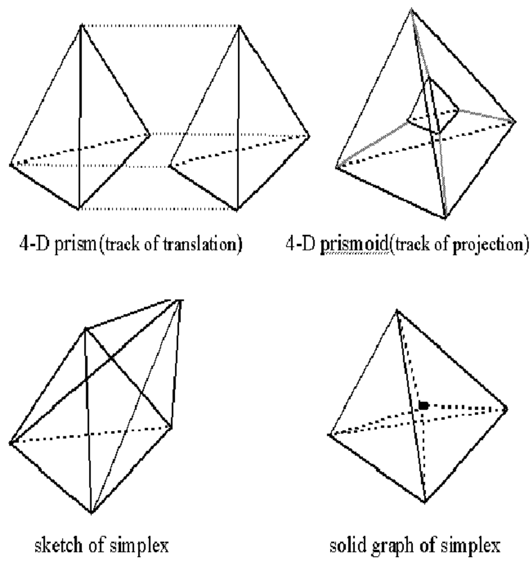

**[Fig. 5] the basic ideas of constructing 4-D structures**

He supposed that the minimal number of $k$-dim figures which meet in a $m$ -dim components of $n$-dim figure is ${}_nC_k$ and the minimum of k-dim figures which meet in a m-dim components of n-dim figure is ${}_{(n-m)}C_k$. This idea is very contributable to prove the existence of only 6 regular polytopes which is on the next section.

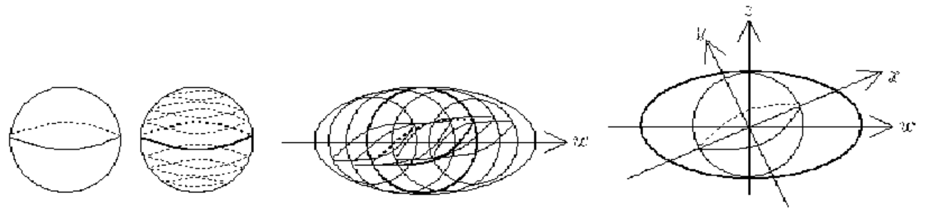
One student proposed several expressions([Fig. 6], [Fig. 7]) on 4-D which was derived by 3 expressions on 3-D(sketch, development figure, and planar graph). He named the 3rd expression of [Fig 6] as 'solid graph', and '4-D prism' and '4-D prismoid' in [Fig. 7]. And another student proposed the sketch of hyper ball([Fig. 8]). They said that their ideas were result from the using 'the principle of the permanence of equivalent forms'.




[Fig. 6] several expressions of 3-D and 4-D figures




[Fig. 7] different expressions on 4-D figure



[Fig. 8] sketch of hyper ball

## 2. Proving the existence of 6 regular polytopes

A student proved that there exists only 6 regular polytopes on the analogy with the proof of the existence of the 5 regular polyhedrons. He used the dihedral angles of faces of the polyhedrons in each edges(which are given on the web information) instead of the angles between the edges of the polygons in each points. This is very important founding in 4-D figures. He got this ideas from the using 'the principle of the permanence of equivalent forms'.

<Table 4> kinds of regular polytopes

| (the polyhedrons in each edges) / Kinds of regular - | Number of - / dihedral angle | 3 | 4 | 5 |
|---|---|---|---|---|
| tetrahedron | 70°32′ − | simplex(5-cell) | 16-cell | 600-cell |
| hexahedron | 90° | hyper cube(8cell) | × | × |
| octahedron | 109°28′ + | 24-cell | × | × |
| dodecahedron | 116°34′ − | 120-cell | × | × |
| icosahedron | 138°11′ + | × | × | × |

## 3. Hyper Volumes and Hyper solid angle

An excellent student who has learned advanced calculus, on himself, he calculated several 4-D cells and hyper ball with integration. He often used to use 'the principle of the permanence of equivalent forms'. The results are already known to the mathematics community, but he was very excited to know the result on the general $k$-dimensional figures. He was satisfied that he could use the 'the principle of the permanence of equivalent forms'. And his mentor would like to use the method of 'guided re invention'.

(1) hyper volumes of each cells( $r$ : the length of each sides)

① 8−cell : $r^4$

② 16−cell : $\frac{r^4}{6}$

③ 5−cell : $\frac{\sqrt{5}r^4}{96}$

(2) hyper volumes of Hyper-ball( $r$ : the radius of the ball)

$$\frac{1}{2}\pi^2 r^4$$

(3) $k$-dim hyper volumes of hyper-balls( $r$ : the radius of the ball)

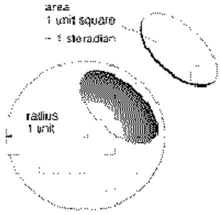
① $k$: even $V=\frac{\pi^{k/2}r^k}{(k/2)!}$

② $k$: odd $V=\frac{\pi^{(k-1)/2}(k+1/2)!2^{k+1}r^k}{(k+1)!}$

(the limit of all V is 0) : (*) these results are from web-surfing

Volume Boundary

$$B=\frac{d}{dr}V$$




| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 2-D disc | : | $\pi r^2$ | → | $2\pi r$ | → radian | $2\pi$ | radian |
| 3-D ball | : | $\frac{4}{3}\pi r^3$ | → | $4\pi r^2$ | → solid angle | $4\pi^2$ | steradian |
| 4-D hyperball | : | $\frac{1}{2}\pi^2 r^4$ | → | $2\pi^2 r^3$ | → hyper solid angle | $2\pi^2$ | ? |

먼저 구의 부피는 $V=\frac{4}{3}\pi r^3$이며, 4차원 축 상의 임의의 지점w에서의 구형의 단체 반지름은 $r'=\sqrt{r^2-w^2}$

이다. 따라서 임의 w지점에서 구형 단체의 부피는 $\frac{4}{3}\pi r'^3=\frac{4}{3}\pi(r^2-w^2)^{\frac{3}{2}}$ 이다.

따라서 초구의 초부피는 $2\int_0^r \frac{4}{3}\pi(r^2-w^2)^{\frac{3}{2}}dw = \frac{8}{3}\pi\int_0^r (r^2-w^2)^{\frac{3}{2}}dw$이며,

$w$는 구간 [-1,1]의 수이므로, $w=r\sin\theta$로 치환가능 하며, 치환하면 $\frac{8}{3}\pi\int_0^r (r^2-r^2\sin^2\theta)^{\frac{3}{2}} rd\sin\theta$

그런데 $\frac{rd\sin\theta}{d\theta}=r\cos\theta$ $\quad\therefore rd\sin\theta = r\cos\theta d\theta$

대입하여 정리하면, $\frac{8}{3}\pi r^4\int_0^{\frac{\pi}{2}}(1-\sin^2\theta)^{\frac{3}{2}}\cos\theta d\theta$

한편, $1-\sin^2\theta=\cos^2\theta$ 이므로 $\frac{8}{3}\pi r^4\int_0^{\frac{\pi}{2}}\cos^4\theta d\theta$

이로부터 배각 공식을 사용하여 다음을 얻는다.

$$\frac{8}{3}\pi r^4\int_0^{\frac{\pi}{2}}\cos^4\theta d\theta = \frac{8}{3}\pi r^4\int_0^{\frac{\pi}{2}}\frac{1}{4}(1+\cos 2\theta)^2 d\theta = \frac{2}{3}\pi r^4\int_0^{\frac{\pi}{2}}(1+\cos 2\theta)^2 d\theta$$

$$= \frac{2}{3}\pi r^4\int_0^{\frac{\pi}{2}}(1+2\cos 2\theta+\cos^2 2\theta)d\theta = \frac{2}{3}\pi r^4\int_0^{\frac{\pi}{2}}(1+2\cos 2\theta+\frac{1}{2}+\frac{1}{2}\cos 4\theta)d\theta$$

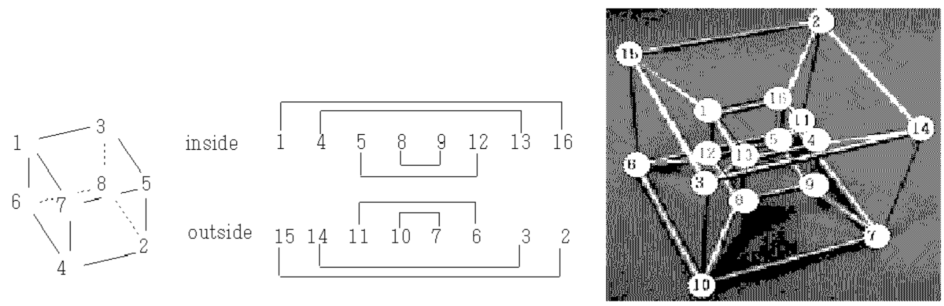
$$= \frac{2}{3}\pi r^4\int_0^{\frac{\pi}{2}}(\frac{3}{2}+2\cos 2\theta+\frac{1}{2}\cos 4\theta)d\theta = \frac{2}{3}\pi r^4[\frac{3}{2}\theta+\sin 2\theta+\frac{1}{8}\sin 4\theta]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{3}\pi r^4(\frac{3\pi}{4}+\sin\pi+\frac{1}{8}\sin 2\pi-\sin 0-\frac{1}{8}\sin 0) = \frac{1}{2}\pi^2 r^4$$

[Fig. 9] from a student's report

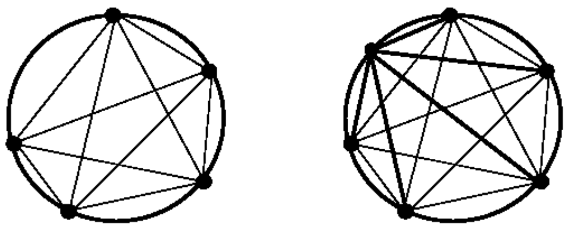
## 4. others

After the period of the study, a curious student who would like to make magic square supposed a 'magic hyper cube' as like [Fig. 10]. Now, he is tying to find another types of magic square with using planar graph of pseudo-regular polyhedron and 'solid graph' of some 4-D figures.




[Fig. 10] magic hypercube

And a student gave a conjecture about the maximal number of 4-D sections to be cut by 3-D figure which was come from the maximal number of 3-D sections to be cut by 2-D figure(plane) and 2-D sections to be cut by 1-D(line) figure, as <Table 5>. He suggested that the pattern is the number of planar sections on [Fig. 11]. And he would like to know that the pattern of the maximal number of 5-D sections to be cut by 4-D figure. But he couldn't solve the suggestion, yet.



[Fig. 11] the number of planar sections

<Table 5> the maximum of sections of $n$-D figure divided by $(n-1)$ D cutter

| the maximal number of sections | | the number of cutter($n-1$ dimensional figure) | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| to be cut object ($n$-dim) | 1-D(line) | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| | 2-D(plane) | 2 | 4 | 7 | 11 | 16 | 22 | 29 |
| | 3-D(space) | 2 | 4 | 8 | 15 | 22 | 33 | 49 |
| | 4-D figure | 2 | 4 | 8 | 16 | 31 | 53 | 86 |
| | 5-D figure | 2 | 4 | 8 | 16 | 32 | 63 | 116 |

# Ⅳ. Conclusion

Korean high school students are very excited in investigating 4-D figures. But they don't know how to start their works. 4 high school students(except only one student) were not attend to accelerated course, but they could imagine 4-D figures and debate about geometrical relations. They could understand and discuss several difficult mathematical theorems by themselves where I had used to guide them by using Freudenthal's 'guided re-invention method'.

The role of mentor is very important There are 3 suggestions to guide students as a mentor.

The first, every mentor has to introduce the essence works, as possible as in detail, to their mentees. It is important to know the sub goals of the study. The goal of this project was to address their reports what they re-invented to the other co-workers and to suggest a mathematical paper, whether it is good or not. The quality of their products was up to each mentee's ability and effort.

The second, mentor has to enhance students' inherent curiosities and encourage their desires. Every students would like to get new things which are not already published. They would like to suggest a new theorems or findings. But it will not be so easy. So  it is necessary to be allowed to re-invent anything, if it is very related to the given subject, and if it is a new finding to the mentees not to their mentees. It is necessary to encourage them to propose their conjecture and to produce anything what they are thinking. If students attended in the process of their study by the self-directed learning, they could explain the research result with confidence. The best teaching is not to teach but to study themselves. The method of 'guided re-invention' is one of the useful teaching method. But every mentors have to think and expect that their mentees can find or invent a new product and they have creative abilities and aggressive mathematical attitude.

The third, mentor has to give some exemplifications of the products, and to guide mentees to try suppose more creative ideas or general solution. They would like to know the relations of the properties among all dimensions, and how to expand the given examples. 'The principle of the permanence of equivalent forms' is one of the very useful to expand the student's ideas which is already known to the unknown results. High school students could find and invent some meaningful mathematical properties and conjectures in the 4-D figures by using the 'the principle of the permanence of equivalent forms'.

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## 국문초록

# 수학 영재들을 4차원 도형에 대한 탐구로 안내하는 사례 연구

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이 연구는 경기과학고등학교 1학년 학생 5명을 대상으로 사사연구를 진행하면서 학생들이 탐구한 수학적인 내용에 대한 분석과 그 결과가 나오기까지 멘토링을 하는 지도교수의 역할을 설명하고 있다. 학생들이 탐구한 수학적인 내용은 4차원 도형의 모양과 그 도형들에 나타나는 수학적인 성질이다. 지도교수는 연구에 익숙하지 않은 학생들을 위하여 수학자 피코크가 제안했던 '형식불역의 원리'를 모델로 삼도록 했고, 지도교수는 학생들의 창조적인 산출물 생산을 격려하기 위해 수학교육학자 프로이덴탈의 '안내된 재발명의 방법'을 사용하였다.

학생들은 지도교수의 안내에 의한 (재)발명의 원리에 따라 기존에 이미 알고 있던 수학적 성질을 고차원 도형에 적용시키면서 확장, 일반화시켜나갔는데, 여기에는 '형식불역의 원리'라는 틀이 매우 유용하게 작용하였다. 지도교사는 학생들에게 3차원 도형을 2차원에 표현하는 겨냥도, 전개도, 평면그래프를 응용하여 4차원을 3차원과 2차원에 표현하는 방식을 탐구하도록 하였다. 이 과정에서 학생들은 이미 알려진 파스칼의 삼각형과 이항정리, 오일러 정리, 하세의 다이어그램 등을 4차원 이상의 도형을 탐구할 때에도 적용할 수 있음을 확인하였다. 그리고 몇 가지의 추측과 후속 연구 과제를 제안하였다. 학생들의 산출물들은 형식불역의 원리와 안내된 재발명의 방법의 결과물인 것이다.

이 연구는 사사연구의 과정에 도움이 될 수 있는 3가지의 제안과 그 실 예를 담고 있다.

**주제어** : 수학, 영재, 사사, 차원, 안내된 재발명의 방법, 형식불역