J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math.
Volume 12, Number 2 (May 2005), Pages 133–142

ISSN 1226-0657

HYERS-ULAM-RASSIAS STABILITY OF QUADRATIC FUNCTIONAL EQUATION IN THE SPACE OF SCHWARTZ TEMPERED DISTRIBUTIONS

JAEYOUNG CHUNG

ABSTRACT. Generalizing the Cauchy-Rassias inequality in [Th. M. Rassias: On the stability of the linear mapping in Banach spaces. *Proc. Amer. Math. Soc.* **72** (1978), no. 2, 297–300.] we consider a stability problem of quadratic functional equation in the spaces of generalized functions such as the Schwartz tempered distributions and Sato hyperfunctions.

1. INTRODUCTION

We consider the following quadratic functional equation and its staility in the spaces of distributions and hyperfunctions:

$$f(x+y)+f(x-y)-2f(x)-2f(y)=0. \tag{1.1}$$

The concept of stability for a functional equation arises when the equation (1.1) is replaced by an inequality which acts as a perturbation of the equation, *i. e.*,

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\|_{L^\infty}\le\varepsilon. \tag{1.2}$$

The stability question is that how do the solutions of the inequality (1.2) differ from those of equations (1.1).

The Hyers-Ulam stability of the quadratic functional equation was first proved by Cholewa [2] (see also Skof [17]).

Theorem 1.1 (Cholewa [2]). *Let $f : G \to E$ be a mapping from a group G to a Banach space E satisfying the inequality*

$$|f(x+y)+f(x-y)-2f(x)-2f(y)|\le\varepsilon \tag{1.3}$$

Received by the editors May 24, 2004 and, in revised form, May 17, 2005.
2000 *Mathematics Subject Classification.* 39B22, 47H15, 46F05.
Key words and phrases. Gauss transforms, Heat kernel, distributions, hyperfunctions, stability.

for all $x,\ y \in G$. Then there exists a unique quadratic function $q : G \to E$ such that

$$\|f(x) - q(x)\| \leq \frac{\varepsilon}{2} \tag{1.4}$$

for all $x \in E_1$. Here, a quadratic mapping $q : G \to E$ means that q satisfies the inequality (1.3) *for* $\varepsilon = 0$.

The above result was later extended by Czerwik [9].

Theorem 1.2 (Czerwik [9]). *Let $f : G \to E$ be a mapping from a group G to a Banach space E satisfying the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p), \quad p \neq 2 \tag{1.5}$$

for all $x,\ y \in G$. Then there exists a unique quadratic function $q : G \to E$ such that

$$\|f(x) - q(x)\| \leq \frac{2\varepsilon}{|2^p - 4|}\|x\|^p, \tag{1.6}$$

for all $x \in G$.

Recently, Theorem 1.1 was generalized to the spaces of Schwartz tempered distributions in Chung [3] with the reformulation

$$\|u \circ A + u \circ B - 2u \circ P_1 - 2u \circ P_2\| \leq \varepsilon. \tag{1.3$'$}$$

In this paper, following the same approach as in Chung [4] we generalize the above Theorem 1.2 for the case that p is an even integer greater than 4 in the spaces of generalized functions such as the space $\mathcal{S}'$ of Schwartz tempered distributions which is the dual space of the Schwartz space $\mathcal{S}$ of rapidly decreasing functions and the space $\mathcal{F}'$ of Fourier hyperfunctions which is the dual space of the Sato space $\mathcal{F}$ of analytic functions of exponential decay.

Note that the above inequalities (1.5) makes no sense in the spaces of generalized functions. As in Chung [4] making use of the tensor product and pullback of generalized functions we extend the inequality (1.5) in the spaces of generalized functions:

$$\|u \circ A + u \circ B - 2u \circ P_1 - 2u \circ P_2\| \leq \varepsilon(|x|^p + |y|^p), \tag{1.5$'$}$$

where $A(x,y) = x + y$, $B(x,y) = x - y$, $P_1(x,y) = x$, $P_2(x,y) = y$, $x, y \in \mathbb{R}^n$, and $u \circ A$, $u \circ B$, $u \circ P_1$ and $u \circ P_2$ are the pullbacks of u in $\mathcal{S}'$ or $\mathcal{F}'$ by A, B, P_1 and P_2, respectively. Also $|\cdot|$ denotes the Euclidean norm and the inequality $\|v\| \leq \psi(x,y)$ in (1.5$'$) means that $|\langle v, \varphi\rangle| \leq \|\psi\varphi\|_{L^1}$ for all test functions $\varphi(x,y)$ defined on $\mathbb{R}^{2n}$.

As a result, we prove that every solution u in $\mathcal{S}'$ or $\mathcal{F}'$ of the inequality (1.5′) satisfies

$$\|u - q(x)\| \leq \frac{2\varepsilon}{2^p - 4}|x|^p$$

for a unique quadratic form

$$q(x) := \sum_{1 \leq j \leq k \leq n} a_{jk}\, x_j x_k.$$

2. Distributions and hyperfunctions

We first introduce briefly some spaces of generalized functions such as the space $\mathcal{S}'$ of tempered distributions and the space $\mathcal{F}'$ of Fourier hyperfunctions which is a natural generalization of $\mathcal{S}'$. Here we use the multi-index notations for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ (where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is the set of non-negative integers).

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!, \quad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n},$$

where $\partial_j = \partial/\partial x_j$.

Definition 2.1 (J. Chung, S.-Y. Chung & Kim [5], Hörmander [10], Schwartz [16]). We denote by $\mathcal{S}$ or $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all infinitely differentiable functions φ in $\mathbb{R}^n$ such that

$$\|\varphi\|_{\alpha,\beta} = \sup_x |x^\alpha \partial^\beta \varphi(x)| < \infty \tag{2.1}$$

for all α, $\beta \in \mathbb{N}_0^n$, equipped with the topology defined by the seminorms $\|\cdot\|_{\alpha,\beta}$. The elements of $\mathcal{S}$ are called rapidly decreasing functions and the elements of the dual space $\mathcal{S}'$ are called *tempered distributions.*

As a matter of fact, it is known in [5] that (2.1) is equivalent to

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \varphi(x)| < \infty, \quad \sup_{\xi\in\mathbb{R}^n} |\xi^\beta \hat{\varphi}(\xi)| < \infty \tag{2.1′}$$

for all $\alpha, \beta \in \mathbb{N}_0^n$.

Imposing growth conditions on $\|\cdot\|_{\alpha,\beta}$ in (2.1) Sato and Kawai introduced the space $\mathcal{F}$ of test functions for the Fourier hyperfunctions as follows:

Definition 2.2 (Chung, Chung & Kim [6], Hörmander [10], Schwartz [16]). We denote by $\mathcal{F}$ or $\mathcal{F}(\mathbb{R}^n)$ the Sato space of all infinitely differentiable functions φ in

$\mathbb{R}^n$ such that

$$\|\varphi\|_{A,B} = \sup_{x,\alpha,\beta} \frac{|x^\alpha \partial^\beta \varphi(x)|}{A^{|\alpha|} B^{|\beta|} \alpha!\, \beta!} < \infty \tag{2.2}$$

for some positive constants A, B.

We say that $\varphi_j \to 0$ as $j \to \infty$ if $\|\varphi_j\|_{A,B} \to 0$ as $j \to \infty$ for some A, $B > 0$, and denote by $\mathcal{F}'$ the strong dual of $\mathcal{F}$ and call its elements *Fourier hyperfunctions.*

It is known in Chung, Chung & Kim [6] that the inequality (2.2) is equivalent to

$$\sup_{x \in \mathbb{R}^n} |\varphi(x)| \exp k|x| < \infty, \quad \sup_{\xi \in \mathbb{R}^n} |\hat{\varphi}(\xi)| \exp h|\xi| < \infty \tag{2.2$'$}$$

for some h, $k > 0$. It is easy to see the following topological inclusions:

$$\mathcal{F} \hookrightarrow \mathcal{S}, \quad \mathcal{S}' \hookrightarrow \mathcal{F}'.$$

From now on *a test function* means an element in the Schwartz space $\mathcal{S}$ or the Sato space $\mathcal{F}$ and *a generalized function* means *a tempered distribution* or *a Fourier hyperfunction.*

3. Main theorems

Let $E_t(x) = (4\pi t)^{-n/2} \exp(-|x|^2/4t)$, $t > 0$, be the n-dimensional heat kernel. It is easy to see that the *semigroup property* of the heat kernel

$$(E_t * E_s)(x) = E_{t+s}(x) \tag{3.1}$$

holds for convolution. This semigroup property will be very useful later. Let $u \in \mathcal{S}'$. Then its *Gauss transform*

$$\tilde{u}(x,t) = \langle u_y, E_t(x-y) \rangle, \quad x \in \mathbb{R}^n, \ t > 0$$

is well defined and is a smooth function in $\mathbb{R}^n \times (0,\infty)$ since $E_t(\cdot)$ belongs to the Schwartz space $\mathcal{S}$. Furthermore $\widetilde{u}(x,t) \to u$ as $t \to 0^+$ in $\mathcal{S}'$, that is, for every $\varphi \in \mathcal{S}$,

$$\langle u, \varphi \rangle = \lim_{t \to 0^+} \int \tilde{u}(x,t) \varphi(x)\, dx.$$

Throughout this paper we denote by $H_{2\gamma}$ the heat polynomial of degree 2γ with $|\gamma| > 2$, which is given by

$$H_{2\gamma}(x,t) = [\xi^{2\gamma} * E_t(\xi)](x) = (2\gamma)! \sum_{0 \le \alpha \le \gamma} \frac{t^{|\alpha|} x^{2\gamma - 2\alpha}}{\alpha!\, (2\gamma - 2\alpha)!}. \tag{3.2}$$

We first consider the Hyers-Ulam-Rassias stability of *quadratic-additive type* functional equation.

Lemma 3.1. *Let $f : \mathbb{R}^n \times (0, \infty) \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y, t+s) + f(x-y, t+s) - 2f(x,t) - 2f(y,s)| \le \Phi(x,y,t,s). \tag{3.3}$$

where

$$\Phi(x,y,t,s) = \varepsilon(H_{2\gamma}(x,t) + H_{2\gamma}(y,s)).$$

Then there exists a unique function $Q(x,t)$ satisfying the quadratic-additive functional equation

$$Q(x+y, t+s) + Q(x-y, t+s) - 2Q(x,t) - 2Q(y,s) = 0 \tag{3.4}$$

such that

$$\|f(x,t) - Q(x,t)\| \le 2\varepsilon \sum_{0 \le \alpha \le \gamma} a_\alpha t^{|\alpha|} x^{2\gamma - 2\alpha}, \tag{3.5}$$

for all $x \in \mathbb{R}^n$, $t > 0$, where

$$a_\alpha = (2\gamma)!\, 2^{|\alpha|}[\alpha!\,(2\gamma - 2\alpha)!\,(2^{|2\gamma|} - 2^{|\alpha|+2})]^{-1}, \quad |\alpha| < |\gamma|$$
$$a_\gamma = (2\gamma)!\, \gamma!^{-1}[(2^{|\gamma|-1} - 2)^{-1} + (2^{|\gamma|+1} - 4)^{-1}].$$

Proof. Let $F(x,t) = f(x,t) - f(0,t)$. Then we get the inequality

$$\begin{aligned}|F(x+y, t+s) + F(x-y, t+s) - 2F(x,t) - 2F(y,s)|& \\ \le \Phi(x,y,t,s) + \Phi(0,0,t,s)&\end{aligned} \tag{3.6}$$

Replacing both x and y by $x/2$, t and s by $t/2$ in (3.6) we have

$$\left|F(x,t) - 4F\Big(\frac{x}{2}, \frac{t}{2}\Big)\right| \le 2\varepsilon\left[H_{2\gamma}\Big(\frac{x}{2}, \frac{t}{2}\Big) + H_{2\gamma}\Big(0, \frac{t}{2}\Big)\right].$$

Making use of the induction argument and triangle inequality we have

$$\begin{aligned}\left|F(x,t) - 4^n F\Big(\frac{x}{2^n}, 2^{-n}t\Big)\right| &\le 2\varepsilon \sum_{k=1}^{n} 4^{k-1}\left[H_{2\gamma}\Big(\frac{x}{2^k}, \frac{t}{2^k}\Big) + H_{2\gamma}\Big(0, \frac{t}{2^k}\Big)\right] \\ &\le 2\varepsilon \sum_{0 \le \alpha \le \gamma} b_\alpha t^{|\alpha|} x^{2\gamma - 2\alpha}\end{aligned} \tag{3.7}$$

for all $x \in \mathbb{R}^n$, $t > 0$, where

$$b_\alpha = \begin{cases} (2\gamma)!\, 2^{|\alpha|}[\alpha!\,(2\gamma - 2\alpha)!\,(2^{|2\gamma|} - 2^{|\alpha|+2})]^{-1}, & |\alpha| < |\gamma| \\ 2(2\gamma)!\, 2^{|\alpha|}[\alpha!\,(2\gamma - 2\alpha)!\,(2^{|2\gamma|} - 2^{|\alpha|+2})]^{-1}, & \alpha = \gamma. \end{cases}$$

Replacing x, t by $x/2^m$, $t/2^m$, respectively in (3.7) and multiplying 4^m in the result it follows easily from the fact $|\gamma| > 2$ that

$$g_m(x,t) := 4^m F\Big(\frac{x}{2^m}, \frac{t}{2^m}\Big)$$

is a Cauchy sequence which converges locally uniformly. Now let

$$g(x,t) = \lim_{m\to\infty} g_m(x,t).$$

Then $g(x,t)$ is the unique mapping in $\mathbb{R}^n \times (0,\infty)$ satisfying

$$|F(x,t) - g(x,t)| \le 2\varepsilon \sum_{0\le\alpha\le\gamma} b_\alpha t^{|\alpha|} x^{2\gamma-2\alpha}, \tag{3.8}$$

$$g(x+y,t+s) + g(x-y,t+s) - 2g(x,t) - 2g(y,s) = 0 \tag{3.9}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. Replacing x, y, t, s by $x/2^m$, $y/2^m$, $t/2^m$, $s/2^m$ in (3.6), respectively, multiplying 4^m and letting $m \to \infty$, the inequality (3.9) follows immediately from the fact $|\gamma| > 2$.

On the other hand, putting $x = y = 0$ in (3.3) and dividing the result by 2 we have

$$|f(0,t+s) - f(0,t) - f(0,s)| \le \frac{1}{2}\Phi(0,0,t,s). \tag{3.10}$$

Replacing t, s by $t/2$ in (3.10) we have

$$|f(0,t) - 2f(0,t/2)| \le \varepsilon H_{2\gamma}(0,t/2).$$

By the induction argument we can easily verify that

$$h(t) := \lim_{m\to\infty} 2^m f(0,\, t/2^m)$$

is the unique function satisfying

$$h(t+s) = h(t) + h(s), \tag{3.11}$$

$$|f(0,t) - h(t)| \le \varepsilon(2\gamma)!\ [\gamma!\ (2^{|\gamma|} - 2)]^{-1} t^{|\gamma|} \tag{3.12}$$

for all $t, s > 0$.

Now let $Q(x,t) = g(x,t) + h(t)$. Then $Q(x,t)$ is the function satisfying (3.4) and (3.5).

Finally we prove the uniqueness of Q. Let $Q_0(x,t) = Q(x,t) - Q(0,t)$. Then $Q_0(x,t)$ also satisfies the quadratic–additive functional equation

$$Q_0(x+y,t+s) + Q_0(x-y,t+s) - 2Q_0(x,t) - 2Q_0(y,s) = 0. \tag{3.13}$$

Putting $y = 0$ in (2.19) we have

$$Q_0(x,t+s) = Q_0(x,t)$$

for all $x \in \mathbb{R}^n$, $t, s > 0$. Thus $Q_0(x,t)$ is independent of $t > 0$ and we may write $G_0(x,t) :\equiv Q_0(x)$. Since Q_0 satisfies the quadratic functional equation

$$Q_0(x+y) + Q_0(x-y) - 2Q_0(x) - 2Q_0(y) = 0,$$

and that

$$Q(rx, r^2t) = Q_0(rx) + Q(0, r^2t) = r^2 Q(x,t). \tag{3.14}$$

for all rational numbers r.

Now suppose that $Q^*(x,t)$ also satisfies (3.4) and (3.5). Then we have

$$\begin{aligned} |Q(x,t) - Q^*(x,t)| &= r^{-2}|Q(rx, r^2t) - Q^*(rx, r^2t)| \\ &\le 4\varepsilon\, r^{|2\gamma|-2} \sum_{0\le\alpha\le\gamma} a_\alpha\, t^{|\alpha|} x^{2\gamma-2\alpha}. \end{aligned}$$

Letting $r \to 0^+$ we have $Q = Q^*$. This completes the proof. □

Now we state and prove the main results of this paper.

Theorem 3.2. *Let $u \in \mathcal{S}'$ satisfy the inequality*

$$\|u \circ A + u \circ B - 2u \circ P_1 - 2u \circ P_2\| \le \varepsilon(x^{2\gamma} + y^{2\gamma}). \tag{3.15}$$

for some $\gamma \in \mathbb{N}_0^n, |\gamma| > 2$.

Then there exists a unique quadratic function

$$q(x) := \sum_{1\le j\le k\le n} a_{jk}\, x_j x_k$$

such that

$$\|u - q(x)\| \le \frac{2\varepsilon}{4^{|\gamma|} - 4} x^{2\gamma}. \tag{3.16}$$

Proof. Convolving in each side of (3.15) the tensor product $E_t(x)E_s(y)$ of n-dimensional heat kernels we have in view of the semigroup property (3.1).

$$\begin{aligned} [(u \circ A) * (E_t(\xi)E_s(\eta))](x,\, y) &= \left\langle u_\xi,\, \int E_t(x - \xi + \eta)E_s(y - \eta)\, d\eta \right\rangle \\ &= \langle u_\xi,\, (E_t * E_s)(x + y - \xi)\rangle \\ &= \tilde{u}(x + y, t + s). \end{aligned}$$

Similarly we have

$$\begin{aligned} &[(u \circ B) * (E_t(\xi)E_s(\eta))](x,\, y) = \tilde{u}(x - y, t + s), \\ &[(u \circ P_1) * (E_t(\xi)E_s(\eta))](x,\, y) = \tilde{u}(x, t), \\ &[(u \circ P_2) * (E_t(\xi)E_s(\eta))](x,\, y) = \tilde{u}(y, s), \end{aligned}$$

where $\tilde{u}(x,t)$ is the Gauss transform of u.

Thus the inequality (3.15) is converted to the stability problem of quadratic–additive type functional equation

$$|\tilde{u}(x+y,t+s)+\tilde{u}(x-y,t+s)-2\tilde{u}(x,t)-2\tilde{u}(y,s)| \le \Phi(x,y,t,s)$$

for $x,\ y\in\mathbb{R}^n$, $t,s>0$, where

$$\Phi(x,y,t,s)=\varepsilon(H_{2\gamma}(x,t)+H_{2\gamma}(y,s)).$$

By Lemma 3.1, there exists a unique function $Q(x,t)$ satisfying the quadratic-additive functional equation (3.4) such that

$$\|\tilde{u}(x,t)-Q(x,t)\| \le 2\varepsilon \sum_{0\le\alpha\le\gamma} a_\alpha\, t^{|\alpha|} x^{2\gamma-2\alpha}. \tag{3.17}$$

Since the Gauss transform $\tilde{u}$ a sooth function, $Q(x,t)$ is at least a continuous function as we see in the proof of Lemma 3.1. Thus the solution $Q(x,t)$ has the form Chung & Lee [7].

$$Q(x,t)=\sum_{1\le i\le j\le n} a_{ij}\,x_i\,x_j+bt.$$

Letting $t\to 0^+$ in (3.17) we get (3.16). This completes the proof. □

As a direct consequence of the above result we obtain the Hyers-Ulam-Rassias stability of quadratic functional equation.

Theorem 3.3. *Let $u\in\mathcal{S}'$ or $\mathcal{F}'$ satisfy the inequality*

$$\|u\circ A+u\circ B-2u\circ P_1-2u\circ P_2\| \le \varepsilon(|x|^p+|y|^p). \tag{3.18}$$

for some even integer $p>4$.

Then there exists a unique quadratic function

$$q(x):=\sum_{1\le j\le k\le n} a_{jk}\,x_j x_k$$

such that

$$\|u-q(x)\| \le \frac{2\varepsilon}{2^p-4}|x|^p, \tag{3.19}$$

Proof. Note that we can write for even integer p,

$$|x|^p=\sum_{|\gamma|=p/2}\frac{(p/2)!}{\gamma!}x^{2\gamma}.$$

Thus convolving in each side of (3.18) the tensor product $E_t(x)E_s(y)$ of n-dimensional heat kernels as a function of x, y the inequality (3.18) is converted to the following inequality as in the proof of Theorem 3.2

$$\|\tilde{u}(x+y,t+s)+\tilde{u}(x-y,t+s)-2\tilde{u}(x,t)-2\tilde{u}(y,s)\| \leq \varepsilon \sum_{|\gamma|=p/2} \frac{(p/2)!}{\gamma!}(H_{2\gamma}(x,t)+H_{2\gamma}(y,s))$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$.

Now making use of the same approach as in the proof of above Theorem 3.2 we have

$$\begin{aligned} \|u - q(x)\| &\leq \sum_{|\gamma|=p/2} \frac{(p/2)!}{\gamma!}\left(\frac{2\varepsilon}{4^{|\gamma|}-4}x^{2\gamma}\right) \\ &= \frac{2\varepsilon}{2^p-4}|x|^p. \end{aligned}$$

This completes the proof. □

As a direct consequence of the above result we obtain the following.

Corollary 3.4 (Chung & Lee [7]). *Every solution $u \in \mathcal{S}'$ or $\mathcal{F}'$ of the quadratic functional equation*

$$u \circ A + u \circ B - 2u \circ P_1 - 2u \circ P_2 = 0$$

has the form

$$q(x) := \sum_{1 \leq j \leq k \leq n} a_{jk}\, x_j x_k.$$

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DEPARTMENT OF MATHEMATICS KUNSAN NATIONAL UNIVERSITY, SAN-68 MIRYONG-DONG, GUNSAN, JEONBUK 573-701, KOREA
Email address: jychung@kunsan.ac.kr