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Generating functions for t-norms

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Abstract

We investigate the P-generating functions, L-generating functions, and A-generating function, respectively induced by product t-norms, Lukasiewicz t-norms and additive semi-groups. Furthermore, we investigate the relations among them.

Key words : P-generating functions, L-generating functions, A-generating function, dominated t-norms.

1. Introduction

The basic ideas of triangular norms (t-norms) were introduced in 1961 in a series of papers by Schweizer and Sklar [7] for the purpose of generalizing the triangle inequality in metric spaces to probabilistic metric spaces. Triangular norms were introduced into the fuzzy set community for modeling the logical conjunction and the pointwise intersection of fuzzy sets [1-6, 8-10].

In this paper, we investigate the P-generating functions, L-generating functions, and A-generating function, respectively induced by product t-norms, Lukasiewicz t-norms and additive semi-groups. Furthermore, we investigate the relations among them.

A binary operation $T:[0,1]\times[0,1]\to[0,1]$ is called a t-norm if it satisfies the following conditions:
for each $x, y, z\in[0,1]$,

(T1) $T(x,y)=T(y,z)$,

(T2) $T(x, T(y,z))=T(T(x,y),z)$

(T3) $T(x,1)=x$,

(T4) if $y\le z$, then $T(x,y)\le T(x,z)$.

We denote $T(x,y)=x\odot y$. A t-norm T_1 is called weaker than a t-norm T_2(T_2 is called stronger than a t-norm T_1, denoted by $T_1\le T_2$, if $T_1(x,y)\le T_2(x,y)$.

A function $f:[0,1]\to[0,1]$ is called a P-generator for T_f if $T_f(x,y)=f^{-1}(f(x)f(y)\vee f(0))$ is a t-norm.

A function $g:[0,1]\to[0,\infty]$ is called an A-generator for T_g if $T_g(x,y)=g^{-1}((g(x)+g(y))\wedge g(0))$ is a t-norm.

A function $h:[0,1]\to[0,1]$ is called an L-generator for T_h if $T_h(x,y)=h^{-1}((h(x)+h(y)-1)\wedge 0)$ is a t-norm.

Theorem 1.1 [6] If T is an Archimedean t-norm, then there is an order isomorphism $f:[0,1]\to[f(0),1]$ such that $x\odot y=f^{-1}(f(x)f(y)\vee f(0))$ for all $x,y\in[0,1]$. If $g:[0,1]\to[g(0),1]$ is an order isomorphism, then $x\odot y=g^{-1}(g(x)g(y)\vee g(0))$ for all $x,y\in[0,1]$ iff $f(x)=g(x)^r$ for some $r>0$.

Example 1.2 Let $T(x,y)=\frac{xy}{\sqrt{x^2+y^2-x^2y^2}}$ be a t-norm and a P-generating function f of T.

For $x_n\odot x_n=x_{n-1},\ x_0=\frac{1}{2}$,
since $x_n\odot x_n=f^{-1}(f(x_n)f(x_n)\vee f(0))$, we have $f(x_{n-1})=f(x_n)f(x_n)$. Put $f(x_n)=a_n$ and $f(x_0)=a_0=\frac{1}{2}$. Then $a_{n-1}=a_n^2$, i.e. $\log_{\frac{1}{2}}a_{n-1}=2\log_{\frac{1}{2}}a_n$. We obtain

$$f(x_n)=a_n=(\frac{1}{2})^{2^{-n}}$$

Since $x_1\odot x_1=\frac{x_1^2}{\sqrt{2x_1^2-x_1^4}}=x_0=\frac{1}{2}$ then

$$x_1=\sqrt{\frac{2}{5}},\quad f(\sqrt{\frac{2}{5}})=2^{-\frac{1}{2}}$$

$$x_2\odot x_2=\frac{x_2^2}{\sqrt{2x_2^2-x_2^4}}=x_1=\sqrt{\frac{2}{5}}$$

$$x_2=\sqrt{\frac{4}{7}},\quad f(\sqrt{\frac{4}{7}})=2^{-\frac{1}{4}}$$

$$x_3\odot x_3=\frac{x_3^2}{\sqrt{2x_3^2-x_3^4}}=x_2=\sqrt{\frac{4}{7}}$$

$$x_3=\sqrt{\frac{8}{11}},\quad f(\sqrt{\frac{8}{11}})=2^{-\frac{1}{8}}$$

Put

$$h(1)=\frac{1}{2},\ h(\frac{1}{2})=\sqrt{\frac{2}{5}},\ h(\frac{1}{4})=\sqrt{\frac{4}{7}},\ h(\frac{1}{8})=\sqrt{\frac{8}{11}}$$

$$h(2^{-(n-1)})=\frac{(\sqrt{2})^{n-1}}{\sqrt{2^{n-1}+3}}=\frac{1}{\sqrt{1+3\cdot 2^{-n+1}}}$$

We obtain $h(x)=\frac{1}{1+3x}$ Since $h^{-1}(x)=\frac{1}{3}(\frac{1}{x^2}-1)$ we have $f(x)=2^{\frac{1}{3}(\frac{1}{x^2}-1)}$

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Theorem 1.3 [5] If T is an Archimedean t-norm, then there is an order reversing continuous function $f:[0,1]\to[0,\infty]$ such that $x\odot y=f^{-1}((f(x)+f(y))\wedge f(0))$ for all $x,y\in[0,1]$. If $g:[0,1]\to[0,\infty]$ is an order reversing continuous function, then $x\odot y=g^{-1}((g(x)+g(y))\wedge g(0))$ for all $x,y\in[0,1]$ iff $f(x)=ag(x)$ for some $r>0$.

Example 1.4 Let $x\odot y=\frac{xy}{x+y-xy}$ be a t-norm. Let A-generating function g of $\odot$. Put $g(1)=\frac{1}{2}$. Then we obtain

$$g(2)=\frac{1}{2}\odot\frac{1}{2}=\frac{1}{3},g(3)=\frac{1}{4},\ldots,g(n)=\frac{1}{n+1}$$

Hence $g(x)=\frac{1}{x+1}$ $\quad g(x+y)=g(x)\odot g(y)$

Thus, $f(x)=g^{-1}(x)=\frac{1-x}{x}$.

2. Generating functions

Lemma 2.1 Let g be a differentiable function as

$$\left(\frac{dg}{dx}\right)_{x=1}=a. \text{ Then}$$

(1) If $g(xy)=g(x)+g(y)$, then $g(x)=a\ln x$.
(2) If $g(x+y-1)=g(x)+g(y)$, then $g(x)=ax-a$.
(3) If $g(xy)=g(x)+g(y)-1$, then $g(x)=1+a\ln x$.

Proof (1) Since $g(1)=0$, we have

$$\begin{aligned}\frac{d}{dx}g &= \lim_{t\to0}\frac{g(x+t)-g(x)}{t}\\ &=\lim_{t\to0}\frac{g(x)g(1+\frac{t}{x})-g(x)}{t}\\ &=\lim_{t\to0}\frac{g(1+\frac{t}{x})-g(1)}{t}\\ &=g'(1)\frac{1}{x}=\frac{a}{x}.\end{aligned}$$

Then $g'(x)=\frac{a}{x}$. So, $g(x)=a\ln x+c$.
Since $g(1)=0$, $g(x)=a\ln x$.
(2) Since $g(1)=0$, we have

$$\begin{aligned}\frac{d}{dx}g &= \lim_{t\to0}\frac{g(x+t)-g(x)}{t}\\ &=\lim_{t\to0}\frac{g(x)+g(1+t)-g(x)}{t}\\ &=\lim_{t\to0}\frac{g(1+t)-g(1)}{t}\\ &=g'(1)=a.\end{aligned}$$

Then $g'(x)=a$. Since $g(1)=0$, $g(x)=ax-a$.
(3) Since $g(1)=1$, we have

$$\begin{aligned}\frac{d}{dx}g &= \lim_{t\to0}\frac{g(x+t)-g(x)}{t}\\ &=\lim_{t\to0}\frac{g(x)g(1+\frac{t}{x})-g(x)}{t}\\ &=\lim_{t\to0}\frac{g(1+\frac{t}{x})-g(1)}{t}\\ &=g'(1)\frac{1}{x}=\frac{a}{x}.\end{aligned}$$

Then $g'(x)=\frac{a}{x}$. Since $g(1)=1$, $g(x)=1+a\ln x$.

Theorem 2.2 Let $g:[0,1]\to[0,\infty]$ be a strictly decreasing function as $0<g(0)\le\infty$, Then:

(1) A function $f:[0,1]\to[f(0),1]$ with $f(x)=b^{-g(x)}$ for $b>1$ is a P-generating function iff gf^{-1} is differentiable and

$$\begin{aligned}x\odot y&=f^{-1}(f(x)f(y)\vee f(0))\\ &=g^{-1}((g(x)+g(y))\wedge g(0))\end{aligned}$$

(2) If $g(0)<\infty$, a bijective function $h:[0,1]\to[0,1]$ with $h(x)=-\frac{g(x)}{g(0)}+1$ is an L-generating function iff gh^{-1} is differentiable and

$$\begin{aligned}x\odot y&=h^{-1}((h(x)+h(y)-1)\vee0)\\ &=g^{-1}((g(x)+g(y))\wedge g(0)).\end{aligned}$$

(3) If $g(0+)=\infty$, $h(x)=\log_a(g(x)+1), a>1$ is an A-generating function iff gh^{-1} is differentiable and

$$\begin{aligned}x\odot y&=g^{-1}(g(x)g(y)+g(x)+g(y))\\ &=h^{-1}(h(x)+h(y))\end{aligned}$$

Proof (1) Since $f(x)=b^{-g(x)}$, $f^{-1}(x)=g^{-1}(-\log_b x)$, then $gf^{-1}(x)=-\log_b x$ is differentiable ,

If $f(x)f(y)\ge f(0)$, then

$$\begin{aligned}x\odot y&=f^{-1}(f(x)f(y)\vee f(0))\\ &=f^{-1}(b^{-g(x)}b^{-g(y)})\\ &=g^{-1}(-\log_b b^{-g(x)-g(y)})\\ &=g^{-1}(g(x)+g(y))\end{aligned}$$

If $f(x)f(y)<f(0)$, then

$$\begin{aligned}&f(x)f(y)<f(0)\Leftrightarrow b^{-g(x)}b^{-g(y)}<b^{-g(0)}\\ &\Leftrightarrow -g(x)-g(y)<-g(0)\ \Leftrightarrow g(x)+g(y)>g(0)\end{aligned}$$

Conversely, let $f^{-1}(f(x)f(y))=g^{-1}(g(x)+g(y))$. Then $gf^{-1}(f(x)f(y))=g(x)+g(y)$

Put $k=gf^{-1}$, $f(x)=t,f(y)=s$. Then $k(st)=k(s)+k(t)$. By Lemma 2.1, $k(t)=a\ln t$ Thus, $g(x)=a\ln f(x)$.

(2) Since $h(x)=-\frac{g(x)}{g(0)}+1$, $h^{-1}(x)=g^{-1}(g(0)-g(0)x)$ then $gh^{-1}(x)=g(0)-g(0)x$ is differentiable ,

If $h(x)+h(y)\ge1$, then

$$\begin{aligned}x\odot y&=h^{-1}(h(x)+h(y)-1)\\ &=h^{-1}\left(-\frac{g(x)}{g(0)}+1-\frac{g(y)}{g(0)}+1-1\right)\\ &=h^{-1}\left(1-\frac{g(x)+g(y)}{g(0)}\right)\\ &=g^{-1}(g(x)+g(y))\end{aligned}$$

If $h(x)+h(y)<1$, then

$$h(x)+h(y)<1 \Leftrightarrow -\frac{g(x)}{g(0)}+1-\frac{g(y)}{g(0)}+1<0$$
$$\Leftrightarrow g(x)+g(y)>g(0)$$

Conversely, let $h^{-1}(h(x)+h(y)-1)=g^{-1}(g(x)+g(y))$ Then $gh^{-1}(h(x)+h(y)-1)=g(x)+g(y)$

Put $k=gh^{-1}, h(x)=t, h(y)=s$ Then $k(s+t-1)=k(s)+k(t)$ By Lemma 2.1 , $k(x)=ax-a$. Thus, $g(x)=g(0)-g(0)h(x)$.

(3) Since $h(x)=\log_a(g(x)+1), h^{-1}(x)=g^{-1}(a^x-1)$, then $gh^{-1}(x)=a^x-1$ is differentiable. We obtain

$$\begin{aligned} x\odot y &= g^{-1}(g(x)g(y)+g(x)+g(y)) \\ &= h^{-1}(h(x)+h(y)) \end{aligned}$$

Conversely, Since
$h^{-1}(h(x)+h(y))=g^{-1}(g(x)g(y)+g(x)+g(y))$
put $k=gh^{-1}$, then
$k(x+y)=k(x)k(y)+k(x)+k(y)=(k(x)+1)(k(y)+1)-1$
Since k is increasing, then $k(t)+1=a^t, a>1$;i.e. $g(x)=a^{h(x)}-1$. So, $h(x)=\log_a(g(x)+1), a>1$

Theorem 2.3 Let $f:[0,1]\to[f(0),1]$ be order preserving with $f(0)>0$. A bijective function $h:[0,1]\to[0,1]$ with $h(x)=1-\frac{\ln f(x)}{\ln f(0)}$ is an L-generating function iff hf^{-1} is differentiable and

$$\begin{aligned} x\odot y &= h^{-1}((h(x)+h(y)-1)\vee 0) \\ &= f^{-1}((f(x)f(y))\vee f(0)) \end{aligned}$$

Proof (1) Since $h(x)=1-\frac{\ln f(x)}{\ln f(0)}$ and $h^{-1}(x)=f^{-1}(f(0)^{1-x})$, then $hf^{-1}(x)=1-\log_{f(0)}f(x)$ is differentiable.

If $h(x)+h(y)\geq 1$, then

$$\begin{aligned} x\odot y &= h^{-1}(h(x)+h(y)-1) \\ &= h^{-1}(1-\frac{\ln f(x)}{\ln f(0)}+1-\frac{\ln f(x)}{\ln f(0)}-1) \\ &= g^{-1}(1-\frac{\ln f(x)f(y)}{\ln f(0)}) \\ &= f^{-1}(f(x)f(y)). \end{aligned}$$

If $h(x)+h(y)<1$, then

$$h(x)+h(y)<1 \Leftrightarrow 2-\frac{\ln f(x)f(y)}{\ln f(0)}<1$$
$$\Leftrightarrow \frac{\ln f(x)f(y)}{\ln f(0)}>1 \Leftrightarrow f(x)f(y)>f(0).$$

Conversely, let $f^{-1}(f(x)f(y))=h^{-1}(h(x)+h(y)-1)$ Then $hf^{-1}(f(x)f(y))=h(x)+h(y)-1$. So, $k(st)=k(s)+k(t)-1$. By Lemma 2.1 , $k(x)=1+a\ln x$. Thus, $h(x)=1+a\ln f(x)$ Since $h(0)=1+a\ln f(0)=0$ then $h(x)=1-\frac{\ln f(x)}{\ln f(0)}$ is an L-generating function.

Example 2.4 Let $g(x)=1-x$ be given. Then $f(x)=e^{x-1}$ and $h(x)=x$. We have

$$\begin{aligned} x\odot y &= f^{-1}(f(x)f(y)\vee f(0)) \\ &= g((g(x)+g(y))\wedge g(0)) \\ &= h((h(x)+h(y)-1)\vee 0) \\ &= (x+y-1)\vee 0 \end{aligned}$$

Example 2.5 Let $h(x)=x^p$ be given. Then $g(x)=a(1-x^p), a>0$ and $f(x)=f(0)^{1-x^p}$. We have

$$\begin{aligned} x\odot y &= f^{-1}(f(x)f(y)\vee f(0)) \\ &= g((g(x)+g(y))\wedge g(0)) \\ &= h((h(x)+h(y)-1)\vee 0) \\ &= ((x^p+y^p-1)\vee 0)^{\frac{1}{p}} \end{aligned}$$

Example 2.6 Let $g_s(x)=-\ln\frac{s^x-1}{s-1}$, $0<s<\infty$ where

$$g_1(x)=\lim_{s\to 1} g_s(x)=\lim_{s\to 1}(-\ln\frac{s^x-1}{s-1})=-\ln x$$

Then $f_s=e^{-g_s(x)}=\frac{s^x-1}{s-1}$, $f_1(x)=x$.

$$h_s(x)=e^{g_s(x)}-1=\frac{2s-1-s^x}{s^x-1}, h_1(x)=\frac{1}{x}-1,$$

We obtain:

$$\begin{aligned} x\odot y &= f_s^{-1}(f_s(x)f_s(y)) \\ &= g_s^{-1}((g_s(x)+g_s(y)) \\ &= h_s^{-1}(h_s(x)h_s(y)+h_s(x)+h_s(y)) \\ &= \log_s(1+\frac{(s^x-1)(s^y-1)}{s-1}) \end{aligned}$$

$$\begin{aligned} x\odot y &= f_1^{-1}(f_1(x)f_1(y)) \\ &= g_1^{-1}(g_1(x)+g_1(y)) \\ &= h_1^{-1}(h_1(x)h_1(y)+h_1(x)+h_1(y)) \\ &= xy \end{aligned}$$

Example 2.7. Let g_s be an A-generator as $g_s(x)=1-\ln_{1+s}(1+sx)=1-\frac{\ln(1+sx)}{\ln(1+s)}$, $1<s$

Then $g_s^{-1}(x)=\frac{1}{s}((1+s)^{1-x}-1)$

Put $f_s(x)=(1+s)^{-g(x)}=\frac{1+sx}{1+s}, s>1$ Then $f_s^{-1}(x)=\frac{1}{s}((1+s)x-1)$. We obtain

$$\begin{aligned} x\odot y &= g_s^{-1}((g_s(x)+g_s(y))\wedge g_s(0)) \\ &= g_s^{-1}(2-\frac{\ln(1+sx)(1+sy)}{\ln(1+s)}\wedge g_s(0)) \\ &= \frac{1}{s}((1+s)^{-1+\frac{\ln(1+sx)(1+sy)}{\ln(1+s)}}-1)\vee 0 \\ &= \frac{1}{s}((1+s)^{\ln_{1+s}\frac{(1+sx)(1+sy)}{1+s}}-1)\vee 0 \\ &= \frac{1}{s}(\frac{(1+sx)(1+sy)}{1+s}-1)\vee 0 \\ &= \frac{1}{s}(\frac{s^2xy+sx+sy-s}{1+s})\vee 0 \\ &= (\frac{sxy+x+y-1}{1+s})\vee 0 \end{aligned}$$

We can obtain a t-norm by a generator f_s as

$$\begin{aligned} x\odot y &= f_s^{-1}(f_s(x)f_s(y)\vee f_s(0)) \\ &= f_s^{-1}\left(\frac{1+sx}{1+s}\,\frac{1+sy}{1+s}\vee\frac{1}{1+s}\right). \\ &= \left(\frac{sxy+x+y-1}{1+s}\right)\vee 0 \end{aligned}$$

Since $g_s(0)=\ln(1+s), s>1$, put $g_s(x)=2h_s(x)-2$. Then $h_s(x)=\frac{3}{2}-\frac{1}{2}\ln_{1+s}(1+sx)=1-\frac{1}{2}\frac{\ln(1+sx)}{\ln(1+s)}$, $1<s$ be an L-generator.

Moreover, $h_s^{-1}=\frac{1}{s}((1+s)^{3-2x}-1)$

$$\begin{aligned} x\odot y &= h_s^{-1}(h_s(x)+h_s(y)-1\vee 0) \\ &= h_s^{-1}((2-\frac{1}{2}\log_{1+s}(1+sx)(1+sy))\vee 0) \\ &= \left(\frac{sxy+x+y-1}{1+s}\right)\vee 0 \end{aligned}$$

3. Relations of t-norms

Definition 3.1 Let T_1, T_2 be t-norms. T_1 dominates T_2, denoted by $T_1 \ll T_2$, if for each $x_1, x_2, y_1, y_2 \in [0,1]$
$T_1(T_2(x_1, y_1), T_2(x_2, y_2)) \geq T_2(T_1(x_1, x_2), T_1(y_1, y_2))$.

Theorem 3.2. Let T_f, T_g be a strict t-norm with P-generators f, g. Then we have the following properties:

(1) T_f is stronger than T_g iff it satisfies

$$h(st)\geq h(s)\cdot h(t), \quad \forall s,t\in[0,1], h=g\circ f^{-1}.$$

(2) T_f dominates T_g iff it satisfies, $\forall s_i, t_i\in[0,1], h=g\circ f^{-1}$.

$$h^{-1}(h(s_1)h(t_1))h^{-1}(h(s_2)h(t_2)) \geq h^{-1}(h(s_1t_1))h^{-1}(h(s_2t_2)).$$

(3) If T_f dominates T_g, then T_f is stronger than T_g.

(4) Every t-norm T dominates T.

Proof (1) Since
$T_f(x,y)=f^{-1}(f(x)f(y))\geq T_g(x,y)=g^{-1}(g(x)g(y))$
we have:

$$gf^{-1}(f(x)f(y))\geq g(g^{-1}(g(x)g(y)))=g(x)g(y).$$

Put $s=f(x)$ and $t=f(y)$. Then:

$$h(st)\geq h(s)\cdot h(t), \quad \forall s,t\in[0,1], h=g\circ f^{-1}.$$

The converse can be similarly proved.

(2) For each $x_1, x_2, y_1, y_2\in[0,1]$,

$$\begin{aligned} & f^{-1}(f((g^{-1}(g(x_1)g(y_1))\cdot f(g^{-1}(g(x_2)g(y_2))) \\ &= T_f(g^{-1}(g(x_1)g(y_1)), g^{-1}(g(x_2)g(y_2)) \\ &= T_f(T_g(x_1)y_1), T_g(x_2, y_2) \\ &\geq T_g(T_f(x_1)x_2), T_f(y_1, y_2)) \\ &= g^{-1}(g((f^{-1}(f(x_1)f(x_2))\cdot g(f^{-1}(f(y_1)f(y_2)))). \end{aligned}$$

It implies

$$\begin{aligned} & f((g^{-1}(g(x_1)g(y_1))\cdot f(g^{-1}(g(x_2)g(y_2))) \\ &\geq fg^{-1}(g((f^{-1}(f(x_1)f(x_2))\cdot g(f^{-1}(f(y_1)f(y_2)))) \end{aligned}$$

Put $f(x_1)=s_1, f(x_2)=s_2, f(y_1)=t_1, f(y_2)=t_2$ and $h=g\circ f^{-1}$. Then

$$h^{-1}(h(s_1)h(t_1))h^{-1}(h(s_2)h(t_2)) \geq h^{-1}(h(s_1t_1))h^{-1}(h(s_2t_2)).$$

The converse can be similarly proved.

(3) By (2), put $s_2=1$ and $t_1=1$. Then

$$h^{-1}(h(s_1))h^{-1}(h(t_2))=s_1t_2\geq h^{-1}(h(s_1)h(t_2)).$$

So, $h(s_1t_2)\geq h(s_1)h(t_2)$. By (1), T_f is stronger than T_g.

(4) For each $x_1, x_2, y_1, y_2\in[0,1]$

$$\begin{aligned} T(T(x_1,y_1),T(x_2,y_2)) &= T(T(T(x_1,y_1),x_2),y_2) \\ &= T(T(x_2, T(x_1,y_1)),y_2)) \\ &= T(T(T(x_2,x_1),y_1)),y_2)) \\ &= T(T(x_2,x_1),T(y_1,y_2)) \\ &= T(T(x_1,x_2),T(y_1,y_2)) \end{aligned}$$

Example 3.3 Let T_f and T_g be continuous t-norms with P-generators as $f(x)=x$, $g(x)=2^x-1$

Then we obtain $h(x)=g\circ f^{-1}(x)=2^x-1$.

Since $h(st)\geq h(s)\cdot h(t)$, $\forall s,t\in[0,1]$, T_f is stronger than T_g.

Example 3.4 Let T_f and T_g be continuous t-norms with P-generators as

$$f(x)=e^{1-\frac{1}{x}}, \quad g(x)=\begin{cases} e^{-\frac{1}{2x}} & 0<x\leq\frac{1}{2} \\ e^{1-\frac{1}{x}} & \frac{1}{2}<x\leq 1, \\ 0, & x=0 \end{cases}$$

The we obtain

$$h(x)=g\circ f^{-1}(x)=\begin{cases} \sqrt{\frac{x}{e}} & 0\leq x\leq\frac{1}{e} \\ x & \frac{1}{e}<x\leq 1 \end{cases}$$

Since $h(st)\geq h(s)\cdot h(t)$, $\forall s,t\in[0,1]$, T_f is stronger than T_g.

But T_f does not dominate T_g because

$$\frac{1}{4e^4} = h^{-1}(h(\frac{1}{e})h(\frac{1}{e}))h^{-1}(h(\frac{1}{e})h(\frac{1}{2}))$$
$$\not\geq h^{-1}(h(\frac{1}{e^2})h(\frac{1}{2e})) = \frac{1}{2e^4}.$$

Theorem 3.5 Let T_{f} and T_{g^*} be strict t-norms with A-generators f^* and g^* respectively, then

(1) T_{f} is stronger than T_{g^*} iff $h^* = g^* \circ f^{*-1}$ satisfies $h^*(s+t) \leq h^*(s) + h^*(t), \quad \forall t, s \in [0,1]$

(2) T_{f} dominates T_{g^*} iff for each $s_1, s_2, t_1, t_2 \in [0,1]$

$$h^{*-1}(h^*(s_1) + h^*(t_1)) + h^{*-1}(h^*(s_2) + h^*(t_2)) \geq h^{*-1}(h^*(s_1+s_2) + h^*(t_1+t_2)).$$

Proof. (1)Put $f(x) = e^{-f^*(x)}$, $g(x) = e^{-g^*(x)}$ Then $y = f^{-1}(x) = f^{*-1}(-\ln x)$. By Theorem 3.2, we have $h(x) = g(f^{-1}(x)) = e^{-g^*(f^{-1}(x))} = e^{-g^*(f^{*-1}(-\ln x))}$.

It implies

$$h(st) = e^{-g^*(f^{*-1}(-\ln st))}$$
$$\geq h(s)h(t) = e^{-g^*(f^{*-1}(-\ln s))}e^{-g^*(f^{*-1}(-\ln t))}$$

So,

$$g^*(f^{*-1}(-\ln st)) \leq g^*(f^{*-1}(-\ln s)) + g^*(f^{*-1}(-\ln t))$$

Put $-\ln s = x$, $-\ln t = y$ and $h^* = g^* \circ f^{*-1}$ Then

$$T_g h^*(s+t) \leq h^*(s) + h^*(t), \quad \forall t, s \in [0,1]$$

(2) Since $h(x) = e^{-g^*(f^{*-1}(-\ln x))}$ and $h^{-1}(h(s_1)h(t_1))h^{-1}(h(s_2)h(t_2)) \geq h^{-1}(h(s_1t_1))(h(s_2t_2))$, we have

$$h^{*-1}(-\ln h(s_1)h(t_1)) + h^{*-1}(-\ln h(s_2)h(t_2))$$
$$\leq h^{*-1}(-\ln h(s_1t_1))h(s_2t_2)).$$

Put $-\ln s_i = x_i$, $-\ln t_i = y_i$, $i = 1,2$ and $h^* = g^* \circ f^{*-1}$

$$-\ln h(s_1t_1))h(s_2t_2)$$
$$= -\ln e^{-h^*(-\ln(s_1s_2))}e^{-h^*(-\ln(t_1t_2))}$$
$$= h^*(-\ln(s_1s_2)) + h^*(-\ln(t_1t_2))$$
$$= h^*(x_1+x_2) + h^*(y_1+y_2).$$

Hence

$$h^{*-1}(h^*(x_1) + h^*(y_1)) + h^{*-1}(h^*(x_2) + h^*(y_2)) \geq h^{*-1}(h^*(x_1+x_2) + h^*(y_1+y_2)).$$

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