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SOME BASIC THEOREMS OF CALCULUS ON THE FIELD OF p-ADIC NUMBERS

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ABSTRACT. In this paper, we introduce the concept of derivative of the function $f : \mathbb{Q}_p \to \mathbb{R}$ where $\mathbb{Q}_p$ is the field of the p-adic numbers and $\mathbb{R}$ is the set of real numbers. And some basic theorems on derivatives are given.

1. INTRODUCTION

The field $\mathbb{Q}_p$ of the p-adic numbers is defined as the completion of the field $\mathbb{Q}$ of rational numbers with respect to the p-adic metric induced by the p-adic norm $|\cdot|_p$ (see Vladimirov, Volovich & Zelenov [1]). A p-adic number $x \neq 0$ is uniquely represented in the canonical form

$$x = p^{-r} \sum_{k=0}^{\infty} x_k p^k, \quad |x|_p = p^r \tag{1.1}$$

where p is prime and $r \in \mathbb{Z}$ ($\mathbb{Z}$ is the integer set), $0 \leq x_k \leq p-1$, $x_0 \neq 0$. For x, $y \in \mathbb{Q}_p$, we define $x < y$ if $|x|_p \leq |y|_p$ and there exists an integer j such that

$$x_0 = y_0, \ldots, x_{j-1} = y_{j-1}, x_j < y_j$$

from viewpoint of (1.1). By the *interval* $[a, b]$, we mean the set defined by $\{x \in \mathbb{Q}_p \mid a \leq x \leq b\}$.

It is known that if $x_R = p^r \sum_{k=0}^{n} x_k p^{-k} \in \mathbb{R}^+ \cup \{0\}$ and

$$x_0 \neq 0, \quad 0 \leq x_k \leq p-1, \text{ for } k = 1, 2, \ldots,$$

then there is another expression;

$$x_R = p^r \left(\sum_{k=0}^{n-1} x_k p^{-k} + (x_n - 1) p^{-n} + (p-1) \sum_{k=n+1}^{\infty} p^{-k} \right). \tag{1.2}$$

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Let M_R denote all such expressions.

As introduced in Kozyrev [2], we let $\bar{\rho} : \mathbb{Q}_p \to \mathbb{R}$ be the map defined as follows. For $x = p^{-r} \sum_{k=0}^{\infty} x_k p^k$ in $\mathbb{Q}_p$ where $x_0 \neq 0,\ 0 \leq x_k \leq p-1$ we define

$$\bar{\rho} = p^r \sum_{k=0}^{\infty} x_p^{-k-1} \tag{1.3}$$

and let

$$M_p = \bar{\rho}^{-1} M_R. \tag{1.4}$$

Now define

$$\rho : \mathbb{Q}_p / M_p \to \mathbb{R}^+ \cup \{0\} \tag{1.5}$$

by just passing $\bar{\rho}$ to the quotient $\mathbb{Q}_p/M_p$.

We will denote the real numbers by putting a subscript $\mathbb{R}$ and the p-adic numbers by putting the subscript p, for example, x_R, a_R, b_R will be the real numbers and x_p, a_p, b_p will be the p-adic numbers.

In Cui & Zhang [3], a measure is constructed using the mapping ρ from $\mathbb{Q}_p/M_p$ into $\mathbb{R}^+ \cup \{0\}$ and Lebesgue measure on $R^+ \cup \{0\}$. Let Σ be the set of all compact subsets of $\mathbb{Q}_p$ and S be the σ-ring generated by Σ.

Definition 1.1 (Cui & Zhang [3]). Let $E \in S$, and put $E_P = E \setminus M_p$, and $E_R = \rho(E_P)$. If E_R is a measurable set on $R^+ \cup \{0\}$, the we call E a measurable set on $\mathbb{Q}_p$, and define a set function $\mu_p(E)$ on S by

$$\mu_p(E) = \mu(E_R)$$

where $\mu(E_R)$ is the Lebesgue measure on E_R. This $\mu_p(E)$ is a measure on E.

By Definition 1.1 and by definition of ρ in (1.5), some examples can be given immediately (see Cui & Zhang [3]):

(1) $\rho\{B_r(a_i)\} = [a_i, a_i + p)$
(2) $\rho\{[a_p, b_p)\} = [a_R, b_R)$
(3) $b_R - a_R = \mu_p\{[a_p, b_p]\}$
(4) Let $a_p, b_p \in \mathbb{Q}_p$, then $\mu_p\{[a_p, b_p]\} = (b_R - a_R)/p$
(5) Let $B_r(0) = \{x_p |\ |x_p|_p \leq p^r,\ x_p \in \mathbb{Q}_p\}$, then $\mu_p\{B_r(0)\} = p^r$
(6) Let $S_r(0) = \{x_p |\ |x_p|_p = p^r,\ x_p \in \mathbb{Q}_p\}$, then $\mu_p\{S_r(0)\} = p^r(1 - \frac{1}{p})$
(7) $\mu_p\{M_p\} = 0$

According to Definition 1.1 of measure, we can define integration over a measurable set E of $\mathbb{Q}_p$ by

$$\int_E f(x_p)d\mu_p(x_p) \quad \text{or} \quad \int_E f(x_p)dx_p$$

For the basic properties see Vladimirov, Volovich & Zelenov [1].

2. Lemmas

By Definition (1.1) of measure, we have the following lemmas.

Lemma 2.1. *Let f be a real-value function defined on $\mathbb{Q}_p$ denote*

$$x_p = \varrho^{-1}(x_R),\ x_R \in \mathbb{R}^+ \cup \{0\},\ f_R(x_R) = \left(f \circ \rho^{-1}\right)(x_R)$$

then

$$\int_{a_R}^{b_R} f_R(x_R)\mathrm{d}x_R = \int_{[a,b]} f\mathrm{d}\mu \tag{2.1}$$

where $a_p = \rho(a_R)$, $b_p = \rho^{-1}(b_R)$, and $\mathrm{d}\mu$ denotes a measure on $\mathbb{Q}_p$, and $\mu(B_0(p))$ $= 1$, $B_r(p) = \{x_p \mid |x_p - a| \le p^r, x_p \in \mathbb{Q}_p\}$, $r \in \mathbb{Z}$.

Lemma 2.2. *Let f be a real-value function on the field $\mathbb{Q}_p$ of p-adic numbers and M_R be the set of number given in formula (1.3). Let $M_p = \rho^{-1}(M_R)$. For $x_p \in \mathbb{Q}_p \setminus M_p$, denote $x_R = \rho(x_p), f_R = f \circ \rho^{-1}$. If f_R is a continuous function on $\mathbb{R}^+ \cup \{0\}$, then there exists $\xi \in [a_p, b_p]$, such that*

$$\int_{[a_p,b_p]} f\mathrm{d}\mu_p = f(\xi)\mu_p\left([a_p, b_p]\right).$$

Proof. From Lemma 2.1, we have

$$\int_{[a_p,b_p]} f\mathrm{d}\mu = \int_{a_R}^{b_R} f_R(x_R)\mathrm{d}x_R = f_R(\xi_R)(b_R - a_R), = f(\xi_p)\mu_p\left([a,b]\right),$$

for some $\xi_R \in [a_R, b_R]$ and $\rho^{-1}(\xi_R) = \xi_p$. □

3. The Derivative of Functions on $\mathbb{Q}_p$

In this section, we discuss the derivatives of real-value functions on $\mathbb{Q}_p$.

Definition 3.1. Let $f(x_p)$ be a real-value function on $\mathbb{Q}_p$. If there exists $g \in \mathbb{Q}_p$ such that for $x_p \in \mathbb{Q}_p$

$$f(x_p) = \int_0^{x_p} g(y_p)\mathrm{d}\mu_p, \quad x_p \in \mathbb{Q}_p \tag{3.1}$$

then $g(x)$ is called the *derivative* of $f(x)$ and denote $f'(x) = g(x)$. If $f'(x)$ is continuous then from Lemma 2.2 and formulas (3.1) and (1.4), we have

$$\begin{aligned}\frac{(f\circ\rho^{-1})(x_R+\Delta x_R)-(f\circ\rho^{-1})(x_R)}{\Delta x_R} &= \frac{1}{\Delta x_R}\int_{\rho^{-1}(x_R)}^{\rho^{-1}(x_R+\Delta x_R)} g(y_p)\mathrm{d}\mu_p \\ &= \frac{g(\xi_p)\mu\big([\rho^{-1}(x_R),\rho^{-1}(x_R+\Delta x_R)]\big)}{\Delta x_R} \\ &= g(\xi_p),\end{aligned}$$

where $\xi_p \in [\rho^{-1}(x_R),\rho^{-1}(x_R+\Delta x_R)]$ and, by taking limit of this formula as $\Delta x_R \to 0$, we obtain

$$\big(f_R(x_R)\big)' = \big(f\circ\rho^{-1}\big)'(x_R) = g(x_p), \quad x_p = \rho^{-1}(x_R). \tag{3.2}$$

Remark. Let $f'(x_p)$ be continuous on $[a_p, b_p] \subset \mathbb{Q}_p$. We ave

$$f(b_p) - f(a_p) = \int_a^b f'(x)\mathrm{d}\mu_p = f'(\xi)\mu_p\big([a_p,b_p]\big), \ \xi \in [a_p,\ b_p]. \tag{3.3}$$

Theorem 3.1. *If the derivatives of $f:\mathbb{Q}_p \to \mathbb{R}$ and $h:\mathbb{Q}_p \to \mathbb{R}$ exist, then*

$$(fh)' = f'h + fh'.$$

Proof. Let $f' = g$, then by

$$f(x_p) = \int_0^{x_p} g(x)\mathrm{d}\mu_p = \int_0^{x_R} g_R(x_R)\mathrm{d}\mu_p, \tag{3.4}$$

we have the equality

$$f'(x_p) = g_R(x_R) = (f')_R(x_R). \tag{3.5}$$

Because $f(x_p) = f_R(x_R)$ and formula (3.4), we have

$$\big(f_R(x_R)\big)' = g_R(x_R) = (f')_R(x_R). \tag{3.6}$$

From (3.5) and (3.6), we have

$$f'(x_p) = (f')_R(x_R) = \big(f_R(x_R)\big)'. \tag{3.7}$$

Using formula (3.7) and the derivative rules of real functions on the real number field, we have

$$\begin{aligned}(f(x_p)h(x_p))' &= (f_R(x_R)h_R(x_R))' \\ &= (f_R(x_R))'h_R(x_R) + f_R(x_R)(h_R(x_R))' \\ &= f'(x_p)h(x_p) + f(x_p)h'(x_p).\end{aligned}$$

□

Theorem 3.2. *If* $(f_R')(x_R)$ *is continuous on* $\mathbb{R}^+ \cup \{0\}$, *then*

$$\int_{a_p}^{b_p} f'(x_p)\mathrm{d}\mu_p = f(b_p) - f(a_p).$$

Proof. For $x \in \mathbb{Q}_p \setminus M$, let $\rho(a) = a_R, \rho(b) = b_R$. From (3.7), we have

$$\begin{aligned}\int_{a_p}^{b_p} f'(x_p)\mathrm{d}\mu_p &= \int_{a_R}^{b_R} (f')_R(x_R)\mathrm{d}x_R \\ &= \int_{a_R}^{b_R} (f_R(x_R))'\mathrm{d}x_R \\ &= f_R(b_R) - f_R(a_R) \\ &= f(b_p) - f(a_p).\end{aligned}$$

□

Now we give the principle of the differentiation of composite functions.

Theorem 3.3. *Let* $f'(x_p)$ *be a real valued continuous function on* $\mathbb{Q}_p \setminus M_p$, *where* M_p *is defined in the formula* (1.3), *and the derivative of function* $g : \mathbb{Q}_p \to \mathbb{Q}_p$ *exists on* $\mathbb{Q}_p \setminus M_p$ *and* $g'(x) \neq 0$. *Then for* $x_p \in \mathbb{Q}_p \setminus M_p$

$$\Big(f(g(x_p))\Big)' = f'(g(x_p))\rho(g'(x_p))$$

Remark. The derivative of $g : \mathbb{Q}_p \longrightarrow \mathbb{Q}_p$ is defined as

$$g'(x_p) = \lim_{|\Delta x|_p \to 0} \frac{(g(x_p + \Delta x_p) - g(x_p))}{\Delta x_p}$$

Proof. Let $x \in \mathbb{Q}_p \setminus M_p$. Denote $\rho(x) = x_R$. Without loss of generality, we may assume $g(x) > 0$. We have

$$\begin{aligned}f(g(x_p)) &= (f \circ \rho^{-1})((\rho \circ g)(\rho^{-1}(x_R))) \\ &= F_R(x_R)\end{aligned} \tag{3.8}$$

where

$$F_R(x_R) = (f \circ \rho^{-1})((\rho \circ g \circ \rho^{-1})(x_R)) = (f \circ g)(\rho^{-1}(x_R))$$

Because $f \circ \rho^{-1}$ and $(\rho \circ g)\rho^{-1}$ are real value functions of real number x_R, therefore

$$F_R'(x_R) = (f \circ \rho^{-1})'\Big((\rho \circ g \circ \rho^{-1})(x_R)\Big)(\rho \circ g \circ \rho^{-1})'(x_R) \tag{3.9}$$

Now we will calculate the factors $(f \circ \rho^{-1})'\big((\rho \circ g)\big(\rho^{-1}(x_R)\big)\big)$ and $(\rho \circ g \circ \rho^{-1})'(x_R)$. Since

$$\begin{aligned}&(f \circ \rho^{-1})'\big((\rho \circ g \circ \rho^{-1})(x_R)\big)\\ &= \lim_{\Delta x_R \to 0} \frac{(f \circ \rho^{-1})\big((\rho \circ g \circ \rho^{-1})(x_R + \Delta x_R)\big) - (f \circ \rho^{-1})\big((\rho \circ g \circ \rho^{-1})(x_R)\big)}{(\rho \circ g \circ \rho^{-1})(x_R + \Delta x_R) - (\rho \circ g \circ \rho^{-1})(x_R)}\end{aligned} \tag{3.10}$$

and $f_R = f \circ \rho^{-1}$, we have

$$(f \circ \rho^{-1})'\big((\rho \circ g \circ \rho^{-1})(x_R)\big) = f_R'\big((\rho \circ g \circ \rho^{-1})(x_R)\big) = f'\big(g(x_p)\big)$$

and

$$(\rho \circ g \circ \rho^{-1})'(x_R) = \lim_{\Delta x_R \to 0} \frac{(\rho \circ g \circ \rho^{-1})(x_R + \Delta x_R) - (\rho \circ g \circ \rho^{-1})(x_R)}{\Delta x_R}.$$

Let $\Delta x_R = p^{\,\delta_k}, \delta_k \in \mathbb{Z}$ and $x_R = p^r(x_0 + x_1 p + \cdots)$. Now we let δ_k tend to $-\infty$ (as $k \to \infty$) such that $\rho^{-1}(x_R + p^{\,\delta_k}) = \rho^{-1}(x_R) + \rho^{-1}(p^{\,\delta_k})$. There exists such δ_k, because $x_R \in M_R$. It follows that

$$(\rho \circ g \circ \rho^{-1})'(x_R) = \lim_{\Delta x_R \to 0} \frac{(\rho \circ g)(x_p + p^{-\delta_k}) - (\rho \circ g)(x_p)}{p^{\,\delta_k}} \tag{3.11}$$

On the other hand, we have

$$(\rho \circ g)(x_p + p^{-\delta_k}) - (\rho \circ g)(x_p) = \mu\Big(\big[0, g(x_p + p^{-\delta_k})\big]\Big) - \mu\Big(\big[0, g(x_p)\big]\Big) \tag{3.12}$$

Because $g'(x_p) > 0$, the right hand side of (3.12) equals to

$$\mu\Big(\big[g(x), g(x + p^{-\delta_k})\big]\Big) = \mu\Big(\big[0, g(x + p^{-\delta_k}) - g(x)\big]\Big) = \rho\big(g(x_p + p^{-\delta_k}) - g(x_p)\big).$$

By the continuity of operator ρ (see Lemma 2.1, it follows that

$$(\rho \circ g \circ \rho^{-1})'(x_R) = \lim_{k \to -\infty} \rho\Big\{\frac{g(x_p + p^{-\delta_k}) - g(x_p)}{p^{-\delta_k}}\Big\} = \rho\big(g'(x_p)\big)$$

From (3.9), (3.10) and (3.13), we obtain

$$F_R'(x_R) = f'\big(g(x_p)\big)\rho\big(g'(x_p)\big)$$

Finally, by (3.2) and by the equalities $F_R(x_R) = \big(f \circ g\big)(x_p)$, $x_p = \rho^{-1}(x_R)$, we have

$$\Big(f\big(g(x_p)\big)\Big)' = F_R'(x_R) = f'\big(g(x_p)\big)\rho\big(g'(x_p)\big). \qquad \square$$

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