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Sensitivity Analysis for a Make-to-Order Inventory-Production System with Limited Order Acceptance Level

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■ Abstract ■

This paper considers a make-to-order inventory-production system in which customer orders are admitted only when the number of outstanding customer orders is below a value committed by the system. We deal with general distributions for the customer order inter-arrival, production, and replenishment lead time processes. Monotonicities of the optimal average cost with respect to these distribution parameters are established using sample path coupling arguments. When distributions are given as an exponential one, we implement a sensitivity analysis on the optimal inventory policy and show that it has monotonicities with respect to system costs using dynamic programming.

Keyword : Monotonicity, Sample Path Coupling Argument, Dynamic Programming, Inventory Management, Sensitivity Analysis, Make-to-order

1. Introduction

Make-to-order inventory-production systems where a firm processes customer orders using items which are procured from an outside supplier are quite common in service and manufacturing applications. This paper considers a make-to-order inventory-production system where the

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orders from the firm's customer group are admitted only when the number of outstanding (i.e., waiting for production) orders in the system is below a value committed by the system. This value can be a space limit or considered as a firm's management policy. Limited commitment on outstanding order level is a common practice in the manufacturing and service systems. For example, assembly systems are normally required to hold minimal work-in-process inventories because of space or capital limitations [11, 13].

Most of make-to-order inventory-production models in the literature (see Kim [10] for detail), in contrast with this paper, have studied the situation that firms do not impose any restriction on the admission of customer orders upon their arrival. Hence, every customer order is accepted for production at any time. While this arrangement affords a great deal of flexibility to the customer, it also results in a great deal of uncertainty for the firm's inventory and production management.

The purpose of the limited commitment on the outstanding order level is strike to balance between the flexibility for customer and reduction of uncertainty for the firm. This paper assumes that each rejected customer order incurs a penalty cost. This cost can be considered as a lost sales opportunity, or a compensation for the customer for his rejected order. It is resonable to expect that as the limited outstanding order level increase, the penalty would decline.

In a recent paper, Kim [10] has dealt with the model discussed above. Assuming that production, customer order arrival, and replenishment processes follow exponential distributions, he established a necessary condition for the existence of an optimal inventory replenishment policy and characterized its structure. In contrast with Kim [10], we consider general (non-exponential) distributions for the customer order inter-arrival, production, and replenishment lead time processes. Our contribution can be summarized as follows. First, we analytically show that there are clear monotonic relationships between the problem parameters and the optimal average cost using sample path coupling arguments. For example, we show that reductions in the mean arrival rate decrease the firm's average cost. Second, we implement a sensitivity analysis on the optimal policy characterized in Kim [10] and show that it can be expressed as a monotonic function of system costs.

The rest of paper is organized as follows. We review the relevant literature in the next section. Section 3 presents model assumptions and problem formulation. In section 4, we present a sensitivity analysis on the optimal policy. Section 5 characterizes the monotonicity of the optimal cost with respect to system parameters. The last section contains conclusions.

2. Literature review

The related research works can be classified into three groups : (i) the make-to-order inventory-production management ; (ii) the optimal control of queueing systems ; (iii) the manufacturing contract problem. For the detailed survey of (i) and (ii), the reader is referred to Kim [10].

The existence of manufacturing contracts with various forms of commitments has long been recognized in supply chain management literature. Bassok snd Anulindi [2] considers a single product periodic review inventory model in which, at the beginning of the horizon, the customer makes a commitment to purchase at least a minimum quantity by the end of the horizon and the manufacturer discounts the unit purchase price to all units purchased. Chen and Krass [5] extended Bassok and Anupindi [2] to a more general model where demand distributions are non-stationary and unit prices are different for the as-ordered purchase and the commitment purchase. Cohen and Agrawal [6] compare the desirability of a long-term manufacturing contract with predetermined purchase price against that of a short-term manufacturing contract subject to fluctuating market price. Serel et al. [15] considered a single item periodic inventory model in which the customer purchases a part periodically depending on the part inventory level and proposed a capacity reservation contract in which the customer guarantees a fixed payment to the preferred manufacturer in return for the delivery of any portion of a reserved fixed capacity. Li and Kouvelis [12] analyzed manufacturing contracts providing flexibility with regard to the quantity purchased and the timing of the purchase when the demand during the problem horizon is given as deterministic and the market price for the part purchased is assumed stochastic.

Retailer-supplier relationships based on the VMI contract have been also discussed in several papers. Fry et al. ([7, 8]) studied a (z,Z) type VMI contract with a single retailer and a single supplier under which the supplier pays penalties to the retailer for every unit of the retailer's inventory either less than z or more than Z. The supplier replenishes the retailer from an outside party as well as her on-hand inventory, and produces either once at the beginning of the planning horizon or once every fixed periods. The delivery time from the supplier to the retailer is assumed to be negligible. It is shown that the optimal replenishment and production policies for the supplier are characterized as an order up-to policy, respectively, and outsourcing of inventory should be allowed only when supplier's on-hand inventory is empty. Cachon and Zipkin [3] studied a two-echelon supply chain with a single supplier and a single retailer in which both are willing to pay customer backorder costs and independently choose their base stock policies to minimize their own casts, that is, competitive policies. Based on the numerical comparison with the policy to minimize total system casts, they showed that when the supplier and retailer equally share backorder costs, the competition penalty is small, but if is not the case, the competition penalty can be huge. Cachon [4] studied VMI with one supplier and multiple retailers. He showed that VMI can be a coordinated channel, that is, achieve the minimum of supply chain costs only when both the supplier and the retailers agree to make fixed transfer payments in order to participate in the VMI contract and both wish to share the benefits. Aviv and Federgruen [1] also analyzed a VMI model with a single supplier and multiple retailers in which supplier's objective is to minimize a system-wide cost. They numerically compared the case under both information sharing alone and information sharing in conjunction with VMI, and conclude that VMI is always more beneficial than information sharing alone.

3. Model Assumption and Problem Formulation

A firm produces a single product based on customer orders and each product requires a single type of item provided by an outside supplier. Customer orders, denoted by A, occur with a general arrival process with mean $E(A)$, the expected number of customer order arrivals during the unit time. They are accepted only when the number of outstanding (waiting for production) orders is below M. Whenever each customer order is rejected, a lump-sum cost, C_R, is incurred. The production time, P, follows a general, strictly positive distribution with mean $E(P)$ and a finite second moment. Successive productions are independent and identically distributed (*i.i.d.*) and independent of all else.

Holding cost is assessed at a rate of h_1 and h_2 for each customer order in queue and each item in inventory, respectively. A lead time, L, is incurred at each instant the firm places a replenishment order with size Q to an outside supplier. We assume that successive replenishment orders require strictly positive times which are *i.i.d.*, possess a finite mean $E(L)$ and second moment, and are independent of all else. At each instant a replenishment order is placed, a lump-sum cost, C_S is incurred. A policy specifies that the firm places a replenishment order or not. With $r^+(Z^+)$ denoting the non-negative real values (integers), let $\{x_n(t): t \in r^+\}$, $n = 1, 2,$ be the right continuous queue length process of customer order ($n=1$) and inventory ($n=2$), respectively. The average cost per unit time of policy π, $\bar{J}_\pi$, can now be expressed as

$$\bar{J}_\pi = \lim_{T\to\infty} \frac{1}{T} E\left\{ \int_0^T \sum_{n=1}^{2} h_n x_n(t)\, dt + \sum_{t\in B_1^\pi(T)} C_R + \sum_{t\in B_2^\pi(T)} C_S \right\}$$

where $B_1^\pi(T)$ and $B_2^\pi(T)$ denote the set of random instances that a customer order is rejected and a replenishment order is placed, respectively, under policy π in an interval $[0, T]$. Then, the optimal average cost $\bar{J}^*$ is defined by $\bar{J}^* = \min_\pi \bar{J}_\pi$. We denote by π^* the optimal policy that achieves the optimal average cost in the equation above, that is, $\bar{J}_{\pi^*} = \bar{J}^*$. For convenience, we summarize the key notations introduced in this section:

$E(A)$	: mean of the number of customer order arrivals during the unit time
$E(P)$	: mean of production time P
$E(L)$	: mean of replenishment lead time L
x_1	: number of outstanding customer orders
x_2	: number of items in inventory
h_1	: holding cost per unit time for each outstanding customer order
h_2	: holding cost per unit time for each item in inventory
C_S	: replenishment setup cost
C_R	: customer order rejection cost
M	: limited outstanding order level
Q	: replenishment quantity

4. Marginal Impact of Cost Parameters on the Optimal Replenishment Policy

In this section, we put our attention to the model that has exponentially distributed customer order inter-arrival, production, and replenishment lead time processes. Kim [10] showed that there exists an optimal replenishment policy characterized by a variable reorder point which is a function of the number of outstanding customer orders (x_1), provided that $c_R \geq h_1/(1-\beta)$

where β is an interest (discount) rate. More specifically, the optimal policy is identified as follows:

(i) given x_1, there exists a reorder point, $r(x_1)$, such that it is optimal to order Q units of items if the inventory level, x_2, is equal to or less than $r(x_1)$, and

(ii) the reorder point, $r(x_1)$, is increasing as x_1 increases.

It is interesting to see that the optimal reorder point is expressed as the function of the customer order queue length. The reasoning behind this can be explained as follows. The demand rate on items in inventory is equal to the production rate μ when the queue is not empty but it is equal to the customer order arrival rate λ when it is empty. Hence, it is reasonable to believe that the reorder point becomes adjusted according to the customer order queue length, more precisely the system utilization. For example, the demand rate on items in inventory will be close to the production rate under a heavy traffic while it will be close to the customer order arrival rate under a light traffic condition.

This section is devoted to the sensitivity analysis of how the optimal reorder point, $r(x_1)$, defined in Kim [10], changes as a function of cost parameters. For this, we assume that all the process of customer order inter-arrival, production, and replenishment is exponentially distributed with mean λ^{-1}, μ^{-1}, and d^{-1}, respectively.

We now proceed to characterize the monotonicity of the optimal reorder point with respect to the rejection cost c_R, replenishment cost c_S, and inventory holding cost h_2. Consider two instances of the replenishment scheduling problem. To differentiate each other, we use symbol A and B in the first and second case, respectively, for system parameters and the optimal reorder point. We first show the monotonicity of $r(x_1)$ with respect to the rejection cost c_R.

Theorem 1 : *(i) Suppose that* $\lambda^A=\lambda^B, \mu^A=\mu^B$, $\mu^A=\mu^B, d^A=d^B, c_S^A=c_S^B, h_1^A=h_1^B$, and $h_2^A=h_2^B$. *If* $c_R^A < c_R^B$, *then*, $r^A(x_1)\le r^B(x_1)$ for *all* $x_1\ge 0$.
(ii) The optimal reorder point $r(x_1)$ is non-decreasing as c_R increases, provided that other parameters remain the same.

Proof : See the Appendix.

The explanation behind Theorem 1 is as follows. As long as other parameters remain the same, the increase in the cost of rejecting customer order will force the firm to keep more inventory because stockout can cause the outstanding customer order level to reach M. Hence, it is reasonable to expect that given the state (x_1, x_2), the optimal reorder point will be higher than before.

The monotonicity of $r(x_1)$ with respect to the replenishment cost c_S and inventory holding cost h_2 is also preserved in the following theorem.

Theorem 2
(i) The optimal reorder point $r(x_1)$ is non-increasing as c_S increases, provided that other parameters remain the same.
(ii) The optimal reorder point $r(x_1)$ is non-increasing as h_2 increases, provided that other parameters remain the same.

Proof : It can be shown using an argument similar to the one used in the proof of Theorem 1 and we omit the detailed proof.

The intuition behind the first part of Theorem 2 can be explained using a reasoning similar to the case c_R. As long as other parameters remain the same, the increase in replenishment setup cost will restrain the firm from replenishing inventory. Hence, it is likely that given the state (x_1, x_2), the optimal reorder point will be lowered. In a similar reasoning, the increase in inventory holding cost will push the firm to keep less inventory, which implies the lower optimal reorder point than before.

5. Monotonicity of System Parameters on the Optimal Average Cost

In this section, we show how the optimal average cost, defined in section 3, changes as a function of system parameters when the model has general (i.e., non-exponential) distributions for the customer order inter-arrival time, production time, and replenishment lead time. The first result is straightforward.

Theorem 3 : *If some or all of rejection cost c_R, holding cost $h_i (i=1,2)$, and replenishment setup cost c_R become(s) reduced, it will result in an equal or lesser expected cost, provided that both the original system and the new one adopt an optimal policy.*

Proof : If a new system has a rejection cost $c'_R < c_R$, one can couple the new system to the original by applying the policy π^* that was optimal for the original system to the new one as well. Along every sample path, the cost of the new system is not increased under π^*, therefore an optimal policy will perform at least as well. The same argument is applied to the costs other than a rejection cost. □

We now introduce the definition of stochastic order between two random variables (see Ch9 of Ross [14] for the detail). The random variable X is said to be stochastically larger than the random variable Y, denoted by $X \geq_{st} Y$, if $Pr(X > t) \geq Pr(Y > t)$ for all t where $Pr(X > t)$ is the probability that the random variable X is greater than t. In addition, if $X \geq_{st} Y$, it is known that $E(X) \geq E(Y)$ where $E(X)$ is the mean of the random variable X. The next two theorems give the marginal analysis of the optimal average cost with regard to production time and replenishment lead time, respectively.

Theorem 4 : *If the production time decreases from P to P′ such that $P \geq_{st} P'$, it will result in an equal or lesser expected cost, provided that both the original system and the new one adopt an optimal policy.*

Proof : Let π^* be an optimal replenishment policy for the original system and $F_P(F'_P)$ be the cumulative distribution function of $P(P')$. To show that there exists a policy π' for the new system which performs at least as well as the original system, we employ a coupling argument. Along any sample path w of the stochastic process, assume that the customer order inter-arrival time and replenishment lead time take the same realization in old system as in the new.

Consider the n^{th} production in the new system, which has production time realization denoted by $P'(n,w)$. For the old system, the corresponding production is coupled to the realization of the new system as follows. Let {U(n) : n∈N} denote an *i.i.d.* sequence of uniform distribution random variables on the interval [0, 1]. For n∈ N, let

$$P(n, w) = F_P^{-1}(U(n, w)),$$
$$P'(n, w) = F_{P'}^{-1}(U(n, w))$$

where $F^{-1}(y) \equiv \min\{x \in R : F(x) \geq y\}$. Then, $P'(n, w)$ and $P(n, w)$ respectively have distribution F_P and $F_{P'}$ (see Lemma 9.2.1 of Ross [14]). By Proposition 9.2.2 of Ross [14], $P'(n, w) \leq P(n, w)$ with probability one.

Now to make the evolution of the original system under π^* identical to that of the new system under π', we employ idling along each sample path w. Define the n^{th} idle period by $I'(n, w) = I'(n, w) = P(n, w) - P'(n, w) \geq 0$. Thus, along any sample path w, the policy π' can reconstruct the state of the original system under π^* and generate equal performance. Because π' may not necessary be optimal under the new system, the optimal policy in the new system will perform at least as well π'. □

Theorem 5 : *If the replenishment lead time decreases from L to L′ such that $L \geq_{st} L'$, it will result in an equal or lesser expected cost, provided that both the original system and the new one adopt an optimal policy.*

Proof : As in Theorem 4, a coupling argument can be used here. We present a sketch of the proof instead of the detail one. To mimic the behavior of the original system in the one, we employ the postponement of replenishment order. Let τ_n be the time epoch that the n^{th} replenishment order is placed under π^* in the original system. By placing a replenishment order at $\tau_n + L(n, w) - L'(n, w)$ in the new system, both systems receive the n^{th} order at $\tau_n + L(n, w)$. Therefore, both systems have the same state realization along any sample path. □

Theorem 6 : Suppose that the customer order arrival process follows a Poisson distribution with rate λ will result in an equal or lesser expected cost, provided that both the original system and the new *one adopt an optimal policy.*

Proof : Suppose that in the new system, the customer order arrival rate is decreased to $\lambda' = \lambda/a$, $1 < a < \infty$. For the sake of the discussion, we define system 3, denoted by the double prime ($''$) symbol, in which $\lambda'' = \lambda/a$, $P''(n,w) = aP(n, w)$, and $L''(n, w) = aL(n, w)$, $n \in N$.

It is convenient to think that the unit time scale of system 3 is increased as a times as that of the original system. Then, we see that the stochastic process of system 3 becomes equivalent to that of the original system.

Now we set $h_i'' = h_i$, $i = 1, 2$, $c_R'' = a c_R$, and $c_S'' = a c_S$. Then, the original system and system 3 have the equal average cost per unit time. Note that the average customer order queue length and inventory level remain the same in both systems and the number of customer order rejections and replenishment setups are reduced to $1/a$ in system 3.

From the above discussion, to compare the original system with the new one becomes equivalent to compare system 3 with the new system. It is obvious that compared with System 3, the new system has a reduced production time and replenishment lead time, that is, $P''(n, w) \geq_{st} P'(n, w)$ and $L''(n, w) \geq_{st} L'(n, w)$. Therefore, if we apply Theorem 4 and 5 here, the new system has a lesser cost than system 3, and thus the original system. □

The only parameter of the model not treated in Theorem 1-6 is the limited outstanding order level, M. Unfortunately, a moment of reflection indicates that a decrease in M increases the steady state probability of buffer overflow and hence increases the rejection penalties. Thus, a blanket monotonicity property does not apply to the parameter M. A thorough study of the optimal selection of M is beyond the scope of this paper, which focuses on how best to use available production capacity. It is natural to investigate whether or not the optimal cost per unit time is a convex function of M, which is a useful property in searching for an optimal M^*. Our numerical investigations suggest that such is the case. The rejection costs are proportional to the probability of overflow, which we believe is geometrically decreasing in M.

6. Conclusions

In this paper, we studied a make-to-order inventory-production system with a limited commitment on outstanding order level. We treated the model with general distributions about customer order inter-arrival, production, and replenishment lead time processes. We established monotonicities of distribution parameters on the optimal average cost based on sample path arguments. In particular, when relevant time processes follow an exponential distribution, we show that the optimal reorder point defined in Kim [10] is monotonically changed as each of system costs except for outstanding order holding cost changes using a dynamic programming approach.

Further research is necessary to develop a more complete characterization of the optimal policy for the model with general distributions and design a numerical algorithm to approximate an optimal value of limited outstanding order level. Future research should also address make-to-order inventory-production systems in which multiple customer group exists and inventory rationing occurs.

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Appendix

Proof of Theorem 1 : With the assumption of exponential distributions of customer order inter-arrival, production, and lead time processes, the optimization problem described in Section 2 can be converted into the following discrete Markov decision problem (MDP). We omit the detailed step of deriving this MDP formulation. The reader is referred to Kim [10] for it.

Let $g(x_1, x_2)=\sum_{i=1}^{2} h_i x_i+\lambda c_R 1\{x_1=M\}$, which is the expected cost per unit time where the indicator function $1\{a\}$ = 1 if a is true and 0 if not. Denote $I(x_1, x_2)=(x_1+1, x_2)$ if $x_1<M$; (x_1, x_2) otherwise, and $D(x_1, x_2)=(x_1-1, x_2-1)$ if $x_1>0$ and $x_2>0$; (x_1, x_2) otherwise. The operator I and D represent a state transition corresponding to customer order arrival and production completion event, respectively. Then, the optimal cost function J satisfies the following optimality equation :

$$TJ(x_1, x_2, \delta)=\begin{cases} \min\{T_U J(x_1, x_2, \delta),\ T_P J(x_1, x_2, \delta)\} & \text{if } \delta=0 \\ T_u J(x_1, x_2, \delta) & \text{if } \delta=1 \end{cases}$$

where

$$T_u J(x_1, x_2, \delta)=\begin{cases} g(x_1, x_2)+\lambda J(I(x_1, x_2), \delta)+\mu J(D(x_1, x_2), \delta)+dJ(x_1, x_2, \delta) & \text{if } \delta=0 \\ g(x_1, x_2)+\lambda J(I(x_1, x_2), \delta)+\mu J(D(x_1, x_2), \delta)+dJ(x_1, x_2+Q, \delta-1) & \text{if } \delta=1, \end{cases}$$

$$T_P J(x_1, x_2, , 0)=K+T_u J(x_1, x_2, 1),$$

where δ is an indicator variable such that δ = 0 means no order is in process while δ = 1 implies an order is in process. T_u and T_p are value iteration operators corresponding to *Do not order* and *Order* actions, respectively.

We now introduce the following incremental costs of J: $\triangle_1 J(x_1, x_2, \delta)\equiv J(x_1+1, x_2, \delta)-J(x_1, x_2, \delta)$, $\triangle_2 J(x_1, x_2, \delta)\equiv J(x_1, x_2+1, \delta)-J(x_1, x_2, \delta)$, and $\triangle_{11} J(x_1, x_2, \delta)\equiv J(x_1+1, x_2+1, \delta)-J(x_1, x_2, \delta)$. To differentiate the optimal cost function for two instances, we use symbol A and B in the first and second case, respectively. We express the optimal policy of the first instance as (u^A/p^A) where $u^A(p^A)$ implies the optimal policy of the first instance is *Do not order (Order)*. (u^B/p^B) can be similarly defined. Much of our analysis follows the framework of Ha [11].

We first prove (i) of Theorem 1. Consider the following functional properties established by J^A and J^B:

$$J^B(x_1, x_2, 1)-J^B(x_1, x_2, 0) \leq J^A(x_1, x_2, 1)-J^A(x_1, x_2, 0), \quad (1)$$

$$J^B(x_1, x_2, +Q, 0)-J^B(x_1, x_2, 1) \leq J^A(x_1, x_2, +Q, 0)-J^A(x_1, x_2, 1), \quad (2)$$

$$\triangle_{11} J^B(x_1, x_2, \delta) \geq \triangle_{11} J^A(x_1, x_2, \delta),\ \delta=0,1, \quad (3)$$

$$\triangle_{11} J^B(x_1, x_2, 1) \geq \triangle_{11} J^A(x_1, x_2, 0), \quad (4)$$

$$\triangle_{11} J^B(x_1, x_2+Q, 0) \geq \triangle_{11} J^A(x_1, x_2, 0), \quad (5)$$

$$\triangle_2 J^B(x_1, x_2, \delta) + c_R^B \geq \triangle_2 J^A(x_1, x_2, \delta) + c_R^B, \delta = 0,1. \quad (6)$$

$$\triangle_1 J^B(x_1, x_2, \delta) - c_R^B \leq \triangle_1 J^A(x_1, x_2, \delta) - c_R^B, \delta = 0,1. \quad (7)$$

We first prove the following lemma :

Lemma 1 If (1)~(7) hold,

$$T_P J^B(x_1, x_2, 0) - T_U J^B(x_1, x_2, 0) \leq T_P J^A(x_1, x_2, 0) - T_U J^A(x_1, x_2, 0). \quad (8)$$

Proof:

$$\begin{aligned} & T_P J^B(x_1, x_2, 0) - T_U J^B(x_1, x_2, 0) - T_P J^A(x_1, x_2, 0) - T_U J^A(x_1, x_2, 0). \\ & = \lambda[J^B(I(x_1, x_2),1) - J^B(I,(x_1, x_2),0) - (J^A(I(x_1, x_2),1) - J^A(I(x_1, x_2),0))] \\ & \quad + \mu[J^B(D(x_1, x_2),1) - J^B(D(x_1, x_2),0) - J^A(D(x_1, x_2),1) - J^A(D(x_1, x_2),0))] \\ & \quad + d[J^B(x_1, x_2)+Q,0) - J^B(x_1, x_2),0) - J^A(x_1, x_2)+Q,0) - J^A(x_1, x_2),0))] \leq 0. \end{aligned}$$

(1) is applied to μ and λ terms. The non-negativity of d term follows by (1) and (2). □

We now prove $r^A(x_1) \leq r^b(x_1)$ using contradiction. Suppose $r^A(x_1) > r^b(x_1)$. Then, we have $T_U J^A(r^A(x_1), x_2, 0) > T_P J^A(r^A,(x_1), x_2, 0)$ and $T_U J^B(r^A(x_1), x_2, 0) < T_P J^B(r^A(x_1), x_2, 0)$. It follows that $T_p J^B(r^A(x_1), x_2, 0) - T_u J^B(r^A(x_1), x_2, 0)$ and $T_u J^A(r^A(x_1), x_2, 0) - T_U J^A(r^A(x_1), x_2, 0)$, which is a contradiction by (8).

To complete the proof of this theorem, we show that (1)-(7) are preserved under T.

(i) $W = TJ^B(x_1, x_2, 1) - TJ^B(x_1, x_2, 0) - (TJ^A(x_1, x_2, 1) - TJ^A((x_1, x_2, 0)) \leq 0$: We focus on admissible actions in $(x_1, x_2, 0)^B$ and $(x_1, x_2, 0)^A$. Case (u^B, p^A) is excluded by (8). For (p^B, p^A), W becomes zero. For (u^B, u^A),

$$W \leq TJ^B(x_1, x_2, 1) - T_P J^B(x_1, x_2, 0) - (TJ^A(x_1, x_2, 1) - T_P J^A(x_1, x_2, 0))(\text{by } (8)) = 0.$$

Case (p^B, u^A) can be shown using the result of case (u^B, u^A) because

$$W \leq TJ^B(x_1, x_2, 1) - T_U J^B(x_1, x_2, 1) - (TJ^A(x_1, x_2, 1) - T_U J^A(x_1, x_2, 0)) \leq 0.$$

(ii) $W = TJ^B(x_1, x_2 + Q, 0) - TJ^B(x_1, x_2, 1) - TJ^A(x_1, x_2 + Q, 0) - TJ^A(x_1, x_2, 1)) \leq 0$: We focus on admissible actions in $(x_1, x_2+Q, 0)^B$ and $(x_1, x_2+Q, 0)^A$. Case (u^B, p^A) is excluded by (8). For (u^B, u^A),

$$\begin{aligned} W = & \lambda[J^B(I(x_1, x_2+Q),0) - J^B(I(x_1, x_2),1) - (J^A(I(x_1, x_2+Q),0)) - J^A(x_1, x_2, 1)] \\ & + \mu[J^B(D(x_1, x_2+Q),0) - J^B(I(x_1, x_2, 1) - (J^A(I(x_1, x_2+Q),0)) - J^A(D(x_1, x_2, 1)] \\ & + d[J^B(x_1, x_2+Q, 0) - J^B(x_1, x_2+Q, 0) - (J^A(x_1, x_2+Q, 0) - J^A(x_1, x_2+Q, 0))] \leq 0. \end{aligned}$$

When $x_1 > 0$ and $x_2 = 0$ or when $x_1 = 0$ and $x_2 \geq 0$, μ term is non-negative by (2). When $x_1 > 0$ and $x_2 = 0$, it becomes

$$\begin{aligned} & J^B(x_1 - 1, Q-1, 0) - J^B(x_1, 0, 1) - (J^A(x_1 - 1, Q-1, 0) - J^A(x_1, 0, 1)) \\ & \leq J^B(x_1, Q, 0) - J^B(x_1, 0, 1) - (J^A(x_1, Q, 0)) - J^A(x_1, 0, 1) \ (\text{by}(3)) \leq 0 (\text{by}(2)). \end{aligned}$$

The λ term is non-negative by (2) and the d term is canceled out. For case (p^B, p^B),

$W \leq T_U J^B(x_1, x_2+Q,0) - TJ^B(x_1, x_2+1) - (T_u J^A(x_1, x_2+Q,0) - TJ^A(x_1, x_2,1))$ (by (8))

≤ 0 (by case (u^B,u^A)).

Case (p^B,u^A) can be shown using the result of case (p^B,p^A) bacause

$W \leq T_P J^B(x_1, x_2+Q,0) - TJ^B(x_1, x_2,+1) - (T_P J^A(x_1, x_2+Q,0) - TJ^A(x_1, x_2,1)) \leq 0.$

(iii) $W = \triangle_{11} TJ^B(x_1, x_2, \delta) - \triangle_{11} TJ^A(x_1, x_2, \delta) \geq 0$: Suppose $\delta = 1$.Then,

$W = \lambda[\triangle_{11} J^B(I(x_1, x_2),1) - \triangle_{11} J^A(I(x_1, x_2),1)] + \mu[\triangle_{11} J^B(D(x_1, x_2)),1) - \triangle_{11} J^A(D(x_1, x_2),1)]1\{x_1>0, x_2>0\} + d[\triangle_{11} J^B(x_1, x_2+Q,0) - \triangle_{11} J^A(x_1, x_2+Q,0))] \geq 0.$

μ and d terms follows by (3), λ term follows by (3) if x_1<M-1. If x_1=M-1, it becomes $\lambda c_R^B - \lambda c_R^B + \lambda[\triangle_2 J^B(M, x_2,1) - \triangle_2 J^A(M, x_2,1)] \geq 0$ by (6).

Suppose $\delta = 0$. We focus on admissible actions in states $(x_1+1, x_2+1,0)^B$ $(x_1, x_2,0)^B$, $(x_1+1, x_2+1,0)^A$, and $(x_1, x_2, 0)^A$. Using(8) and the monotonicity and threshold property of the optimal policy, the following 6 cases are feasible. Cases (u^B,u^B,u^A,u^A),and (p^B,p^B,p^A,p^A) be shown in a similar way used in proving case $\delta = 1$. For (p^B,p^B,u^A,u^A),

$W = \lambda[\triangle_{11} J^B(I(x_1, x_2),1) - \triangle_{11} J^A(I(x_1, x_2),0)] + \mu[\triangle_{11} J^B(D(x_1, x_2),1) - \triangle_{11} J^A(D(x_1, x_2),0)]$
$1\{x_1>0, x_2>0\} + d[\triangle_{11} J^B(x_1, x_2+Q,0) - \triangle_{11} J^A(x_1, x_2,0))] \geq 0$ (by (4) to μ and λ terms and (5) to d term)

For (u^B,p^B,u^A,p^A), $W \geq T_U J^B(x_1+1, x_2+1,0) - T_P J^B(x_1, x_2,0) - (T_U J^A(x_1+1, x_2+1,0) - T_P J^A(x_1, x_2,0))$ (by (8)) ≥ 0 (by case (u^B,u^B,u^A,u^A)).

Cases (u^B,p^B,u^A,u^A) and (p^B,$p^B$$u^A$$p^A$) can be shown using the result of(p^B,$p^B$$u^A$,$u^A$)

(iv) $\triangle_{11} TJ^B(x_1, x_2,1) \geq \triangle_{11} TJ^A(x_1, x_2,0)$:

$\triangle_{11} TJ^B(x_1, x_2,1) \geq \triangle_{11} TJ^B(x_1, x_2,0)$(by (3.8)) in Kim [10]) $\geq \triangle_{11} TJ^A(x_1, x_2,0)$ (by (iii)).

(v) $\triangle_{11} TJ^B(x_1, x_2+Q,0) \leq \triangle_{11} TJ^A(x_1, x_2,0)$:

$\triangle_{11} TJ^B(x_1, x_2+Q,0) \geq \triangle_{11} TJ^B(x_1, x_2,1)$(by(3.9)) in Kim [10]) $\geq \triangle_{11} TJ^A(x_1, x_2,0)$(by (iv)).

(vi) $W = \triangle_2 TJ^B(x_1, x_2, \delta) - \triangle_2 TJ^A(x_1, x_2, \delta) \geq c_R^A - c_R^B$: Suppose $\delta = 1$.Then,

$W = \lambda[\triangle_2 J^B(I(x_1, x_2), 1) - \triangle_2 J^A(I(x_1, x_2), 1)] + \mu[\triangle_2 J^B(D(x_1, x_2),1) - \triangle_2 J^A(D(x_1, x_2),1)]$
$+ d[\triangle_2 J^B(x_1, x_2+Q, 0) - \triangle_2 J^A(x_1, x_2+Q, 0))] \geq (\lambda + \mu + d)(c_R^A - c_R^B) \geq c_R^A - c_R^B$

(since $\lambda + \mu + d < 1$ and $c_R^A < c_R^B$).

The first inequality comes as follows. λ and d terms are equal to or greater than $c_R^A - c_R^B$ by (3), respectively. When either $x_1 > 0$ and $x_2 > 0$ or $x_1 = 0$ and $x_2 > 0$, μ term is equal to or greater than $c_R^A - c_R^B$ by (3). When $x_1 > 0$ and $x_2 = 0$, it becomes

$\triangle_1 J^A(D(x_1-1,0),1) - \triangle_1 J^B(D(x_1-1,0),1) \geq c_R^A - c_R^B$ (by (7)).

Suppose δ =0. We focus on admissible actions in states $(x_1, x_2+1,0)^B$, $(x_1, x_2,0)^B$, $(x_1, x_2+1,0)^A$ and

$(x_1, x_2, 0)^A$ Using (8) and the monotonicity and threshold property of the optimal policy, the following 6 cases are feasible. Case (u^B,u^B,u^A,u^A) and (p^B,p^B,p^A,p^A) be shown in a similar way used in proving case δ = 1. For (p^B,p^B,u^A,u^A),

$$W = \lambda[\triangle_2 J^B(I(x_1,x_2),1) - \triangle_2 J^A(I(x_1,x_2),0)] + \mu[\triangle_2 J^B(D(x_1,x_2),1) - \triangle_2 J^A(D(x_1,x_2),0)]$$
$$+d[\triangle_2 J^B(x_1,x_2+Q,0) - \triangle_2 J^A(x_1,x_2,0))] \geq \lambda[\triangle_2 J^B(I(x_1,x_2,0) - \triangle_2 J^A(I(x_1,x_2,0)]$$
$$+\mu[\triangle_2 J^B(D(x_1,x_2),0) - \triangle_2 J^A(D(x_1,x_2),0)] + d[\triangle_2 J^B(x_1,x_2,0) - \triangle_2 J^A(x_1,x_2,0))]$$ (by (3.3) and (3.5) in Kim) $\geq (\lambda+\mu+d)(c_R^B - c_R^A) \geq c_R^A - c_R^B$.

The second and third inequalities can be explained in the same way as in δ =1. For (u^B,p^B,u^A,p^A),

$W \geq TuJ^B(x_1+1, x_2+1, 0) - TuJ^B(x_1, x_2, 0) - (TuJ^A(x_1+1, x_2+1, 0) - TuJ^A(x_1, x_2, 0))$

(by (8)) $\geq c_R^A - c_R^B$ (by case (u^B,u^B,u^A,u^A)).

Cases (u^B,p^B,u^A,u^A) and (p^B,p^B,u^A,p^A) can be shown using the result of (p^B,p^B,u^A,u^A).

(vii) $W = \triangle_1 TJ^B(x_1, x_2, \delta) - \triangle_1 TJ^A(x_1, x_2, \delta) \leq c_R^B - c_R^A$: Suppose δ = 1. Then,

$$W = \lambda(c_R^B - c_R^A)1\{x_1=M-1\} + \lambda[\triangle_1 J^B(I(x_1,x_2,1) - \triangle_1 J^A(I(x_1,x_2,1)]1\{x_1 < M-1\}$$
$$+\mu[\triangle_1 J^B(D(x_1,x_2,1) - \triangle_1 J^A(D(x_1,x_2,1)] + d[\triangle_1 J^B(x_1,x_2,+Q,0) - \triangle_1 J^A(x_1,x_2,+Q,0))]$$
$$\leq (\lambda+\mu+d)(c_R^B - c_R^A) \leq c_R^A - c_R^B$$ (since $\lambda+\mu+d < 1$ and $c_R^B - c_R^A$).

The first inequality comes as follows. λ and d terms are equal to or less than $c_R^B - c_R^A$ by (7). respectively. When either X1>0 and $x_2 > 0$ or $x_1 \geq 0$ and $x_2 = 0$, μ term is equal to or less than $c_R^B - c_R^A$ by (7), When $x_1 = 0$ and $x_2 > 0$, it becomes $\triangle_2 J^A(D(0, x_2-1, 0), 1) - \triangle_2 J^B(D(0, x_2-1, 0), 1) \leq c_R^B - c_R^A$ (by (7)).

Suppose $\delta = 0$. We focus on admissible actions in states $(x_1+1, 0)^B$, $(x_1, x_2, 0)^B$, $(x_1+1, x_2, 0)^A$, and $(x_1, x_2, 0)^A$. Using (8) and the monotonicity and threshold property of the optimal policy, the following 6 cases are feasible. Case (u^B,u^B,u^A,u^A) and (p^B,p^B,p^A,p^A) be shown in a similar way used in proving case $\delta = 1$. For (p^B,p^B,u^A,u^A),

$$W = \lambda(c_R^B - c_R^A)1\{x_1=M-1\} + \lambda[\triangle_1 J^B(I(x_1,x_2,1) - \triangle_1 J^A(I(x_1,x_2,0)]1\{x_1 < M-1\}$$
$$+\mu[\triangle_1 J^B(D(x_1,x_2,1) - \triangle_1 J^A(D(x_1,x_2,0)] + d[\triangle_1 J^B(x_1,x_2+Q,0) - \triangle_1 J^A(x_1,x_2,0))]$$
$$\leq \lambda(c_R^B - c_R^A)1\{x_1=M-1\} + \lambda[\triangle_1 J^B(I(x_1,x_2,0) - \triangle_1 J^A(I(x_1,x_2,0)]1\{x_1 < M-1\}$$
$$+\mu[\triangle_1 J^B(D(x_1,x_2,0) - \triangle_1 J^A(D(x_1,x_2,0)] + d[\triangle_1 J^B(x_1,x_2,0) - \triangle_1 J^A(x_1,x_2,0))]$$ (by (3.2) and (3.4) in Kim) $\leq (\lambda+\mu+d)(c_R^B - c_R^A) \leq c_R^B - c_R^A$.

The second and third inequalities can be explained in the same way as in δ = 1.

For (p^B,u^B,p^A,u^A),

$W \leq TuJ^B(x_1+1, x_2+1, 0) - TuJ^B(x_1, x_2, 0) - (TuJ^A(x_1+1, x_2+1, 0) - TuJ^A(x_1, x_2, 0))$

(by (8)) $\leq c_R^B - c_R^A$ (by case (u^B,u^B,u^A,u^A)).

Cases (p^B,u^B,u^A,u^A) and (p^B,p^B,p^A,u^A) can be shown using the result of (p^B,p^B,u^A,u^A). □

The part (ii) of Theorem 1 directly follows from the recursive application of (i). □