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LICHNEROWICZ CONNECTIONS IN ALMOST COMPLEX FINSLER MANIFOLDS

Nany Lee and Dae Yeon Won

Abstract. We consider the connections ∇ on the Rizza manifold (M, J, L) satisfying $\nabla G = 0$ and $\nabla J = 0$. Among them, we derive a Lichnerowicz connection from the Cartan connection and characterize it in terms of torsion. Generalizing Kähler condition in Hermitian geometry, we define a Kähler condition for Rizza manifolds. For such manifolds, we show that the Cartan connection and the Lichnerowicz connection coincide and that the almost complex structure J is integrable.

1. Introduction

1.1. History

In [6, 7], G. B. Rizza introduced the so-called Rizza condition (R) for almost complex manifolds (M, J) with Finsler metric L. We call the triple (M, J, L) satisfying the Rizza condition a Rizza manifold. Shortly after the work of Rizza, E. Heil[3] noticed that if the fundamental tensor g_{ij} of the Finsler metric L is compatible with the almost complex structure J, then the Finsler metric L is *a priori* a Riemannian metric. Thus it is necessary to consider a weaker assumption on the Finsler metric like the Rizza condition.

The notion of Rizza manifolds was taken up by Y. Ichijyō. He ([4, 5]) showed that every tangent space to a Rizza manifold is a complex Banach space and that Rizza condition does not necessarily reduces the Finsler metric to a Riemannian metric. He also defined a notion of Kähler

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Finsler metric and showed that for Kähler Finsler manifold, the almost complex structure is automatically integrable.

Recently, S. Kobayashi[9] derived various connections on almost Hermitian manifolds. He maintained that there exists a unique connection preserving the metric and the almost complex structure with prescribed $(1,1)$-component of the torsion. Then he produced various connections on almost Hermitian manifolds by varying $(1,1)$-component of the torsion.

Among such connections, the one with vanishing $(1,1)$-component of the torsion plays a very important role in hyperbolic complex analysis. The holomorphic sectional curvature of almost complex manifolds with such connections has non-increasing property for submanifolds. With the aid of such a connection, Kobayashi[8] obtained hyperbolicity criterion for almost complex manifolds. This theme is extended to Rizza manifolds by the authors [10].

Here we are interested in a connection with non-vanishing $(1,1)$-component of the torsion. As a real Finsler manifold (M, L), we can consider the Cartan connection on it. Its partial information gives rise to a new connection which is called the Lichnerowicz connection.

1.2. Summary of contents

In section 2, we begin with (real) Finsler manifolds. We recall the Cartan connection on Finsler manifolds which can be characterized by the system of axioms (C1–C4). Given an almost complex structure, a certain metric compatibility condition-Rizza condition (R)-is explained. Then we define a generalized Finsler structure to induce an almost Hermitian structure on the induced bundle.

In section 3, we define the Lichnerowicz connection on a Rizza manifold from the Cartan connection. Then we characterize the Lichnerowicz connection in terms of its torsion. Finally, we define a notion of Kähler condition for Rizza manifolds. Under this Kähler condition, we prove that the Lichnerowicz connection and the Cartan connection coincide. Furthermore, we can also deduce that the almost complex structure is integrable.

1.3. Acknowledgment

S. Kobayashi pointed out to us the possibility of extending the choice of connections of almost Hermitian manifolds to almost complex Finsler manifolds. The authors would like to express their deep gratitude to him.

2. Preliminaries

2.1. Finsler metric

Let M be a n-dimensional differentiable manifold with a local coordinate system $(x^1, \cdots, x^n)$. And let $(x^1, \cdots, x^n, y^1, \cdots, y^n)$ be the local coordinate system of the tangent bundle TM of M induced by $(x^1, \cdots, x^n)$.

A (real) Finsler metric L on the manifold M is a function $L : TM \to \mathbb{R}$ satisfying

(F1) L is smooth away from the zero section of TM,
(F2) $L(x, y) \geq 0$ and $L(x, y) = 0$ if and only if $y = 0$,
(F3) $L(x, \lambda y) = |\lambda| L(x, y)$ for all $\lambda \in \mathbb{R}$, and
(F4) L is strongly convex, i.e., $\left[\frac{\partial^2 L^2}{\partial y^i \partial y^j}\right]$ is positive definite.

We consider the pull-back bundle $\tilde{\pi} : p^*TM \to \widetilde{TM}$ of the tangent bundle $\pi : TM \to M$ by the projection $p : \widetilde{TM} \to M$. Here $\widetilde{TM} = TM \setminus \{\text{zero section of } \pi : TM \to M\}$ is the slit tangent bundle.

$$\begin{array}{ccc} p^*TM & \xrightarrow{\tilde{p}} & TM \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \widetilde{TM} & \xrightarrow[p]{} & M \end{array}$$

Let $g_{ij} = \frac{1}{2}\partial L^2/\partial y^i \partial y^j$. Then the strong convexity (F4) of L implies that the function g_{ij} on the slit tangent bundle $\widetilde{TM}$ of M defines a Riemannian structure on $\tilde{\pi} : p^*TM \to \widetilde{TM}$.

2.2. Cartan connection

As a generalization of Levi-Civita connection on the Riemannian manifold, we have Cartan connection on the Finsler manifold. Cartan connection $\overset{c}{\nabla}$ on a Finsler manifold (M, L) can be characterized by the following axioms:

(C1) $\overset{c}{\nabla} g = 0$,
(C2) $\eta(v, w) = 0$ for all $v, w \in \mathcal{V}$,
(C3) $\eta(v, w) \subset \mathcal{V}$ for all $v, w \in \mathcal{H}$,
(C4) the deflection tensor D of $\overset{c}{\nabla}$ vanishes,

where η is the torsion of the linear connection on $\widetilde{TM}$ induced by $\overset{c}{\nabla}$.

Note that the axiom (C4) determines a non-linear connection N_i^j of $T\widetilde{TM}$ uniquely. Then $T\widetilde{TM} = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{H}$ is the horizontal subspace of $T\widetilde{TM}$ with the basis $\left\{\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}\right\}_{i=1}^n$ and $\mathcal{V}$ is the vertical subspace of $T\widetilde{TM}$ with the basis $\left\{\frac{\partial}{\partial y^i}\right\}_{1=1}^n$. These subspaces $\mathcal{H}$ and $\mathcal{V}$ are identified with p^*TM by the isomorphisms $\chi^{\mathcal{H}} : p^*TM \to \mathcal{H}$ and $\chi^{\mathcal{V}} : p^*TM \to \mathcal{V}$ defined by $\chi^{\mathcal{H}}(\frac{\partial}{\partial x^i}) = \frac{\delta}{\delta x^i}$ and $\chi^{\mathcal{V}}(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i}$, respectively.

2.3. Rizza manifold

Let (M, J) be an almost complex manifold: an even dimensional manifold M with an almost complex structure J. If the fundamental tensor g_{ij} of the Finsler metric L is compatible with the almost complex structure J, i.e., $g_{ij} = g_{kl}J_i^k J_j^l$, then the Finsler metric is *a priori* a Riemannian metric [3]. Thus we should impose a weaker compatibility condition on L. In [6], Rizza proposed the following condition:

(R) $\quad L(x, \phi_\theta(y)) = L(x, y)$ for all $x \in M$, $y \in T_xM$ and $\theta \in \mathbb{R}$,

where $\phi_\theta(y) = (\cos\theta)y + (\sin\theta)Jy$. We call (R) the Rizza condition and the triple (M, J, L) satisfying the Rizza condition a Rizza manifold after G. B. Rizza. The Rizza condition guarantees that every tangent space to a Rizza manifold is a complex Banach space.

Now we define a generalized Finsler structure G_{ij} by $G_{ij} = g_{ij} + g_{kl}J_i^k J_j^l$. By definition, $G_{ij} = G_{kl}J_i^k J_j^l$. Thus G_{ij} defines an almost Hermitian structure on $\tilde{\pi} : p^*TM \to \widetilde{TM}$. Here p^*TM is equipped with the almost complex structure $J(x, y) = J(x)$. As in [8], it is natural to consider connections ∇ satisfying $\nabla G = 0$ and $\nabla J = 0$. In [10], authors maintained that with respect to a local unitary frame field, such a connection is given by a locally defined 1-form $\omega = (\omega_b^a)$ with values in the Lie algebra $\mathfrak{u}(n)$ of $\mathcal{U}(n)$. This connection 1-form $\omega = (\omega_b^a)$ is skew-Hermitian.

3. Theorems

3.1. Lichnerowicz connection

Let (M, J, L) be a Rizza manifold. Given a local orthonormal frame $\{e_a(x, y), Je_a(x, y)\}$ of p^*TM with respect to g, the connection form of the Cartan connection $\overset{c}{\nabla}$ is

$$\begin{pmatrix} A & B \\ -B^t & C \end{pmatrix}$$

with $A^t = -A$ and $C^t = -C$.

Note that $\overset{c}{\nabla}(Je_a) = (\overset{c}{\nabla}J)e_a + J(\overset{c}{\nabla}e_a) = (\overset{c}{\nabla}J)e_a + A_a^b(Je_b) + {B^t}_a^b e_b$ and that $\overset{c}{\nabla}(Je_a) = B_a^b e_b + C_a^b Je_b$. Thus $(\overset{c}{\nabla}J)e_a = (B_a^b - {B^t}_a^b)e_b + (C_a^b - A_a^b)Je_b$. Therefore $\overset{c}{\nabla}J = 0$ if and only if $B = B^t$ and $A = C$.

By the axiom (C4), we have a non-linear connection $\mathcal{H}$ such that $T(\widetilde{TM}) = \mathcal{H} + \mathcal{V}$. Henceforth we fix a non-linear connection $\mathcal{H}$. Consider the isomorphisms $\chi^{\mathcal{H}} : p^*TM \to \mathcal{H}$ and $\chi^{\mathcal{V}} : p^*TM \to \mathcal{V}$ defined in §2.2. Then

$$\{\chi^{\mathcal{H}}(e_a), \chi^{\mathcal{H}}(Je_a), \chi^{\mathcal{V}}(e_a), \chi^{\mathcal{V}}(Je_a)\}$$

is a local basis of $T(\widetilde{TM})$. Let $\{p^a, q^a, r^a, s^a\}$ be its dual basis of $T^*(\widetilde{TM})$.

Now the torsion η of the Cartan connection $\overset{c}{\nabla}$ is

$$\begin{aligned}
\eta_1^a &= dp^a + A\,p^a + Bq^a, \\
\eta_2^a &= dq^a - B^t p^a + Cq^a, \\
\eta_3^a &= dr^a + A\,r^a + Bs^a, \\
\eta_4^a &= ds^a - B^t r^a + Cs^a.
\end{aligned}$$

Axiom (C2) is equivalent to $\eta_i(\mathcal{V}, \mathcal{V}) = 0$ for $i = 1, \cdots, 4$. And axiom (C3) is equivalent to $\eta_i(\mathcal{H}, \mathcal{H}) = 0$ for $i = 1, 2$.

Considering the almost complex structure J on p^*TM, we have a complexification $p^*TM^{\mathbb{C}}$ of p^*TM and we extend G sesqui-linearly to $p^*TM^{\mathbb{C}}$. Let $\mathbf{e}_a = (e_a - \sqrt{-1}Je_a)/\sqrt{2}$ and $\bar{\mathbf{e}}_a = (e_a + \sqrt{-1}Je_a)/\sqrt{2}$ be a unitary basis of $p^*TM^{\mathbb{C}}$ with respect to G. Then the Cartan connection form with respect to the basis $\{\mathbf{e}_a, \bar{\mathbf{e}}_a\}$ is given by

$$\begin{pmatrix} \lambda & \mu \\ \bar{\mu} & \bar{\lambda} \end{pmatrix},$$

where

$$\begin{aligned}
\lambda &= (A + C + \sqrt{-1}(B + B^t))/2, \\
\mu &= (A - C + \sqrt{-1}(B - B^t))/2.
\end{aligned}$$

Note that $\bar{\lambda}^t = -\lambda$ and $\mu^t = -\mu$.

And we have almost complex structures J on $\mathcal{H}$ and on $\mathcal{V}$ satisfying $J\circ\chi^{\mathcal{H}}=\chi^{\mathcal{H}}\circ J$ and $J\circ\chi^{\mathcal{V}}=\chi^{\mathcal{V}}\circ J$. If we define

$$\begin{aligned}
e_a^{\mathcal{H}} &= (\chi^{\mathcal{H}}(e_a)-\sqrt{-1}\,J\chi^{\mathcal{H}}(e_a))/\sqrt{2}, & \theta^a &= p^a+\sqrt{-1}\,q^a,\\
e_a^{\mathcal{V}} &= (\chi^{\mathcal{V}}(e_a)-\sqrt{-1}\,J\chi^{\mathcal{V}}(e_a))/\sqrt{2}, & \phi^a &= r^a+\sqrt{-1}\,s^a,\\
\bar e_a^{\mathcal{H}} &= (\chi^{\mathcal{H}}(e_a)+\sqrt{-1}\,J\chi^{\mathcal{H}}(e_a))/\sqrt{2}, & \bar\theta^a &= p^a-\sqrt{-1}\,q^a,\\
\bar e_a^{\mathcal{V}} &= (\chi^{\mathcal{V}}(e_a)+\sqrt{-1}\,J\chi^{\mathcal{V}}(e_a))/\sqrt{2}, & \bar\phi^a &= r^a-\sqrt{-1}\,s^a,
\end{aligned}$$

then $\{e_a^{\mathcal{H}},\bar e_a^{\mathcal{H}},e_a^{\mathcal{V}},\bar e_a^{\mathcal{V}}\}$ is dual to $\{\theta^a,\bar\theta^a,\phi^a,\bar\phi^a\}$.

Now the torsion of $\overset{c}{\nabla}$ can be expressed

$$\begin{aligned}
d\theta+\lambda\wedge\theta+\mu\wedge\bar\theta &= \eta_1+\sqrt{-1}\,\eta_2,\\
d\phi+\lambda\wedge\phi+\mu\wedge\bar\phi &= \eta_3+\sqrt{-1}\,\eta_4.
\end{aligned}$$

Since 1-form λ is skew-Hermitian, it defines a connection ∇ on the Rizza manifold (M,J,L) such that $\nabla G=0$ and $\nabla J=0$. We call ∇ the Lichnerowicz connection of the Rizza manifold.

We now derive conditions on the torsion of the Lichnerowicz connection ∇ from the axioms (C2) and (C3) on the torsion of $\overset{c}{\nabla}$. The torsion of the Lichnerowicz connection ∇ is

$$\begin{aligned}
\Theta^a &= \eta_1^a+\sqrt{-1}\,\eta_2^a-\mu_{\bar b}^a\wedge\bar\theta^b,\\
\Phi^a &= \eta_3^a+\sqrt{-1}\,\eta_4^a-\mu_{\bar b}^a\wedge\bar\phi^b.
\end{aligned}$$

Let $\mu_{\bar b}^a=\mu_{\bar bc}^a\theta^c+\mu_{\bar b\bar c}^a\bar\theta^c+\mu_{\bar b:c}^a\phi^c+\mu_{\bar b:\bar c}^a\bar\phi^c$. Then axiom (C2) implies

$$\begin{aligned}
&\Theta^a(e_b^{\mathcal{V}},e_c^{\mathcal{V}})=0, && \Theta^a(e_b^{\mathcal{V}},\bar e_c^{\mathcal{V}})=0, && \Theta^a(\bar e_b^{\mathcal{V}},\bar e_c^{\mathcal{V}})=0,\\
&\Phi^a(\bar e_b^{\mathcal{V}},e_c^{\mathcal{V}})=\mu_{\bar b:c}^a, && \Phi^a(\bar e_b^{\mathcal{V}},\bar e_c^{\mathcal{V}})=\mu_{\bar b:\bar c}^a, && \Phi^a(e_b^{\mathcal{V}},e_c^{\mathcal{V}})=0
\end{aligned}$$

and axiom (C3) implies

$$\Theta^a(\bar e_b^{\mathcal{H}},e_c^{\mathcal{H}})=\mu_{\bar bc}^a,\quad \Theta^a(\bar e_b^{\mathcal{H}},\bar e_c^{\mathcal{H}})=\mu_{\bar b\bar c}^a,\quad \Theta^a(e_b^{\mathcal{H}},e_c^{\mathcal{H}})=0.$$

Summarizing, we have

THEOREM 3.1. *If λ is the Lichnerowicz connection on a Rizza manifold (M,J,L), then its torsion Θ^a and Φ^a has the following properties:*
(3.1)

$$\begin{aligned}
&\Theta^a(\bar e_b^{\mathcal{H}},e_c^{\mathcal{H}})=\Theta^b(\bar e_a^{\mathcal{H}},e_c^{\mathcal{H}}), && \Theta^a(\bar e_b^{\mathcal{H}},\bar e_c^{\mathcal{H}})=\Theta^b(\bar e_a^{\mathcal{H}},\bar e_c^{\mathcal{H}}), && \Theta^a(e_b^{\mathcal{H}},e_c^{\mathcal{H}})=0,\\
&\Theta^a(e_b^{\mathcal{V}},e_c^{\mathcal{V}})=0, && \Theta^a(e_b^{\mathcal{V}},\bar e_c^{\mathcal{V}})=0, && \Theta^a(\bar e_b^{\mathcal{V}},\bar e_c^{\mathcal{V}})=0,\\
&\Phi^a(\bar e_b^{\mathcal{V}},e_c^{\mathcal{V}})=\Phi^b(\bar e_a^{\mathcal{V}},e_c^{\mathcal{V}}), && \Phi^a(\bar e_b^{\mathcal{V}},\bar e_c^{\mathcal{V}})=\Phi^b(\bar e_a^{\mathcal{V}},\bar e_c^{\mathcal{V}}), && \Phi^a(e_b^{\mathcal{V}},e_c^{\mathcal{V}})=0.
\end{aligned}$$

And conversely, we have

THEOREM 3.2. *Let* $\phi = (\phi^a_b)$ *be a connection* 1*-form of a connection preserving* G *and* J*. If its torsion satisfies*

$$
\begin{aligned}
\Theta^a &= \psi^a_{\bar{b}c}\bar{\theta}^b \wedge \theta^c + \psi^a_{\bar{b}\bar{c}}\bar{\theta}^b \wedge \bar{\theta}^c + 0 \cdot \theta^b \wedge \theta^c \\
&\quad + 0 \cdot \bar{\phi}^b \wedge \phi^c + 0 \cdot \bar{\phi}^b \wedge \bar{\phi}^c + 0 \cdot \phi^b \wedge \phi^c + \text{ mixed terms}, \\
\Phi^a &= \psi^a_{\bar{b}:c}\bar{\phi}^b \wedge \phi^c + \psi^a_{\bar{b}:\bar{c}}\bar{\phi}^b \wedge \bar{\phi}^c + 0 \cdot \phi^b \wedge \phi^c + \text{ other terms}
\end{aligned}
\tag{3.2}
$$

and $\psi^a_{\bar{b}c} = -\psi^b_{\bar{a}c}$, $\psi^a_{\bar{b}\bar{c}} = -\psi^b_{\bar{a}\bar{c}}$, $\psi^a_{\bar{b}:c} = -\psi^b_{\bar{a}:c}$ *and* $\psi^a_{\bar{b}:\bar{c}} = -\psi^b_{\bar{a}:\bar{c}}$, *then* ϕ *is a connection* 1*-form of the Lichnerowicz connection.*

Proof. Define $\psi^a_{\bar{b}} = \psi^a_{\bar{b}c}\theta^c + \psi^a_{\bar{b}\bar{c}}\bar{\theta}^c + \psi^a_{\bar{b}:c}\phi^c + \psi^a_{\bar{b}:\bar{c}}\bar{\phi}^c$. Then $\bar{\phi}^t = -\phi$, since ϕ preserves G and J. And $\psi^t = -\psi$. Let $A = Re(\phi + \psi)$, $B = Im(\phi + \psi)$ and $C = Re(\phi - \psi)$.

We maintain that $\begin{pmatrix} A & B \\ -B^t & C \end{pmatrix}$ is the Cartan connection by showing that it satisfies the axioms (C1)–(C3).. Note that $A^t = -A$ and $C^t = -C$. By (3.1), $\eta^a_1 + \sqrt{-1}\,\eta^a_2 = \alpha\,\theta \wedge \phi + \beta\,\bar{\theta} \wedge \phi + \gamma\,\theta \wedge \bar{\phi} + \delta\,\bar{\theta} \wedge \bar{\phi}$ has only mixed components. Thus $\eta_i(\mathcal{H}, \mathcal{H}) = \eta_i(\mathcal{V}, \mathcal{V}) = 0$ for $i = 1, 2$. Similarly $\eta^a_3 + \sqrt{-1}\,\eta^a_4 = 0 \cdot \phi \wedge \phi + 0 \cdot \bar{\phi} \wedge \phi + 0 \cdot \bar{\phi} \wedge \bar{\phi} +$ other terms. Thus $\eta_i(\mathcal{V}, \mathcal{V}) = 0$ for $i = 3, 4$. □

REMARK . Combining Theorem 3.1 and Theorem 3.2, we have following characterization of the Lichnerowicz connection. The Lichnerowicz connection ∇ on the Rizza manifold (M, J, L) is uniquely determined by the following axioms:

1. $\nabla G = 0$,
2. $\nabla J = 0$,
3. the torsion satisfies (3.1),
4. the deflection tensor $D = 0$.

3.2. Kähler Rizza manifold

Consider the 2-form Ω on $\widetilde{TM}$ defined by

$$\Omega = \theta^a \wedge \bar{\theta}^a + \phi^a \wedge \bar{\phi}^a.$$

We call Ω the Kähler form.

DEFINITION 3.1. A Rizza manifold (M, J, L) is called a Kähler Rizza manifold if the Kähler form Ω is closed.

THEOREM 3.3. *If* (M, J, L) *is a Kähler Rizza manifold, then the Lichnerowicz connection and the Cartan connection coincide.*

Proof. First we compute $d\Omega$:

$$\begin{aligned} d\Omega &= d\theta^a \wedge \bar{\theta}^a - \theta^a \wedge d\bar{\theta}^a + d\phi^a \wedge \bar{\phi}^a - \phi^a \wedge d\bar{\phi}^a \\ &= (\Theta^a - \lambda^a_b \wedge \theta^b) \wedge \bar{\theta}^a - \theta^a \wedge (\bar{\Theta}^a - \bar{\lambda}^a_b \wedge \bar{\theta}^b) \\ &\quad + (\Psi^a - \lambda^a_b \wedge \phi^b) \wedge \bar{\phi}^a - \phi^a \wedge (\bar{\Psi}^a - \bar{\lambda}^a_b \wedge \bar{\phi}^b) \\ &= \Theta^a \wedge \bar{\theta}^a - \theta^a \wedge \bar{\Theta}^a + \Psi^a \wedge \bar{\phi}^a - \phi^a \wedge \bar{\Psi}^a. \end{aligned}$$

Note here that the terms containing λ^a_b's cancel out pairwise because $\bar{\lambda}^t = -\lambda$ and that Θ^a and Ψ^a can be replaced by (3.2) with $\psi^a_{\bar{b}c} = \mu^a_{\bar{b}c}$, etc.

For a Kähler Rizza manifold, $d\Omega = 0$. Since $(d\Omega)^{0,3} = 0$, $\mu^a_{\bar{b}\bar{c}} = \mu^a_{\bar{b}:\bar{c}} = 0$. And since $(d\Omega)^{1,2} = 0$, $\mu^a_{\bar{b}c} = \mu^a_{\bar{b}:c} = 0$. Therefore $\mu = 0$. □

Note that $\mu = 0$ if and only if $B = B^t$ and $A = C$. Thus $\overset{c}{\nabla} J = 0$, where $\overset{c}{\nabla}$ is the Cartan connection. In [5], Y. Ichijyō defined a Kähler condition for a Rizza manifold by requiring $\overset{c}{\nabla}^{\mathcal{H}} J = 0$, where $\overset{c}{\nabla}^{\mathcal{H}}$ is the horizontal Cartan covariant derivative. Thus our Kähler condition implies that of Ichijyō. By Theorem 6.3 in [5], the complex structure J is integrable for Ichijyō's Kähler Rizza manifolds. Thus the complex structure J is integrable for our Kähler Rizza manifolds. Summarizing, we have

THEOREM 3.4. *For a Kähler Rizza manifold (M, J, L), the complex structure J is integrable.*

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