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Learning Mathematics with CAS Calculators: Integration and Partnership Issues

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Computer algebra system (CAS) calculators are becoming increasingly common in schools and universities. While they offer quite sophisticated mathematical capability to teachers and students, it is not clear at present how they may best be employed. In particular their integration into students' learning and problem-solving remains an issue. In this paper we address this issue through the lens of a study that considered the introduction of the TI-89 CAS calculator to students about to enter university. We describe a number of different aspects of the partnership they formed with the calculator as they began the process of instrumentation of the CAS in their learning.

Ⅰ. BACKGROUND

Since Heid's (1988) groundbreaking study, research on the use of computer algebra system (CAS) calculators in the learning of mathematics has often tended to concentrate on specific content, such as aspects of algebra or calculus. However, until more recently there has been less emphasis on the processes by which students (and teachers) decide whether to use CAS, and if so, how and when to use it in learning. This is a major area of study since the process of integration of CAS into learning is not a minor consideration but the formation of suitable schemes involves numerous decisions and interactions (Thomas, 2001) with thetechnology.

A number of studies have described how technological tools may be employed in qualitatively different ways. For example, Doerr and Zangor (2000) have listed property investigation; computational; transformational; data collection and analysis; visualizing; and checking as ways in which technology may be useful. Goos, Galbraith, Renshaw and Geiger (2000) describe a hierarchy of technology interactions, where the student may be subservient to the technology, the technology can be a replacement forpen and paper, can be a partner in explorations, or an extension of self, integrated into mathematical working. A particularly useful approach, based on the ideas of Rabardel (1995), distinguishes between the use of technology as a tool or artefact and as an instrument. Transforming a CAS tool into an instrument involves actions and decisions based on the adapting it to a particular task via a consideration of what it can do and how it might do it. According to the theory of Rabardel and

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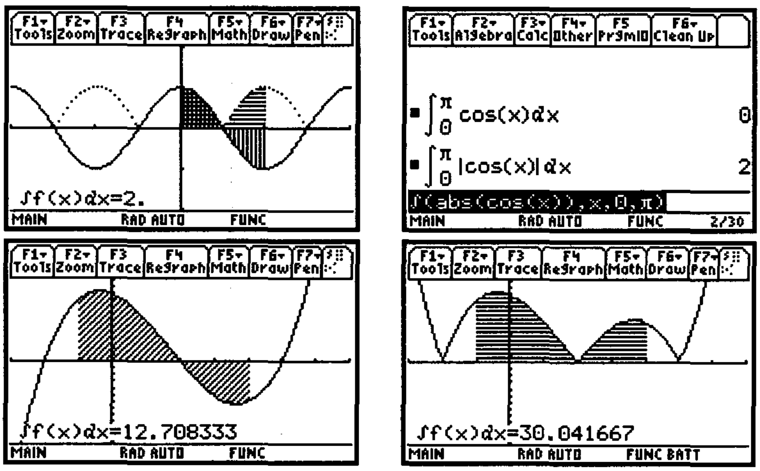
Samurcay (2001), for this to take place the subject must be intentionally engaged in both productive and constructive activity. The latter involves the development of internal and external resources, including instruments, for the productive activity. They explain how theprocess of learning via an instrument, called *instrumental genesis*, proceeds through three types of mediation that the artefact provides:

> Epistemic mediation-oriented towards an awareness of the object [of the activity], its properties, and its changes in line with the subject's actions
> Pragmatic mediation-oriented towards action on the object [of the activity], transformation, regulation management, etc
> Reflexive mediation-where the subject's relation to himself is mediated by the instrument

Instrumental genesis has two dimensions, described as instrumentalization and instrumentation. Instrumentalization charts the emergence and evolution of the instrument's artefact components, the selection of pertinent parts, choice, grouping, elaboration of function, transformation of function, etc. This may be summarised as the subject adapting the tool to himself. In contrast the process of instrumentation is where the subject adapts himself to the tool, and it involves the emergence and development of private schemes and the appropriation of social utilisation schemes (Rabardel & Samurcay, 2001). Thinking of these ideas in a mathematical context Lagrange (1999a) discusses the relationship between an individual's *internal instrument utilisation schemes* that involve decisional, pragmatic, and interpretive dimensions, and techniques, or the mathematical activity, general methods, that sit between tasks and theories. He maintains that classroom discussion on techniques is essential to help students develop suitable schemes, since "techniques without schemes are ineffective because they are not likely to evolve and cannot produce knowledge" (ibid, p. 197). The CAS techniques students require are based on mathematical conceptions such as: *equations can be solved for different letters and a formula can be considered as an object*

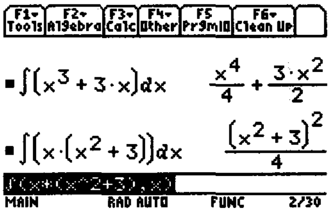


[Figure Ⅰ-1] A CAS-specific technique for finding the area under the graph of a function.

rather than a process. Drijvers and van Herwaarden (2000) used an *Isolate-Substitute-Solve*(ISS) instrumentation scheme in their research. This comprised isolating a variable in one equation, substituting it into a second and then solving that equation. An example of a CAS-specific technique based on Hong and Thomas (2004) may be seen in [Figure Ⅰ-1]. This makes use of the absolute value of the function to calculate $\int_a^b |f(x)|\,dx$ for area. This technique provides greater certainty for the area compared with the integral of the function, which may produce an incorrect value for the area when the function has a zero in the integration range (see [Figure Ⅰ-1c, d]).

As well as Lagrange (1999b), Trouche (2000) and Guin and Trouche (1999), among others, have applied the tool/instrument theory to CAS use in the mathematics classroom. They explain that the instrumentation and conceptualisation processes are dependent on each other, and that for instrumentation to occur classroom activity must be directed at specific conceptions. The study of Drijvers and Herwaarden (2000) concluded that both the technical and conceptual aspects of instrumentation need explicit attention, and that integration of CAS with pen and paper substitution and isolation techniques will lead to improved results. In view of the above it is easy to see that instrumentation is a personal process, and Lagrange (1999a) stresses that each student who uses CAS has to work out its role in their own learning. They have to learn to decide what CAS is useful for, and what might be better done by hand, and how to integrate the two (Thomas, Monaghan, & Pierce, 2004). When controlling the machine they have to be aware of possibilities and constraints, of possible differences between mathematical and CAS functioning, of symbolic notations and internal algorithms. Then there is the issue of monitoring the operation of the CAS (e.g., the syntax and semantics of the input/output, the algebraic expectation, etc), and the difficulties of navigating between screens and between menu operations.

These issues have given rise to some problems with CAS use. For example, Monaghan, Sun and Tall (1994) record how, for some students, CAS can become a mere button pushing process that obscured deeper understanding. In turn, Hunter et al.(1993) found that not only did CAS not motivate students, but they became dependent on it, performing worse than a control group on factorising and expanding when not using CAS. A similar outcome is recorded by Hong, Thomas and Kiernan (2000) who showed weaker students becoming reliant on CAS as a problem-solving support, resulting in a negative effect on their learning. An example of the kind of problems that students may experience was demonstrated by a student who approached one of the authors with a concern about the output from his CAS. He had worked out $4\int x^3 + 3x dx$ by hand and got $x^4 + 6x^2$ and was confident that it was correct. However, when he used the CAS (here a TI-89) to check the answer he entered $\int 4x(x^2+3)dx$ and it gave him $(x^2+3)^2$. Hence he was puzzled, even though he knew about the '+C' that he had habitually added on to the end of his integration calculations, he had not understood the concept of the arbitrary constant well enough to rationalise the two apparently different functions as equivalent within a constant. As [Figure Ⅰ-2] shows the CAS deals with integration of certain factorised forms of functions that it can 'recognise' in a manner different from the standard format polynomial. Students need to be actively encouraged to investigate such CAS-specific outputs.



[Figure Ⅰ-2] Different outputs from CAS depending on the input format.

On the other hand, Drijvers (2000) encourages taking a positive approach to the obstacles that students may encounter during instrumentation, and how these may be overcome. In turn, both he, Kidron (2001) and Heid (2002) have encouraged positive use of CAS for investigations, such as the understanding of parameter use in mathematics. For example, Heid gives as an example the benefits of understanding the effects of a, b, and c in functions such as $f(x)=\frac{a}{1+be^{cx}}$. It seems to us that the key lies in the students' ability to engage in instrumentation, to form a partnership with the CAS whereby they are comfortable with integrating it into their learning, problem-solving and mathematical practice. Doing this the CAS may also enable new types of dynamic representations and interactions with representations that challenge understanding, as well as promoting inter-representational thinking. It is a little surprising that Burrill et al. (2002, p. 22) in their survey report how they "found little research on students' spontaneous use (individual choice of solution strategy with or without the technology) of handheld graphing technology" expressing a need for research that gives some idea of whether students choose to use the graphing calculator on problems that educators believe they will. This paper addresses this and the genesis of the instrumentation process, looking at the ways in which a small group of students begin choosing to use CAS in their mathematical work.

Ⅱ. METHOD

A one week workshop was arranged for students taking a standard first year mathematics course at The University of Auckland that uses the TI-89 calculator. Eight students enrolled for the workshop, 5 females and 3 males, aged 18 to 26, with the exception of two older females, who were 51 and 55. None of the 8 students in the study had ever used a CAS calculator before, but all except one had used a scientific calculator. Each student was given their own TI-89 CAS calculator during the workshop which they kept for the whole week. The workshop covered basic functional aspects of the TI-89 along with use of the CAS calculator's more advanced features when solving problems in calculus and linear algebra. There was also discussion on the learning of core mathematical concepts using the calculators.

The second named researcher taught on the workshop for five two-hour sessions, demonstrating some points using a viewscreen while students followed and copied her working onto their own calculator. Afterwards the students spent the rest of the time working on problems and tackling exercises as a group, while the researcher circulated and assisted with any difficulties. The students were given a pretest prior to the workshop to ascertain their knowledge of calculus and algebra, and four different posttests during the workshop, one after each two-hour section based on that day's material. The tests (of 5 to 7 questions) comprised procedural and conceptual questions (see the results section for some of the questions and the Appendix for the format).

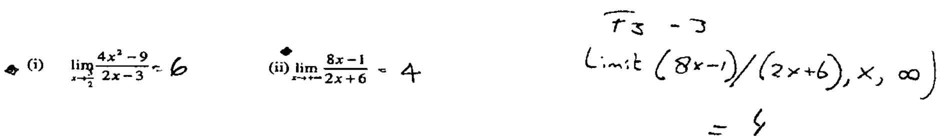
One of the aspects of the students' work we were particularly interested in, and which forms the focus of this paper, was both the manner and the timing of the students' CAS calculator use. In order to have some idea of the use they were making of the technology we asked them to mark it by putting a ◆ symbol alongside the point at which they used the TI-89 calculator to help them answer a question, and to give some idea of how or why they used it. The analysis which follows comprises a discussion of these uses.

III. RESULTS

Since none of the students had used a CAS calculator before, they were all at the genesis of instrumentation of this particular tool, beginning to form the partnership necessary to integrate its use into their mathematics learning and problem solving. What our analysis of the data revealed was a *number of qualitatively different categories* of CAS use, each of which is considered below.

1. Direct Use of CAS for Straightforward and Complex Procedures

As Thomas (2001) has described, interactions with CAS representations can be procedural or conceptual. A number of students chose simply to use the CAS to perform direct procedural calculations (i.e. a single command mapping directly to the mathematicaloperation) instead of doing them by hand. Sometimes they did so when the calculation was relatively straightforward and probably could have been done by hand, and sometimes when it appeared that the calculation was either too long or too complex for themto do it by hand. In [Figure Ⅲ-1] we see examples of the former type for a limit question. These two students did not do the limit question in the pre-test and so may not have been able to do these by hand. In [Figure Ⅲ-1] student 8 writes the ◆ symbol alongside her solution to show that she had simply entered the limits, while student 2 shows what she did by re-writing the

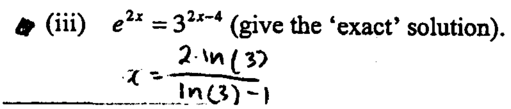


Student 8 Student 2

[Figure Ⅲ-1] Direct procedural use of the CAS, replacing by-hand working for accessible calculations.

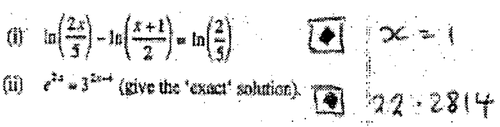
command entered into the calculator (the F3 is the menu selected and 3 the item in the menu).

[Figures Ⅲ-2, 3] show examples of direct procedural use of the CAS when the by-hand procedure may be too complex for the student to carry it out. In [Figure Ⅲ-2] student 8 has used the 'Solve'function of the CAS to solve an equation where the variable x is the index. The lack of intermediate steps, the brackets around the 3s, and the use of the ◆ shows the use of a direct CAS command.



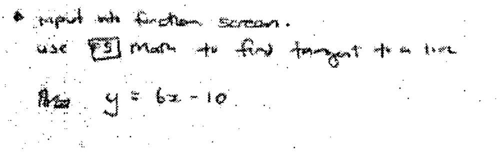
[Figure Ⅲ-2] Direct procedural use of the CAS, replacing by-hand working for complex calculations.

Similarly in [Figure Ⅲ-3] student 9 has written down the final answers and also uses the ◆ symbol to show the direct use of CAS (this includes a second use of ◆ to obtain a decimal answer to part (ii) even though the exact solution was asked for).



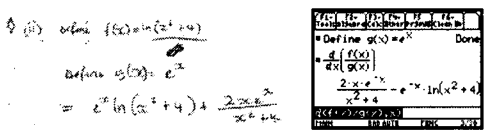
[Figure Ⅲ-3] Direct procedural use of the CAS, replacing by-hand working for complex calculations.

Occasionally even though the student is using the CAS in a direct procedural fashion they are also working in an inter-representational fashion. [Figure Ⅲ-4] gives an example of this, where student 10 has approached the question of finding the gradient of the tangent to the graph of $y = 3x^3 - 5x^2 + 7x - 9$ at x = 1, in a graphical manner. First the function is entered and the graph drawn. A direct command is then available in the F5 Maths menu on the graph screen that will find the tangent at a point on the graph, and this has been used to obtain $y = 6x - 10$.



[Figure Ⅲ-4] Direct use of a CAS procedure, replacing by-hand work but linking graph and algebra.

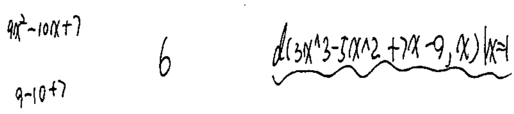
Sometimes when the students want to use the CAS to do a direct procedural calculation they will *implement* a new CAS-based *technique* for doing so. This is illustrated by the working of student 13 in [Figure Ⅲ-5]. Here he turned a single step calculation into a three step technique; firstly has defined the numerator $f(x)$ and denominator $g(x)$ as separate functions and then used the CAS to differentiate $\frac{f(x)}{g(x)}$. Unfortunately he has then made a copying error, missing the three negative signs on the screen in his attempt to reverse the terms (see the CAS screen on the right). This demonstrates a potential problem with calculators, namely copying or transcribing errors.



[Figure Ⅲ-5] Use of a CAS technique in a direct procedure.

2. Using CAS to Check Procedural By-Hand Work

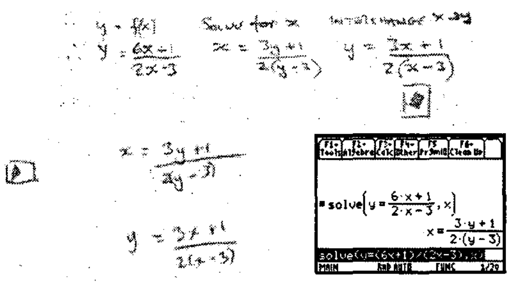
It was expected that our students would use CAS to check pen and paper working. [Figure Ⅲ-6] shows such a strategy employed in question 2 from the second posttest, "Find the gradient of the tangent to the graph of $y = 3x^3 - 5x^2 + 7x - 9$ at x=1". We see student 3's by-hand working to the left, finding the derivative of y and substituting x = 1, giving 9-10+7, or 6. On the right we see where he has checked with the CAS calculator whether the answer is correct. Here again the process can be carried out directly by employing a single command to differentiate the function with respect to x (using d(3x^35x^2 +7x9, x)) as well as calculating the value of the derivative at the point where x=1 (using |x=1).



[Figure Ⅲ-6] Student 3's direct use of CAS for checking by-hand working.

3. Direct CAS Use Within a Mathematical Process

One of the types of activity that we noticed with CAS was the transferring of a well-known by-hand scheme into a CAS version of the scheme. An example of this was the conversion of the scheme for finding the inverse of a function given as y = f(x). This procedure involves making x the subject of the equation and then replacing x with y, and vice-versa, in the result. [Figure Ⅲ-7] shows two examples of how students have integrated the use of CAS into this scheme, using a direct command to perform the first step of making x the subject of the equation. The question was: Given $f(x) = \frac{6x+1}{2x-3}$ is invertible on $x \neq \frac{3}{2}$, find $f^{-1}(x)$, the inverse of f. Student 9 ([Figure Ⅲ-7]-top) has indicated use of the CAS with the ◆ symbol, and is able to write down $x = \frac{3y+1}{2(y-3)}$, using precisely the form given by the CAS (see the screen provided in [Figure Ⅲ-7]bottom right). As can be seen from this CAS screen a CAS technique is needed for solving an equation for x and this requires an appreciation of the equivalence of this with the by-hand method of rearranging to make x the subject of an equation. She has also written down the details of the scheme used above each step. Similarly student 10 ([Figure Ⅲ-7]-bottom left) has been through the same process.

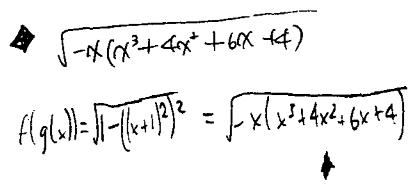


[Figure Ⅲ-7] Integration of CAS in a by-hand scheme.

Another category of CAS calculator use at first seemed to be another direct use. However, further consideration showed that students were not simply using the CAS to perform the whole calculation, as seen above, but there was a partnership evolving with CAS assigned a defined role within the overall solution process. In some questions students appeared to reach a point where the mental load required to keep the mathematical concepts in mind along with the overall process appeared to be sufficiently large that the they decided to resort to the CAS to handle a procedural aspect, possibly to reduce cognitive load (Sweller, 1994). One example occurred in question 4 of the first post-test:

If $f(x)=\sqrt{1-x^2}$ and $g(x)=(x+1)^2$, find $f\circ g(x)$.

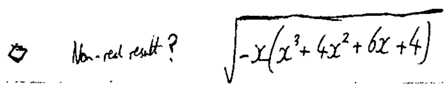
In order to use CAS for this students first have to undertake some preliminary CAS activity, requiring them to define two functions, f and g, and to understand the need to enter f(g(x)) into the calculator for the composite function. Understanding the composite function by-hand working would produce $\sqrt{1+((x+1)^2)^2}=\sqrt{1-(x+1)^4}$, but 4 of the 8 students chose CAS use, possibly either because the procedure was too complex, or they feared errors. They obtained answers like those shown in [Figure Ⅲ-8a]. The extent of their conceptual understanding of the concept of a composite function remains unclear, and we cannot decide from the answers whether or not they used the CAS to avoid cognitive overload. Other students clearly understood the conceptual part of the composite function and produced $\sqrt{1-((x+1)^2)^2}$ by hand. However when student 6 (see [Figure Ⅲ-8b]) decided to take this further and simplify it she chose the CAS to do so, presenting her working in a way that we can see this. Again she is using CAS within the mathematical process, and decisions on when and how to do so form a significant part of the instrumentation process.


8a

8b

[Figure Ⅲ-8] Student 3 and 6's answers to question 4 using CAS commands.

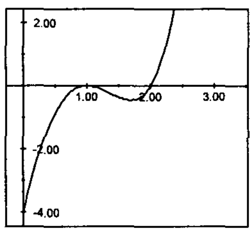
Using CAS to carry out complex procedural calculations can raise difficulties, for example through notation constraints. This is exemplified by the working of student 4 (see [Figure Ⅲ-9]), who was challenged by the format of the answer provided by the CAS. An observation of the surface structure (Thomas & Hong, 2001) of the function under the square root sign leads him to believe that the function is negative, and hence his comment that there is "No real result". He is not able to rationalise the root and negative signs with the domain of x in order to consider whether the function can be positive for some xvalues. This demonstrates that the format in which CAS gives answers can lead to problems which challenge understanding.



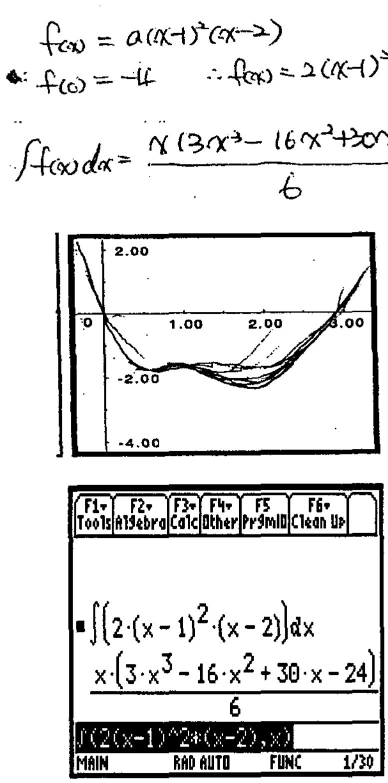
[Figure Ⅲ-9] Student 4 uses the CAS procedurally and meets a challenge with the answer format.

Question 2 of post-test 3, asked for a sketch of an antiderivative function:

The graph of a function is shown in the figure⋯ Make a rough sketch of an *antiderivative function* F alongside, given that F(0)=0.



In response to this [Figure Ⅲ-10] shows the working of student 3. He has first moved from the given graphical representation to an algebraic representation, working by hand to get $f(x) = a(x-1)^2(x-2)$ and then using f(0)=-4 to find the value of a. At this point in the solution process he resorted to the CAS to integrate and find an antiderivative function. He then moved back to the graph mode, sketching the graph of this function by hand (it appears), rather than using CAS and copying the graph. Again it appears that complex interplay between known algebraic schemes, cognitive load, and ability to perform procedures are driving decisions about CAS use.



[Figure Ⅲ-10] Student 3 uses the CAS procedurally within a mathematical solution.

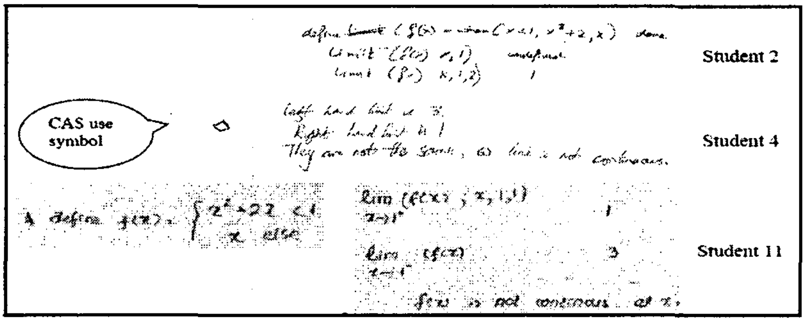
4. Using CAS to Investigate Conceptual Ideas

All of the above have involved direct use of CAS, a single procedural command calculating or evaluating some function or expression, which forms the answer. However, in some of the questions there was more interaction with mathematical concepts. For example, question 3 of the first post-test asked:

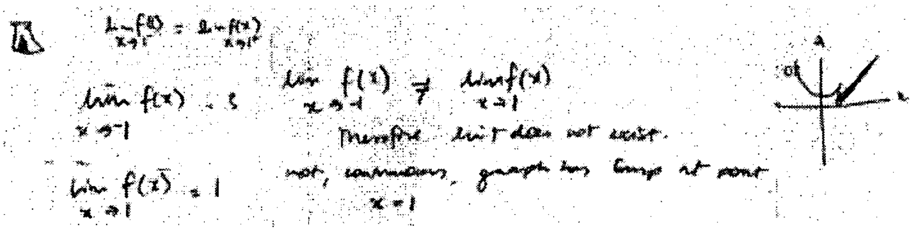
If $f(x) = \begin{cases} x^2 + 2 & \text{for } x < 1 \\ x & \text{for } x \geq 1 \end{cases}$, then using properties of limits, find out whether or not f(x) is continuous at $x = 1$.

Rather than simply asking for a result this question was assessing the concept of continuity. Students 2, 4 and 11 (see [Figure Ⅲ-11]), and others, integrated the CAS into their approach, deciding to use it to find the left and right limits of f(x) at x=1. They seem to have the beginnings of a scheme that runs: try to use CAS to find the limit at the point; if the CAS says that it is 'undefined' then it does not exist; hence use the CAS to find the left and right limits; if these are different then the function is not continuous at the point, otherwise it is. While the CAS performs a procedure each time, to embark on this method they needed to know that these limits were relevant and fundamental to the definition of continuity, and fitted into a scheme something like that above. Student 2 found that CAS would not give the limit at x=1 directly (undefined), but then only found the right-hand limit. Student 4 has answered the question completely (CAS use again shown by the ◆), combining conceptual knowledge with CAS procedural results, while student 11's working shows the additional step of first defining the function $f(x)=\begin{cases} x^2+2 & \text{for } x<1 \\ x & \text{for } x\geq 1 \end{cases}$ on the CAS before proceeding with finding both the left and right hand limits in order to answer the question.

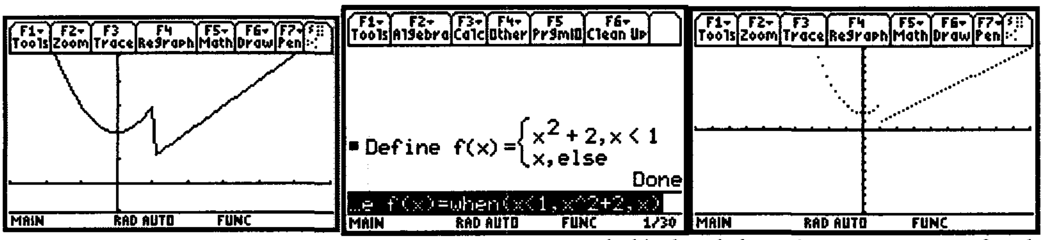
Student 12's approach to this same question (see [Figure Ⅲ-12]) again involved the calculation of the left and right limits using CAS (see the ◆). However, the CAS with its readily available graphing mode has encouraged the student to consider an inter-representational approach, using the graph to confirm the conclusion from the limits. This is also a use of the CAS as a visualizing tool, a use that, not surprisingly arises in the literature (Drijvers & Doorman, 1996; Dahland, & Lingefjard, 1996). While the graph is hand drawn it has the characteristic √shape that CAS graphs have at a discontinuity if the graph style is not changed from a line to dots, and this has led to the wrong reason for the



[Figure Ⅲ-11] Use of CAS procedures within a test of the definition of continuity.



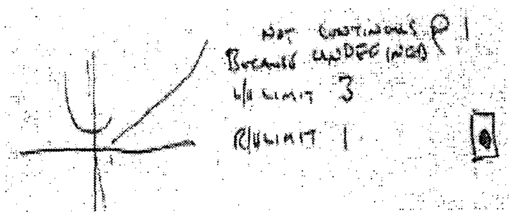
[Figure Ⅲ-12] Use of inter-representational CAS procedures within a test of the definition of continuity.



[Figure Ⅲ-13] A CAS technique using dot graphs to avoid a jump in a discontinuous graph.

discontinuity, the student believing that it "has cusp at point". [Figure Ⅲ-13] confirms how this graph was obtained using the CAS, and how a technique involving the use of dots can alleviate the disconcerting jump effect.

The work of student 10 in [Figure Ⅲ-14] shows that this student was able either to understand that there was a discontinuity in the graph, or drew the graph on the CAS using dots so that they could draw the correct looking graph.

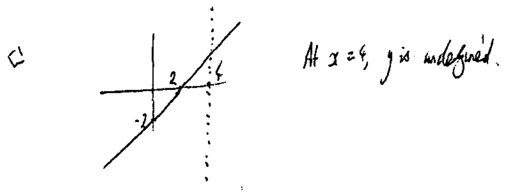


[Figure Ⅲ-14] Using CAS with dot graphs or interpreting the graph to obtain a discontinuous graph.

Another question where integrating the CAS into conceptual thinking seems to have been useful to students was question 6 of post-test 1. It asked:

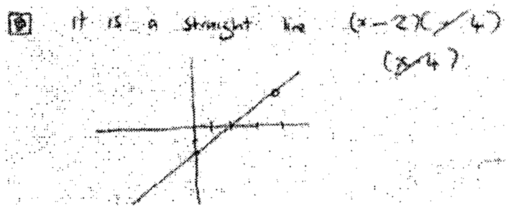
Let $f(x)=\frac{x^2-6x+8}{x-4}$. Sketch the graph of f(x). Can you explainwhy the graph has this form?

There were several approaches possible here using the CAS. Some students chose to use the CAS immediately to draw the graph of the function. In [Figure Ⅲ-15] we see that student 4 indicates that he used the CAS to draw the graph. While the discontinuity at x=4 is not shown on the CAS screen, he is able to combine the graph with his understanding of the function to state that "At x=4, y is undefined."



[Figure Ⅲ-15] Combining CAS with conceptual understanding to answer a question.

Student 10 (see [Figure Ⅲ-16]) has also combined the CAS with by-hand working, first factorising the numerator and then removing the common factor to obtain a linear graph. She has then used the CAS to draw this graph, even though it reduced to a linear function.On copying it from the screen of the CAS the student has then been able to again combine her conceptual understanding in order to insert correctly the point of discontinuity at x = 4.



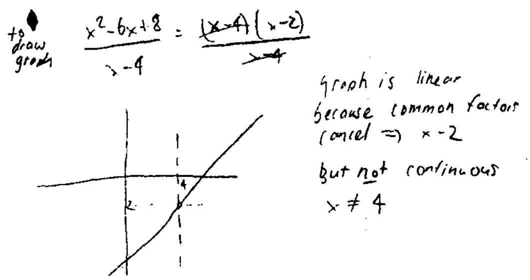
[Figure Ⅲ-16] Combining CAS with conceptual understanding to answer a question.

Similarly student 6 (see [Figure Ⅲ-17a]), appears to have chosen to simplify the rational function by hand and then use the CAS to draw the graph (she shows the by-hand working and explicitly states the use of CAS 'to draw [the] graph'). Having obtained the graph she then was able to combine her understanding of rational functions to show the 'missing' point at x=4 and to say that the function was "not continuous x≠ 4. Unfortunately the line at x = 4 is on the wrong side of the x-intercept, giving y as 2 instead of +2. This, along with student 7's answer in [Figure Ⅲ-17b], illustrates that since the CAS graphs do not show the scale values on the axes, one of the necessities of successful integration is care when transferring attention from CAS mode to by-hand working.

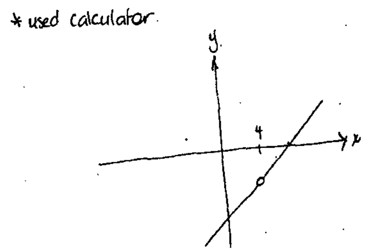
This difficulty could have been surmounted by a CAS technique involvingrecourse to the table mode of the CAS (as seen in [Figure Ⅲ-18]), but it appears that, at this early stage, none of the students had a sufficiently developed instrumentation of the CAS to consider this.

Ⅳ. CONCLUSION

In this paper we have considered the instrumental genesis of the CAS calculator as students begin to use it in solving mathematics problems. The results are consistent with the view that such instrumentation is not a short, easy process, but rather its development takes time, and there are a number of factors influencing it. For example, Guin and Trouche (2000) identify a sufficient mathematical background as a significant factor in instrumentation. In addition, studentshave to form the techniques and utili-

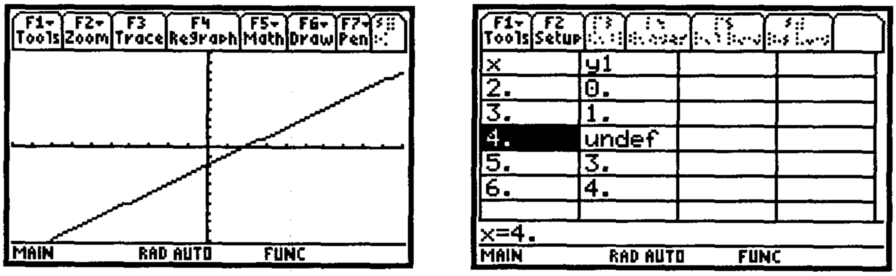


17a



17b

[Figure Ⅲ-17] Integrating CAS into a conceptual approach to a question.



[Figure Ⅲ-18] Use of tablesA CAS technique for spotting a discontinuity.

sation schemes (Lagrange, 1999b) required. It seems that some students, those who become dependent on the CAS, have difficulty reaching an initial instrumentation stage, and the key cross-over point may be when studentsaccept the symbolic register as taking priority over the graphical register. There is some evidence in this study that some of the students were using both graphical and algebraic representations as the approach of first choice. However, our research also showed that the students were more likely at first to learn the use of buttons and menus for entering direct single procedures into the CAS, often to check their by-hand working. Previous research has produced "conflicting evidence, however about whether students used handheld technology for checking or confirming algebraic work"(Burrill et al., 2002, p. 25). Our study confirms the findings of Forster and Mueller (2001) that students do use the CAS to check algebraic work. The process of making decisions about when and how to use the CAS in longer or more difficult mathematical problem solving raises obstacles that come later (Drijvers, 2000). This process may begin with procedural use within a question and then later proceed to cases where the CAS is used to explore conceptual ideas, using several procedures and representations, including, of course, its use as a visualization tool.

The specific categories of CAS use that we have identified are:

- Performing a direct, straightforward procedure,
- Checking of procedural by-hand work,
- Performing a direct complex procedure, for ease of use, or because the procedure is too difficult by hand,
- Performing a procedure within a more complex process, possibly to reduce cognitive load,
- Integrating CAS into an investigation of a conceptual idea.

Thus we found that students were interacting with CAS representations both procedurally (most commonly with direct commands), and conceptually (eg to test for continuity). In addition we have found some evidence that students are relatively quick to acquire some CAS-specific techniques useful for solving some mathematical problems. In particular they acquired the ability to find areas using the absolute value of the function on the CAS, to use the 'solve' function as a means of making a variable the subject of a formula, and they learned the value of the technique of defining functions in terms of, say, f(x) and g(x) before proceeding to use these shorthands in procedures. However, Weigand and Weller (2001, p. 99) note that in their research project on the use of CAS for understanding quadratic and trigonometric functions they found that an integrated working style was rare. Our research supports this observation, although we found some students were already starting to integrate CAS into by-hand working, some of which was procedural in nature, but only a little of which was conceptual. This integrated style of working often requires the students to transfer results from CAS techniques to pen and paper, and vice-versa, and there was some evidence of this.

Of course we have simply made a start in analysing types of usage when CAS is beginning to be integrated into mathematical work. The final category above is where much of the value of instrumentation lies, and it will no doubt yield a number of subcategories of its own. In future research we intend to provide the students with richer problem solving activities in order to investigate the nature of the thinking elicited, and the decisions which lead to integration of CAS. It is in these kinds of situations that we believe a real partnership with CAS will emerge.

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CAS계산기를 활용한 수학학습

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컴퓨터 대수 체계(CAS) 계산기는 점차 학교와 대학에서 보편화되어가고 있다. 이 계산기는 교사와 학생들에게 꽤 정교한 수학적 기능을 제공하지만, 현재로서는 어떻게 잘 활용될 수 있는지에 대해서 명확하지 않다. 특히 학생들의 학습과 문제 해결의 통합은 문제로 남아 있다. 본 논문에서 우리는 TI-89 CAS 계산기를 대학에 입학하려는 학생들에게 도입하는 연구에서 이러한 문제를 제기하고 학생들이 그들의 학습에서 CAS 계산기를 조작하면서 형성하는 협력 관계의 여러 측면을 기술하였다.

* **Key words** : CAS(컴퓨터 대수 체계), instrumental genesis(도구적 발생), epistemic mediation(인식적 매개), pragmatic mediation(활동적 매개), reflexive mediation(반성적 매개)

논문접수 : 2005. 3. 30.
심사완료 : 2005. 4. 30.

<APPENDIX>

CALCULUS AND LINEAR ALGEBRA QUESTIONNAIRE

NAME:________________________________AGE:_____________

PAPER NUMBER: MATHS________

INSTRUCTIONS: Please answer all the following questions.

Whenever you use the TI-89 calculator to help you answer a question, please mark it by putting a ◆ alongside the point at which you used it.

Show all your working clearly.

1. (a) Solve for x

   (i) $|4x - 1| \geq 2$

   (give your answer as an interval)

   (ii) $\ln\left(\frac{2x}{5}\right) - \ln\left(\frac{x+1}{2}\right) = \ln\left(\frac{2}{5}\right)$

   (iii) $e^{2x} = 3^{2x-4}$ (give the 'exact' solution).

2. Find the following limits:

   (i) $\lim_{x \to \frac{3}{2}} \frac{4x^2 - 9}{2x - 3}$ (ii) $\lim_{x \to +\infty} \frac{8x - 1}{2x + 6}$

3. If $f(x) = \begin{cases} x^2 + 2 & \text{for } x < 1 \\ x & \text{for } x \geq 1 \end{cases}$, then using properties of limits, find out whether or not $f(x)$ is continuous at $x = 1$.

4. If $f(x) = \sqrt{1 - x^2}$ and $g(x) = (x+1)^2$, find $f \circ g(x)$.

5. If $f(x) = \begin{cases} 2x^2 + a & \text{for } x \leq 2 \\ 13 - x & \text{for } x > 2 \end{cases}$, find the value of a for which f(x) is continuous at x=2.

6. Let $f(x) = \frac{x^2 - 6x + 8}{x - 4}$. Sketch the graph of f(x). Can you explain why the graph has this form?

7. Given $f(x) = \frac{6x+1}{2x-3}$ is invertible on $x \neq \frac{3}{2}$, find $f^{-1}(x)$, the inverse of f.

<Additional questions used:>

1. Use the rules of differentiation to find the derivative of each of the following.

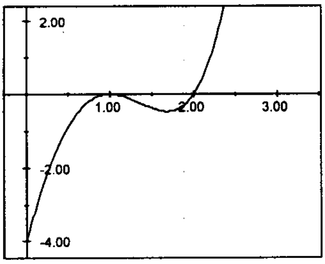
   (i) y = 3cos2x

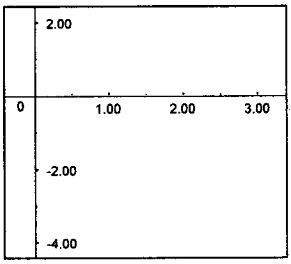
   (ii) $f(x) = \dfrac{\ln(x^2+4)}{e^x}$

   (iii) $f(x) = (4x-2)^3\sqrt{8x+3}$

2. Find the gradient of the tangent to the graph of $y = 3x^3 - 5x^2 + 7x - 9$ at x = 1.

3. The graph of a function is shown in the figure, below left. Make a rough sketch of an antiderivative function F alongside, given that F(0)=0.





Can you find an algebraic model for the function F whose graph is above left?