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(Max, +)-대수를 이용한 3-노드 유한 버퍼 일렬대기행렬에서의 대기시간 분석*

서 동 원**

Application of (Max, +)-algebra to the Waiting Times in Deterministic 3-node Tandem Queues with Blocking*

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■ Abstract ■

In this paper, we consider characteristics of waiting times in single-server 3-node tandem queues with finite buffers, a Poisson arrival process and deterministic service times at all nodes. There are three buffers: one at the first node is infinite and the others are finite. We obtain the fact that sojourn time or departure process is independent of the capacities of the finite buffers and does not depend on the order of service times, which are the same results in the literature. Moreover, the explicit expressions of stationary waiting times in all areas of the systems can be derived as functions of the finite buffer capacities. We also disclose a relationship of waiting times in subareas of the systems between two blocking policies : communication and manufacturing. Some numerical examples are also provided.

Keyword : Tandem Queues, Finite Buffers, Waiting Times, (Max, +)-Linear Systems

1. Introduction

As common models of communication and manufacturing systems, stochastic networks with finite buffers have been widely investigated such as tandem queues, fork-join type queues, and so on. Since the computational complexity and difficulty are growing fast as the size and the ran-

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domness of system are growing, most researches in the literature are focused on very restrictive and small size of systems over the past decades. In literature readers can see several results for characteristics in infinite buffered stochastic systems, whereas there are no explicit results for finite buffered systems. For those stochastic systems, several approximation methods have been studied (see Seo [8] and the references therein). In our best knowledge, moreover, there are no results on characteristics in the subarea of stochastic networks with finite buffers.

Recently, more generous stochastic queueing networks which are called (max, +)-linear systems are studied. (Max, +)-linear systems cover various types of queueing networks which are prevalent in telecommunication, manufacturing and production systems. Various instances of (max, +)-linear systems can be represented by stochastic event graphs, a special type of stochastic Petri net. Petri nets allow one to analyze and model (max, +)-linear systems which are nonconcurrent (choice-free) nets, do not allow overtaking and consist of single server queues under FIFO service discipline. Complex discrete event systems (DESs) can be properly modeled by this method involving only 'max' and '+' operations, but unfortunately it is usually very hard to obtain closed form expressions for performance measures of these complex systems.

Baccelli and Schmidt [6] derived a Taylor series expansion for the expected value of stationary waiting times with respect to the arrival rate in Poisson driven (max, +)-linear systems. This expansion approach was generalized to other characteristics (such as higher order moments, Laplace transform, tail probability) of stationary waiting times and transient waiting times by Baccelli, Hasenfuss and Schmidt [4, 5], Ayhan and Seo [1, 2] and Seo [8].

Their results make ones be able to investigate certain properties of deterministic tandem queueing networks with a Poisson arrival process and blocking. Recently, Seo [8] derived explicit expressions for characteristics of stationary waiting times in all areas of 2-node tandem queues with a finite buffer under two blocking policies: communication and manufacturing. He also disclosed a relationship of stationary waiting times in all areas of the systems between the blocking policies. The methods used in Seo [8] are still valid for more complex (max, +)- linear systems. His results, however, may or may not be applicable to multi-node tandem queues having more than 2 nodes. This motivates our research, an extended version of his study. Similarly to Seo [8], the goal of this paper is deriving explicit expressions for characteristics of stationary waiting times in deterministic 3-node tandem queues with finite buffers. These expressions are functions of finite buffer capacities and are immediately applicable forms to compute the characteristics of stationary waiting times in all areas of the systems. To disclose any relationship of waiting times between the two blocking rules is also the aim of this research. Furthermore, those results will lay a cornerstone of analyzing for more complex (max, +)-linear systems such as multi-node tandem queues and fork-and-join type queueing networks and so forth.

Reader can refer on basic (max, +)-algebra to Baccelli et al. [3] and on some preliminaries on waiting times in (max, +)-linear systems to Seo [8] (see also [6, 7]), we omit them here. The paper is organized as follows. Section 2 contains our main results and some numerical examples are given in section 3. Conclusion and some future research topics are mentioned in Section 4.

2. Waiting Times in 3-node Tandem Queues with Deterministic Service Times

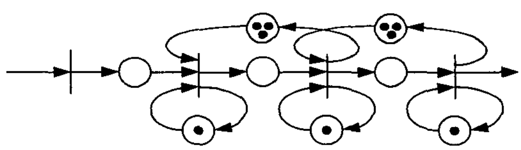
In this study, we investigate on waiting times in single-server 3-node tandem queues with finite buffers, a Poisson arrival process and deterministic service times. The system has three buffers: one at the first node is infinite and the others are finite.

Let σ^i and K_i be the deterministic service time and the capacity of buffer at node i ($i=1,2,3$). The buffer capacities include a room for a customer in service. We first mention about the waiting times in 3-node tandem queues with infinite buffers ($K_1=K_2=K_3=\infty$). From the definition of random vector D_n, one can obtain the expressions of the components of D_n as

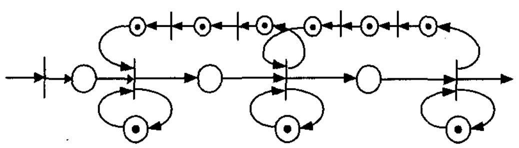
$$D_n^1 = n\sigma^1 \quad for \quad n \geq 0,$$
$$D_n^2 = \sigma^1 + n\max\{\sigma^1, \sigma^2\} \quad for \quad n \geq 0,$$
$$D_n^3 = \sigma^1 + \sigma^2 + n\max\{\sigma^1, \sigma^2, \sigma^3\} \quad for \quad n \geq 0. \quad (2.1)$$

For 3-node tandem queues with finite buffers, we consider waiting times under two blocking policies : communication and manufacturing. Under communication blocking a customer at node j cannot begin his service unless there is a vacant space in the buffer at node $j+1$. For manufacturing blocking, a customer served at node j moves to node $j+1$ only if the buffer of node $j+1$ is not full;otherwise the blocked customer stays in node j until a vacancy is available. During that time, node j is blocked from serving other customers.

One can obtain recursive equations for stationary waiting times in (max, +)-linear systems with finite buffers by using the same way as done in Seo [8], which draws a corresponding event graph and then converts it to an event graph with infinite buffers by inserting dummy nodes with zero service times. For example, the following [Figure 1] shows the event graph of a 3-node tandem queue with finite buffers of size 3 at node 2 and 3 while [Figure 2] shows the event graph of the one with infinite buffers at all nodes by inserting dummy nodes having zero service times.



[Figure 1] 3-node tandem queue with finite buffers of size 3 at node 2 and 3



[Figure 2] 3-node tandem queue with infinite buffers and dummy nodes

We assume that we are interested in the cases : $K_1 = \infty$, $3 \leq K_2$, $K_3 \leq \infty$ for communication blocking and $K_1 = \infty$, $2 \leq K_2$, $K_3 \leq \infty$ for manufacturing blocking. For other cases explicit expressions for the components of D_n can be easily obtained so that we omit them here. Similarly as done in the infinite buffer case, one is able to obtain the following explicit expressions for the components of the random vector D_n as functions of finite buffer capacities K_2 and K_3 after some (tedious) algebra.

Proposition 1 : Under communication blocking, when $K_2 \geq 3, K_3 \geq 3$,

$$D_n^1 = n\sigma^1 \ for\ 0 \leq n < K_2,$$
$$D_n^1 = \max\{n\sigma^1, \sigma^1 + (n - K_2 + 1)\sigma^2\}$$
$$for\ K_2 \leq n < K_2 + K_3,$$
$$D_n^1 = \max\{n\sigma^1, \sigma^1 + (n - K_2 + 1)\sigma^2, \sigma^1 + 2\sigma^2 + [n - (K_2 + K_3) + 1]\sigma^3\}\ for\ n \geq K_2 + K_3,$$
$$D_n^2 = \sigma^1 + \max\{n\sigma^1, n\sigma^2\}\ for\ 0 \leq n < K_3, \quad (2.2)$$
$$D_n^2 = \sigma^1 + \max\{n\sigma^1, n\sigma^2, \sigma^2 + (n - K_3 + 1)\sigma^3\}$$
$$for\ n \geq K_3, \quad (2.3)$$
$$D_n^3 = \sigma^1 + \sigma^2 + n\max\{\sigma^1, \sigma^2, \sigma^3\}\ for\ n \geq 0. \quad (2.4)$$

For systems with manufacturing blocking, the explicit expressions for random vector D_n as functions of K_2 and K_3 are given the following Proposition.

Proposition 2 : Under manufacturing blocking, when $K_2 \geq 2, K_3 \geq 2$,

$$D_n^1 = n\sigma^1 \ for\ 0 \leq n \leq K_2,$$
$$D_n^1 = \max\{n\sigma^1, \sigma^1 + (n - K_2)\sigma^2\}$$
$$for\ K_2 < n \leq K_2 + K_3,$$
$$D_n^1 = \max\{n\sigma^1, \sigma^1 + (n - K_2)\sigma^2, \sigma^1 + \sigma^2 + [n - (K_2 + K_3)]\sigma^3\}\ for\ n > K_2 + K_3,$$
$$D_n^2 = \sigma^1 + \max\{n\sigma^1, n\sigma^2\}\ for\ 0 \leq n \leq K_3, \quad (2.5)$$
$$D_n^2 = \sigma^1 + \max\{n\sigma^1, n\sigma^2, \sigma^2 + (n - K_3)\sigma^3\}$$
$$for\ n > K_3, \quad (2.6)$$
$$D_n^3 = \sigma^1 + \sigma^2 + n\max\{\sigma^1, \sigma^2, \sigma^3\}\ for\ n \geq 0. \quad (2.7)$$

From the above expressions, one is able to disclose three facts. One is that the expressions of D_n^3 for all $n \geq 0$ in deterministic tandem queues with finite buffer capacities under both blocking mechanisms are the same as those in the systems with infinite buffer capacities (see (2.1), (2.4) and (2.7)). It shows the same result in Wan and Wolff [9] that when the first node's buffer capacity is infinite, a customer's sojourn time is not dependent of the finite buffer capacities and the order of nodes (service times) does not affect the sojourn time of a customer (see also Whitt [10]).

The second one is that waiting times at node 2 in systems with manufacturing blocking have the same expressions in systems with communication blocking, except for one difference in the value of the finite buffer capacity at node 3. In other words, by substituting K_3 in (2.2) and (2.3) for $K_3 + 1$ can derive the same expressions for waiting times at node 2 as (2.5) and (2.6). It means that waiting times at the second node in deterministic 3-node tandem queues under manufacturing blocking with K_3 buffer at the second node are the same as those in the system under communication blocking with $K_3 + 1$ buffer.

The third one is that waiting times in all subareas under manufacturing blocking are always smaller than or equal to those under communication blocking in systems with equal buffer capacities since all components of D_n under each blocking policy are nonincreasing in K_2 and K_3, and thus W^i is also stochastically nonincreasing in K_2 and K_3 (see equations (2.2) and (2.3) in Seo [8]). Therefore, we can conclude the following Theorem.

Theorem 1 : In a Poisson driven deterministic 3-node tandem queue with D_n^i satisfying the structure given in (2.5) in Seo [8], when $K_1 = \infty$ and $3 \leq K_i \leq \infty$ ($i = 2, 3$), then

$E[G(W^i_{Communication\ Blocking\ with\ K_2, K_3+1})] =$
$E[G(W^i_{Manufacturing\ Blocking\ with\ K_2, K_3})]$ for $i = 2, 3$
and
$E[G(W^i_{Communication\ Blocking\ with\ K_2, K_3})] \geq$
$E[G(W^i_{Manufacturing\ Blocking\ with\ K_2, K_3})]$ for $i = 1, 2, 3$

where $G(\cdot)$ is an integrable, nonnegative, and differentiable function defined in (2.4) in Seo [8].

Remark : when $K_2 = \infty$ or σ^3 is not a maximum service time, no blocking occurs at the second node. Then, the effect of K_3 disappears from the explicit expression of D^i_n, ($i = 1, 2$) which will become a function of K_2. Then, this Theorem shows the same results in Seo [8].

The following Lemma shows the explicit expressions of D^i_n for waiting times at node i, W^i (the time interval from the arrival until the beginning of the service at node i) which allows us to use Theorem 1 for computing characteristics of waiting times. Now, we only consider explicit expressions of D^i_n under communication blocking since we already know the relationship of waiting times under two blocking policies so that explicit expressions of waiting times under manufacturing blocking policy can be derived by the similar way.

Lemma : when $K_2, K_2 \geq 3$

i. if $\sigma^1 \geq \sigma^2 \geq \sigma^3$ or $\sigma^1 \geq \sigma^3 \geq \sigma^2$, then

$$D^1_n = n\sigma^1 \ for\ n \geq 0, D^2_n = (n+1)\sigma^1 \ for\ n \geq 0, \quad (2.8)$$

ii. if $\sigma^2 \geq \sigma^1 \geq \sigma^3$ or $\sigma^2 \geq \sigma^3 \geq \sigma^1$ then

$$\begin{aligned} D^1_n &= n\sigma^1 && for\ \ 0 \leq n < \xi \\ D^1_n &= \sigma^1 + (n - K_2 + 1)\sigma^2 && for\ \ n = \xi \\ D^1_n &= D^1_\xi + (n - \xi)\sigma^2 && for\ \ n \geq \xi + 1 \end{aligned} \quad (2.9)$$

$$D^2_n = \sigma^1 + n\sigma^2 \qquad for\ \ n \geq 0, \quad (2.10)$$

where an integer $\xi = \left\lceil 1 + (K_2 - 2)\frac{\sigma^2}{\sigma^2 - \sigma^1} \right\rceil$, $\lceil x \rceil$ is the smallest integer greater than or equal to x.

iii. if $\sigma^3 \geq \sigma^1 \geq \sigma^2$, then

$$\begin{aligned} D^1_n &= n\sigma^1 && for\ 0 \leq n < \xi_1 \\ D^1_n &= \sigma^1 + 2\sigma^2 + [n - (K_2 + K_3) + 1]\sigma^3 && for\ \ n = \xi_1 \\ D^1_n &= D^1_{\xi_1} + (n - \xi_1)\sigma^3 && for\ \ n \geq \xi_1 + 1 \end{aligned} \quad (2.11)$$

$$\begin{aligned} D^2_n &= (n+1)\sigma^1 && for\ 0 \leq n < \xi_2 \\ D^2_n &= \sigma^1 + \sigma^2 + (n - K_3 + 1)\sigma^3 && for\ \ n = \xi_2 \\ D^2_n &= D^2_{\xi_2} + (n - \xi_2)\sigma^3 && for\ \ n \geq \xi_2 + 1 \end{aligned} \quad (2.12)$$

where an integer $\xi_i = \lceil x_i \rceil$, and $x_1 = 1 + (K_2 + K_3 - 2)\frac{\sigma^3 - 2\sigma^2}{\sigma^3 - \sigma^1}$ and $x_2 = (K_3 - 1)\frac{\sigma^3 - \sigma^2}{\sigma^3 - \sigma^1}$

iv. if $\sigma^3 \geq \sigma^2 \geq \sigma^1$, then

$$\begin{aligned} D^1_n &= n\sigma^1 && for\ 0 \leq n < \zeta \\ D^1_n &= \sigma^1 + (n - K_2 + 1)\sigma^2 && for\ \ n = \zeta \\ D^1_n &= D^1_\zeta + (n - \zeta)\sigma^2 && for\ \zeta + 1 \leq n < \xi_1 \\ D^1_n &= \sigma^1 + 2\sigma^2 + [n - (K_2 + K_3) + 1)]\sigma^3 && for\ \ n = \xi_1 \\ D^1_n &= D^1_{\xi_1} + (n - \xi_1)\sigma^2 && for\ \ n \geq \xi_1 + 1 \end{aligned} \quad (2.13)$$

$$\begin{aligned} D^2_n &= \sigma^1 + n\sigma^2 && for\ 0 \leq n < \xi_2 \\ D^2_n &= \sigma^1 + \sigma^2 + (n - K_3 + 1)\sigma^3 && for\ \ n = \xi_2 \\ D^2_n &= D^1_{\xi_2} + (n - \xi_2)\sigma^3 && for\ \ n \geq \xi_2 + 1 \end{aligned} \quad (2.14)$$

where an integer $\zeta = \left\lceil 1 + (K_2 - 2)\frac{\sigma^2}{\sigma^2 - \sigma^1} \right\rceil$, $\xi_i = \lceil x_i \rceil$, $x_1 = (K_2 + 1) + (K_3 - 2)\frac{\sigma^3}{\sigma^3 - \sigma^2}$ and $x_2 = 1 + (K_3 - 2)\frac{\sigma^3}{\sigma^3 - \sigma^2}$.

Proof : In 3-node tandem queues with deterministic service times there exist 6 exclusive cases. For each case, one can determine the maximum values of D_n^i and ξ_i given in (2.5) in Seo [8]. When $\sigma^1 \geq \sigma^2 \geq \sigma^3$ or $\sigma^1 \geq \sigma^3 \geq \sigma^2$, no blocking occurs at all nodes of the system. So, expressions in (2.8) from Proposition 1 can be obtained easily. When $\sigma^2 \geq \sigma^1 \geq \sigma^3$ or $\sigma^2 \geq \sigma^3 \geq \sigma^1$, blocking only occurs at the first node but no blocking happens thereafter. In this case, one can determine $D_n^2 = \sigma^1 + n\sigma^2, n \geq 0$ in (2.10) and a unique value $x = 1 + (K_2 - 2)\frac{\sigma^2}{\sigma^2 - \sigma^1}$ for $n \geq x$ such that $(n-1)\sigma^1 \leq (n - K_2 + 1)\sigma^2$. Then, letting $\xi = \lceil x \rceil$ shows the expressions of D_n^1 given in (2.9). Also, we know $x \geq K_2$ because

$$x = 1 + (K_2 - 2)\frac{\sigma^2}{\sigma^2 - \sigma^1} > 1 + (K_2 - 2) = K_2 - 1.$$

Similarly, one can obtain the expressions of D_n^i, $i = 1, 2$, in (2.11)~(2.14) for other cases. We omit the detail of the proof.

Now we are ready to use Theorem 1 to compute moments of waiting times in a Poisson driven 3-node tandem queues with finite buffers. Note that when $\xi = 0$ in (2.5) in Seo [8], it becomes a simple M/D/1 queue with arrival rate λ and constant service time a_i. Some numerical examples are given in the next section to illustrate our results.

3. Examples

To illustrate our results we consider a deterministic 3-node tandem queue with blocking. Let $\sigma^1 = 1$, $\sigma^2 = 3$, $\sigma^3 = 5$ be the constant service times at each node and $K_2 = 5, K_3 = 5$. In this particular example, the maximum of service times (Lyapunov maximum value) a is 5 and we assume that we are only interested in the mean value of waiting time W^i, the elapsed time from the arrival until the beginning of service at node i. From the explicit expressions of the random vector D_n^i together with Theorem 1 given in Seo [8] we are able to compute the exact value of mean waiting times. One can see that $\zeta = 6$, $\xi_1 = 14$ and $\xi_2 = 9$ under communication blocking policy and that $\zeta = 7, \xi_1 = 16$ and $\xi_2 = 11$ under manufacturing blocking policy. Therefore, the explicit expressions for D_n^i are given as followings:

- under communication blocking policy

$D_n^1 = n$ *for* $n = 0, \cdots, 5$,

$D_n^1 = 7 + 3(n-6)$ *for* $6 \leq n < 14$,

$D_n^1 = 32 + 5(n-14)$ *for* $n \geq 14$,

and

$D_n^2 = 1 + 3n$ *for* $n = 0, \cdots, 8$,

$D_n^2 = 29 + 5(n-9)$ *for* $n \geq 9$.

- under manufacturing blocking policy

$D_n^1 = n$ *for* $n = 0, \cdots, 6$,

$D_n^1 = 7 + 3(n-7)$ *for* $7 \leq n < 16$,

$D_n^1 = 34 + 5(n-16)$ *for* $n \geq 16$,

and

$D_n^2 = 1 + 3n$ *for* $n = 0, \cdots, 10$,

$D_n^2 = 34 + 5(n-11)$ *for* $n \geq 11$.

<Table 1> and <Table 2> show exact and simulation values of the expected waiting times (just before the beginning of service) at node 1 and 2 for various values of traffic intensity.

From the numerical results, we can see that our expressions of D_n^i are accurate and that the mean values of waiting times under manufacturing blocking are smaller than or equal to those under communication blocking.

〈Table 1〉 Waiting Times under Communication Blocking with $K_1=\infty, K_2=5, K_3=5$

Traffic Intensity ρ	$E(W^1)$		$E(W^2)$	
	Exact Solution	Simulation	Exact Solution	Simulation
0.1	0.01020	0.01041 ∓ 0.00122	1.09574	1.0921 ∓ 0.00749
0.2	0.02083	0.02013 ∓ 0.00137	1.20455	1.1980 ∓ 0.00808
0.5	0.05594	0.05445 ∓ 0.00175	1.65546	1.6417 ∓ 0.01754
0.8	0.47410	0.4861 ∓ 0.11565	4.20172	4.1143 ∓ 0.23868
0.9	4.75394	4.5113 ∓ 0.79959	12.79257	12.3161 ∓ 1.09133

〈Table 2〉 Waiting Times under Manufacturing Blocking with $K_1=\infty, K_2=5, K_3=5$

Traffic Intensity ρ	$E(W^1)$		$E(W^2)$	
	Exact Solution	Simulation	Exact Solution	Simulation
0.1	0.01020	0.01034 ∓ 0.00132	1.09574	1.0919 ∓ 0.00789
0.2	0.02083	0.02005 ∓ 0.00140	1.20455	1.1980 ∓ 0.00806
0.5	0.05560	0.05465 ∓ 0.00073	1.64603	1.6372 ∓ 0.00852
0.8	0.28457	0.3039 ∓ 0.05227	3.55408	3.5639 ∓ 0.13519
0.9	3.44201	3.6750 ∓ 0.46990	10.89937	11.1558 ∓ 0.66525

〈Table 3〉 Waiting Times under Communication Blocking with $K_1=\infty, K_2=5, K_3=6$

Traffic Intensity ρ	$E(W^1)$		$E(W^2)$	
	Exact Solution	Simulation	Exact Solution	Simulation
0.1	0.01020	0.00965 ∓ 0.00056	1.09574	1.0929 ∓ 0.00356
0.2	0.02083	0.02053 ∓ 0.00057	1.20455	1.2009 ∓ 0.00393
0.5	0.05583	0.05501 ∓ 0.00082	1.64603	1.6387 ∓ 0.00711
0.8	0.34415	0.3649 ∓ 0.05629	3.55408	3.5612 ∓ 0.12978
0.9	3.88841	4.1139 ∓ 0.47854	10.89937	11.1359 ∓ 0.65770

In addition, from <Table 2> and <Table 3> we can obtain the exactly same values of mean waiting times at node 2 for the systems with one difference buffer capacity at node 3 under manufacturing and communication blocking policies.

That is, $E(W^2_{Communication\ Blocking\ with\ 5,6}) = E(W^2_{Manufacturing\ Blocking\ with\ 5,5})$,

which is addressed in Theorem 1 of section 2.

4. Conclusion

In this study, we studied waiting times in Poisson driven 3-node tandem queues with finite buffers and deterministic service times, which is an extended version of the previous study (see [8]). Recursive expressions for waiting times in stochastic system with finite buffers under communication or manufacturing blocking rules can be obtained in (max, +)-algebra notation. From these explicit expressions we can show the following facts. One is that when the capacity of buffer at the first node is infinite, the system sojourn times or departure processes are independent of the capacities of finite buffers and do not depend on the order of nodes (service times), which is the same results as the previous studies. Moreover, we were able to derive a relationship on waiting times in the system with finite buffers under two blocking policies.

These results can be extended to more complex (max, +)-linear systems with finite buffers such as m-node tandem queues, fork-and-join type queues (a special case of tandem queues) and so on. Even though it is much difficult to derive an explicit expression on waiting times in all areas of the deterministic system with $K_1 = \infty$, $K_j < \infty (j = 2, \cdots, m)$, one may be able to find certain common patterns of the expressions of the random vector D_n^i under various types of blocking, which will be studied later. Furthermore, these results can be applied to a buffer allocation problem, an optimization problem to determine optimal buffer capacities satisfying predetermined probabilistic constraints on stationary waiting times at each node.

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