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THE STRUCTURE JACOBI OPERATOR ON REAL HYPERSURFACES IN A NONFLAT COMPLEX SPACE FORM

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ABSTRACT. Let M be a real hypersurface with almost contact metric structure (ϕ, ξ, η, g) in a nonflat complex space form $M_n(c)$. In this paper, we prove that if the structure Jacobi operator R_ξ commutes with both the structure tensor ϕ and the Ricc tensor S, then M is a Hopf hypersurface in $M_n(c)$ provided that the mean curvature of M is constant or $g(S\xi, \xi)$ is constant.

0. Introduction

An n-dimensional complex space form $M_n(c)$ is a Kaehlerian manifold of constant holomorphic sectional curvature c.

As is well known, complete and simply connected complex space forms are isometric to a complex projective space $P_n\mathbb{C}$, a complex Euclidean space $\mathbb{C}_n$ or a complex hyperbolic space $H_n\mathbb{C}$ according as $c > 0, c = 0$ or $c < 0$.

Let M be a real hypersurfaces of $M_n(c)$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the Kaehlerian metric and complex structure J of $M_n(c)$. The structure vector ξ is said to be *principal* if $A\xi = \alpha\xi$, where A is the shape operator in the direction of the unit normal N and $\alpha = \eta(A\xi)$. In this case, it is known that α is locally constant [10] and that M is called a *Hopf hypersurface* [15]. We denote by ∇, the Levi-Civita connection with respect to the Riemannian metric tensor g. Takagi[18] classified all homogeneous real hypersurfaces of $P_n\mathbb{C}$ as six model spaces which are said to be A_1, A_2, B, C, D, and E, and Cecil-Ryan[2] and Kimura[11] proved that they are realized as

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the tubes of constant radius over Kaehlerian submanifolds. Namely, he proved the following

THEOREM T. [18] *Let M be a homogeneous real hypersurface of $P_n\mathbb{C}$. Then M is a tube of radius r over one of the following Kaehlerian submanifolds:*

(A$_1$) *a hyperplane $P_{n-1}\mathbb{C}$, where $0 < r < \frac{\pi}{2}$,*
(A$_2$) *a totally geodesic $P_k\mathbb{C}(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$,*
(B) *a complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$,*
(C) *$P_1\mathbb{C} \times P_{(n-1)/2}\mathbb{C}$, where $0 < r < \frac{\pi}{4}$ and $n(\geq 5)$ is odd,*
(D) *a complex Grassmann $G_{2,5}\mathbb{C}$, where $0 < r < \frac{\pi}{4}$ and $n = 9$,*
(E) *a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{4}$ and $n = 15$.*

Also Berndt[1] showed that all real hypersurfaces with constant principal curvatures of a complex hyperbolic space $H_n\mathbb{C}$ are realized as the tubes of constant radius over certain submanifolds when the structure vector ξ is principal. Nowadays in $H_n\mathbb{C}$ they are said to be of type A$_0$, A$_1$, A$_2$, and B. He proved the following

THEOREM B. [1] *Let M be a real hypersurface of $H_n\mathbb{C}$. Then M has constant principal curvatures and ξ is principal if and only if M is locally congruent to one of the followings:*

(A$_0$) *a self-tube, that is, a horosphere,*
(A$_1$) *a geodesic hypersphere or a tube over a hyperplane $H_{n-1}\mathbb{C}$,*
(A$_2$) *a tube over a totally geodesic $H_k\mathbb{C}(1 \leq k \leq n-2)$,*
(B) *a tube over a totally real hyperbolic space $H_n\mathbb{R}$.*

Let M be a real hypersurface of type A$_1$ or type A$_2$ in a complex projective space $P_n\mathbb{C}$ or that of type A$_0$, A$_1$, or A$_2$ in a complex hyperbolic space $H_n\mathbb{C}$. Then M is said to be of *type A* for simplicity. By a theorem due to Okumura[16] and to Montiel and Romero[14] we have

THEOREM O-MR. [14, 16] *If the shape operator A and the structure tensor ϕ commute to each other, then a real hypersurface of a complex space form $M_n(c), c \neq 0$ is locally congruent to be of type* A.

Characterization problems for a real hypersurface of type A in a complex space form were studied by many authors (cf. [5], [6], [7], [9], [12] and [14] etc.).

The curvature tensor field R on M is defined by

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

where X and Y are vector fields on M. We define the Jacobi operator field $R_X = R(\cdot, X)X$ with respect to a unit vector filed X. Then we see that R_X is a self-adjoint endomorphism of the tangent space. It is related with (the Jacobi vector equation) $\nabla_{\dot\gamma}(\nabla_{\dot\gamma}Y) + R(Y,\dot\gamma)\dot\gamma = 0$ along a geodesic γ. It is well-known that the notion of Jacobi vector fields involve many important geometric properties. Some works have recently studied several conditions on the structure Jacobi operator R_ξ and given some results on the classification of real hypersurfaces in a complex space form([3], [4], [6], [8], and [15] etc). One of them, Cho and one of the present authors proved the following:

THEOREM CK. [3] *Let M be a connected real hypersurface of $P_n\mathbb{C}$. If M satisfies $R_\xi\phi = \phi R_\xi$ and at the same time satisfies $R_\xi A = AR_\xi$. Then M is a Hopf hypersurface. Further if $\eta(A\xi) \neq 0$, then M is locally congruent to one of the following spaces:*

(A_1) *a geodesic hypersphere (that is, a tube of radius r over a hyperplane $P_{n-1}\mathbb{C}$, where $0 < r < \frac{\pi}{2}$ and $r \neq \frac{\pi}{4}$),*
(A_2) *a tube of radius r over a totally geodesic $P_k\mathbb{C}(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$ and $r \neq \frac{\pi}{4}$.*

In this paper we study a real hypersurface of a nonflat complex space form $M_n(c)$ which satisfies $R_\xi\phi = \phi R_\xi$ and at the same time $R_\xi S = SR_\xi$, where S denotes the Ricci tensor of the hypersurface. The main purpose of the present paper is to improve Theorem CK.

All manifolds in the present paper are assume to be connected and of class C^∞ and the real hypersurfaces supposed to be orientable.

1. Fundamental facts of real hypersurfaces

Let M be a real hypersurfaces of $M_n(c)$ and N be a unit normal vector field on M. By $\tilde\nabla$ we denote the Levi-Civita connection with respect to the Fubini-Study metric $\tilde g$ of $M_n(c)$. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde\nabla_X Y = \nabla_X Y + g(AX,Y)N, \ \tilde\nabla_X N = -AX$$

for any vector fields X and Y on M, where g denotes the Riemannian metric of M induced from $\tilde g$ and A is the shape operator of M in $M_n(c)$. For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N, \ JN = -\xi.$$

Then we may see that the structure (ϕ, ξ, η, g) is an almost contact metric structure on M, that is, we have

$$\phi^2 X = -X + \eta(X)\xi,\ g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),$$
$$\eta(\xi) = 1,\ \phi\xi = 0,\ \eta(X) = g(X,\xi)$$

for any vector fields X and Y on M.

From the fact $\tilde{\nabla} J = 0$ and making use of the Gauss and Weingarten formulas, we obtain

$$(1.1)\qquad (\nabla_X \phi)Y = \eta(Y)AX - g(AX,Y)\xi,\ \nabla_X \xi = \phi AX.$$

Since the ambient space is of constant holomorphic sectional curvature c, we have the following Gauss and Codazzi equations:

$$(1.2)\qquad \begin{aligned} R(X,Y)Z = {} & \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X \\ & - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ & + g(AY,Z)AX - g(AX,Z)AY, \end{aligned}$$

$$(1.3)\qquad (\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\},$$

where R is the Riemann-Christoffel curvature tensor on M.

In what follows, to write our formulas in convention forms, we denote by $\alpha = g(A\xi,\xi), \beta = g(A^2\xi,\xi), \gamma = g(A^3\xi,\xi)$ and $h = \mathrm{Tr}A$, and for a function f we denote by ∇f the gradient vector field of f.

If we put $U = \nabla_\xi \xi$, then U is orthogonal to the structure vector ξ. We get

$$(1.4)\qquad \phi U = -A\xi + \alpha\xi,$$

which shows that $g(U,U) = \beta - \alpha^2$. Thus we easily see that ξ is a principal curvature vector, that is, $A\xi = \alpha\xi$ if and only if $\beta - \alpha^2 = 0$. From the Gauss structure equation (1.2), the Ricci tensor S of M is given by

$$(1.5)\qquad S = \frac{c}{4}\{(2n+1)I - 3\eta \otimes \xi\} + hA - A^2,$$

I is an identity map, which implies

$$S\xi = \frac{c}{2}(n-1)\xi + hA\xi - A^2\xi. \tag{1.6}$$

We put

$$A\xi = \alpha\xi + \mu W, \tag{1.7}$$

where W is a unit vector field orthogonal to ξ. Then we have $U = \mu\phi W$, and W is also orthogonal to U. Further we have $\mu^2 = \beta - \alpha^2$.

By the definition of U and the second equation of (1.1) and (1.7), it is verified that

$$g(\nabla_X \xi, U) = \mu g(AW, X). \tag{1.8}$$

Now, differentiating (1.4) covariantly along M and using (1.1), we find

$$\begin{aligned} &\eta(X)g(AU + \nabla\alpha, Y) + g(\phi X, \nabla_Y U) \\ &= g((\nabla_Y A)X, \xi) - g(A\phi AX, Y) + \alpha g(A\phi X, Y), \end{aligned} \tag{1.9}$$

which shows that

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha \tag{1.10}$$

because of (1.3). From (1.9) we also have

$$\nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha, \tag{1.11}$$

where we have used (1.1) and (1.8). Since W is orthogonal to U, we see, using (1.1), that

$$\mu g(\nabla_X W, \xi) = g(AU, X). \tag{1.12}$$

Because of (1.2), the structure Jacobi operator R_ξ is given by

$$R_\xi X = R(X,\xi)\xi = \frac{c}{4}(X - \eta(X)\xi) + \alpha AX - \eta(AX)A\xi \tag{1.13}$$

for any vector field X on M.

2. The Jacobi operator of real hypersurfaces

Let M be a real hypersurface in a complex space form $M_n(c), c \neq 0$ satisfying $R_\xi \phi = \phi R_\xi$, which means that the eigenspace of R_ξ is invariant by the structure operator ϕ. Then by (1.13) we have

$$\alpha(\phi AX - A\phi X) = g(A\xi, X)U + g(U, X)A\xi. \tag{2.1}$$

We set $\Omega = \{p \in M : \mu(p) \neq 0\}$, and suppose that Ω be nonvoid, that is, ξ is not a principal curvature vector on M. In the sequel we discuss our arguments on the open set Ω of M unless otherwise stated. Then, it is, using (2.1), clear that $\alpha \neq 0$ on Ω. So a function λ given by $\beta = \alpha\lambda$ is defined. Thus, replacing X by U in (2.1) and using (1.4), we find

$$\phi AU = \lambda A\xi - A^2\xi. \tag{2.2}$$

In what follows we assume that

$$A^2\xi = \rho A\xi + \sigma\xi \tag{$*$}$$

for certain scalars ρ and σ on M. Then we have

$$\sigma = \alpha(\lambda - \rho). \tag{2.3}$$

Combining $(*)$ with (2.2), it is seen that

$$AU = (\rho - \lambda)U. \tag{2.4}$$

From $(*)$ and (1.7) we also have

$$AW = \mu\xi + (\rho - \alpha)W \tag{2.5}$$

and hence

$$A^2W = \rho AW + \alpha(\lambda - \rho)W \tag{2.6}$$

by virtue of $\mu \neq 0$.

Differentiating $(*)$ covariantly along Ω and taking account of (1.1), we find

$$\begin{aligned} &((\nabla_X A)A\xi, Y) + g(A(\nabla_X A)\xi, Y) \\ &\qquad + g(A^2\phi AX, Y) - \rho g(A\phi AX, Y) \\ &= g(\nabla\rho, X)g(A\xi, Y) + \rho g((\nabla_X A)\xi, Y) \\ &\qquad + g(\nabla\sigma, X)\eta(Y) + \alpha(\lambda - \rho)g(\phi AX, Y) \end{aligned} \tag{2.7}$$

for any vector fields X and Y on M, which together with (1.3) and (1.10) implies that

$$(\nabla_\xi A)A\xi = \rho AU - \frac{c}{4}U + \frac{1}{2}\nabla\beta.$$

We put $X = \xi$ in (2.7) and use (1.10), (2.4), and the last equation. Then we obtain

$$\begin{aligned}\frac{1}{2}\nabla\beta = & -A\nabla\alpha + \rho\nabla\alpha + g(\nabla\rho,\xi)A\xi + g(\nabla\sigma,\xi)\xi \\ & -\{(\rho-\lambda)(\rho+\alpha-3\lambda) - \frac{c}{4}\}U,\end{aligned} \tag{2.8}$$

which connected to (2.4) and $(*)$ gives

$$\begin{aligned}&\frac{1}{2}(A\nabla\beta - \rho\nabla\beta) \\ &= -A^2\nabla\alpha + 2\rho A\nabla\alpha - \rho^2\nabla\alpha + g(\nabla\sigma,\xi)A\xi \\ &\quad + g(\sigma\nabla\rho - \rho\nabla\sigma,\xi)\xi + \lambda\{(\rho-\lambda)(\rho+\alpha-3\lambda) - \frac{c}{4}\}U.\end{aligned} \tag{2.9}$$

Because of (2.3), we see, using (2.8), that

$$\frac{1}{2}d\beta(\xi) = \alpha d\alpha(\xi) + \mu d\alpha(W),$$

where d denotes the exterior differential operator, which together with $\beta = \alpha\lambda$ implies that

$$\alpha d\lambda(\xi) = (2\alpha - \lambda)d\alpha(\xi) + 2\mu d\alpha(W). \tag{2.10}$$

We verify also, making use of (2.5) and (2.8), that

$$\frac{1}{2}d\beta(W) - \alpha d\alpha(W) = \mu(d\rho(\xi) - d\alpha(\xi)),$$

which enables us to obtain

$$\alpha d\lambda(W) = (2\alpha - \lambda)d\alpha(W) + 2\mu(d\rho(\xi) - d\alpha(\xi)). \tag{2.11}$$

Now, define a 1-form u by $u(X) = g(U, X)$ for any vector field X, it is, using (1.3) and (2.7), seen that

$$\begin{aligned}&\frac{c}{4}\{u(Y)\eta(X) - u(X)\eta(Y)\} + \frac{c}{2}(\rho - \alpha)g(\phi Y, X)\\&- g(A^2\phi AX, Y) + g(A^2\phi AY, X) + 2\rho g(\phi AX, AY)\\&- \alpha(\lambda - \rho)\{g(\phi AY, X) - g(\phi AX, AY)\}\\=\ & g(AY, (\nabla_X A)\xi) - g(AX, (\nabla_Y A)\xi) + d\rho(Y)g(A\xi, X)\\&- d\rho(X)g(A\xi, Y) + d(\beta - \rho\alpha)(Y)\eta(X) - d(\beta - \rho\alpha)(X)\eta(Y).\end{aligned} \tag{2.12}$$

On the other hand, differentiating (2.5) covariantly along Ω, we find

$$(\nabla_X A)W + A\nabla_X W = d\mu(X)\xi + \mu\nabla_X\xi + d(\rho - \alpha)(X)W + (\rho - \alpha)\nabla_X W.$$

By taking the inner product this with W, we get

$$g((\nabla_X A)W, W) = -2(\rho - \lambda)u(X) + d\rho(X) - d\alpha(X) \tag{2.13}$$

with the aid of (2.4) and the fact that W is a unit orthogonal to ξ. We also have by applying ξ

$$\mu g((\nabla_X A)W, \xi) = (\rho - \lambda)(\rho - 2\alpha)u(X) + \frac{1}{2}d\beta(X) - \alpha d\alpha(X), \tag{2.14}$$

where we have used (1.12) and (2.4), which together with the Codazzi equation (1.3) gives

$$\mu(\nabla_W A)\xi = \{(\rho - \lambda)(\rho - 2\alpha) - \frac{c}{2}\}U + \frac{1}{2}\nabla\beta - \alpha\nabla\alpha, \tag{2.15}$$

$$\mu(\nabla_\xi A)W = \{(\rho - \lambda)(\rho - 2\alpha) - \frac{c}{4}\}U + \frac{1}{2}\nabla\beta - \alpha\nabla\alpha. \tag{2.16}$$

Replacing X by μW in (2.12) and making use of (1.7), (1.10), (2.4), (2.5), (2.6), (2.14), and (2.15), we find

$$\begin{aligned}&\alpha A\nabla\alpha - \frac{1}{2}A\nabla\beta + \frac{1}{2}(\rho - \alpha)\nabla\beta\\&+ \alpha(\lambda - \rho)\nabla\alpha - \alpha(\lambda - \alpha)\nabla\rho\\=\ & \{(\rho - \lambda)(\alpha\lambda - 2\rho\lambda + 2\rho\alpha + \alpha^2) + \frac{c}{2}(\lambda - \alpha)\}U\\&- \mu d\rho(W)A\xi - \mu d(\beta - \rho\alpha)(W)\xi.\end{aligned} \tag{2.17}$$

If we replace X by $A\xi$ in (2.7) and take account of (1.7), (1.10), (2.4), (2.13)–(2.16), and $(*)$, then we obtain

$$\begin{aligned}&\frac{1}{2}(A\nabla\beta-\rho\nabla\beta)+\alpha^2\nabla\lambda+\mu^2\nabla\rho\\&=g(A\xi,\nabla\rho)A\xi+g(A\xi,\nabla\sigma)\xi\\&\quad+\{(\rho-\lambda)(2\rho\lambda-3\alpha\rho+2\alpha\lambda)+\frac{c}{4}(3\alpha-2\lambda)\}U.\end{aligned}$$

Substituting (2.9) into this, we find

$$\begin{aligned}&\alpha^2\nabla\lambda+\mu^2\nabla\rho-A^2\nabla\alpha+2\rho A\nabla\alpha-\rho^2\nabla\alpha\\&=\{g(A\xi,\nabla\rho)-d\sigma(\xi)\}A\xi\\&\quad+\{g(A\xi,\nabla\sigma)+\rho d\sigma(\xi)-(\beta-\rho\alpha)d\rho(\xi)\}\xi\\&\quad+\{(\rho-\lambda)(\rho\lambda-3\alpha\rho+\alpha\lambda+3\lambda^2)+\frac{c}{4}(3\alpha-\lambda)\}U.\end{aligned}\tag{2.18}$$

Now, it is, using (2.1), verified that

$$\alpha\phi A\phi AX+\alpha A^2X=\rho g(A\xi,X)A\xi+\eta(X)A\xi-g(AU,X)U\tag{2.19}$$

because of properties of almost contact metric structure.

On the other hand, we have from (1.9)

$$\nabla_XU+g(A^2\xi,X)\xi=\phi(\nabla_XA)\xi+\phi A\phi AX+\alpha AX,$$

which together with $(*)$ and (2.19) yields

$$\begin{aligned}&\nabla_XU+\{\rho g(A\xi,X)+\alpha(\lambda-\rho)\eta(X)\}\xi\\&=\phi(\nabla_XA)\xi+\alpha AX-A^2X\\&\quad+\frac{1}{\alpha}\{\rho g(A\xi,X)+\alpha(\lambda-\rho)\eta(X)\}A\xi-\frac{\rho-\lambda}{\alpha}g(AU,X)U.\end{aligned}\tag{2.20}$$

If we put $X=U$ in (2.20) and take account of (2.4), then we get

$$\nabla_UU=\phi(\nabla_UA)\xi+(\rho-\lambda)(2\alpha-\rho)U.\tag{2.21}$$

Using (1.7) and (2.4), we can write the equation (1.11) as

$$\nabla_\xi U=\mu(3\lambda-3\rho+\alpha)W-\alpha(\lambda-\alpha)\xi+\phi\nabla\alpha.\tag{2.22}$$

Since the exterior derivative du of a 1-form u is given by

$$du(Y,X)=\frac{1}{2}\{Yu(X)-Xu(Y)-u([Y,X])\},$$

we verify, using (1.8), (2.22), and (*), that

$$du(\xi,X)=(3\lambda-2\rho)\mu w(X)+g(\phi\nabla\alpha,X), \tag{2.23}$$

where a 1-form w is defined by $w(X)=g(W,X)$, which shows that

$$du(\xi,U)=\mu d\alpha(W). \tag{2.24}$$

Now, differentiating (2.4) covariantly, we find

$$(\nabla_X A)U+A(\nabla_X U)=d(\rho-\lambda)(X)U+(\rho-\lambda)\nabla_X U. \tag{2.25}$$

If we take the inner product this with ξ and make use of (1.3) and (2.22) then we obtain

$$\begin{aligned}&(\nabla_U A)\xi\\ &=\frac{c}{4}\mu W+\varepsilon U-\mu(3\lambda-3\rho+\alpha)\{AW-(\rho-\lambda)W\}\\ &\quad+\alpha(\lambda-\alpha)\{A\xi-(\rho-\lambda)\xi\}-A\phi\nabla\alpha+(\rho-\lambda)\phi\nabla\alpha,\end{aligned} \tag{2.26}$$

where we have put $\varepsilon=d(\rho-\lambda)(\xi)$. Thus, it follows, using (2.1), that

$$\begin{aligned}&\phi(\nabla_U A)\xi\\ &=\{3(\lambda-\rho)(\lambda-\alpha)-\frac{c}{4}-\frac{1}{\alpha}d\alpha(U)\}U+\mu\varepsilon W\\ &\quad+(\rho-\lambda)(\nabla\alpha-d\alpha(\xi)\xi)-A\nabla\alpha+\frac{1}{\alpha}g(A\xi,\nabla\alpha)A\xi.\end{aligned} \tag{2.27}$$

Substituting this into (2.21), we get

$$\begin{aligned}&\nabla_U U\\ &=\{(\rho-\lambda)(3\lambda-\alpha-\rho)+\frac{c}{4}\\ &\quad+\frac{1}{\alpha}d\alpha(U)\}U+A\nabla\alpha-(\rho-\lambda)\nabla\alpha\\ &\quad+\{(\rho-\lambda)d\alpha(\xi)-g(A\xi,\nabla\alpha)\}\xi-\mu\{\varepsilon+\frac{1}{\alpha}g(A\xi,\nabla\alpha)\}W,\end{aligned} \tag{2.28}$$

which tells us that

$$
\begin{aligned}
&A(\nabla_U U) - (\rho - \lambda)\nabla_U U \\
&= A^2\nabla\alpha - 2(\rho - \lambda)A\nabla\alpha + (\rho - \lambda)^2\nabla\alpha \\
&\quad + \{(\rho - \lambda)d\alpha(\xi) - g(A\xi, \nabla\alpha)\}\{A\xi - (\rho - \lambda)\xi\} \\
&\quad - \mu(\varepsilon + \frac{1}{\alpha}g(A\xi, \nabla\alpha))\{AW - (\rho - \lambda)W\}.
\end{aligned} \tag{2.29}
$$

Because of (1.3) and (1.4), the relationship (2.25) implies that

$$
\begin{aligned}
&\frac{c}{4}\mu\{(\eta(Y)w(X) - \eta(X)w(Y)\} \\
&\qquad + g(AX, \nabla_Y U) - g(AY, \nabla_X U) \\
&= d(\rho - \lambda)(Y)u(X) - d(\rho - \lambda)(X)u(Y) \\
&\qquad + (\rho - \lambda)\{(\nabla_Y u)(X) - (\nabla_X u)(Y)\}.
\end{aligned} \tag{2.30}
$$

If we replace X by U in (2.30) and make use of (2.4), then we obtain

$$
A(\nabla_U U) - (\rho - \lambda)\nabla_U U = \mu^2(\nabla\lambda - \nabla\rho) + d(\rho - \lambda)(U)U,
$$

which together with (2.29) gives

$$
\begin{aligned}
&A^2\nabla\alpha - 2\rho A\nabla\alpha + \rho^2\nabla\alpha + 2\lambda(A\nabla\alpha - \rho\nabla\alpha) + \lambda^2\nabla\alpha \\
&= \{g(A\xi, \nabla\alpha) - (\rho - \lambda)d\alpha(\xi)\}\{A\xi - (\rho - \lambda)\xi\} \\
&\quad + \mu\{\varepsilon + \frac{1}{\alpha}g(A\xi, \nabla\alpha)\}\{AW - (\rho - \lambda)W\} \\
&\quad + \mu^2(\nabla\lambda - \nabla\rho) + d(\rho - \lambda)(U)U.
\end{aligned} \tag{2.31}
$$

Substituting (2.18) into (2.31) and using (2.9), we find

$$
\begin{aligned}
&2\mu^2(\nabla\rho - \nabla\lambda) + d(\lambda - \rho)(U)U - 3(\lambda - \alpha)\{(\rho - \lambda)^2 - \frac{c}{4}\}U \\
&= \{g(A\xi, \nabla\alpha) - d\sigma(\xi) - 2\lambda d\rho(\xi)\}A\xi \\
&\quad + \{g(A\xi, \nabla\sigma) + (\rho - 2\lambda)d\sigma(\xi) - \sigma d\rho(\xi)\}\xi \\
&\quad + \{g(A\xi, \nabla\alpha) - (\rho - \lambda)d\alpha(\xi)\}\{A\xi - (\rho - \lambda)\xi\} \\
&\quad + \mu\{\varepsilon + \frac{1}{\alpha}g(A\xi, \nabla\alpha)\}\{AW - (\rho - \lambda)W\}.
\end{aligned} \tag{2.32}
$$

Since $A\xi$ and AW are orthogonal to U, it follows that

$$
d(\rho - \lambda)(U) = 3(\lambda - \alpha)\{(\rho - \lambda)^2 - \frac{c}{4}\}. \tag{2.33}
$$

Using this, (1.7) and (2.5), the equation (2.32) can be written as

$$\mu^2(\nabla\rho - \nabla\lambda) = \mu^2(a\xi + bW) + 3(\lambda - \alpha)\{(\rho-\lambda)^2 - \frac{c}{4}\}U$$

for some functions a and b, which shows that $a = \varepsilon$ and $b = d(\rho-\lambda)(W)$. Since $\lambda - \alpha$ does not vanish on Ω, it follows that

$$\alpha(\nabla\rho - \nabla\lambda) = \alpha(\varepsilon\xi + bW) + \{3(\rho-\lambda)^2 - \frac{3}{4}c\}U. \tag{2.34}$$

On the other hand, if we take the inner product (2.32) with W, and straitforward calculation, then we obtain

$$\alpha^2 d\rho(W) = 3\alpha\mu d\rho(\xi) + \alpha(4\alpha - 3\lambda)d\alpha(W) - \mu(4\alpha-\lambda)d\alpha(\xi),$$

where we have used (2.3), (2.10) and (2.11). Comparing this with (2.10) and (2.11), it is seen that

$$\alpha d(\rho-\lambda)(W) = \mu d(\rho-\lambda)(\xi),$$

that is, $b\alpha = \mu\varepsilon$. From this and (1.7), the equation (2.34) becomes

$$\alpha(\nabla\rho - \nabla\lambda) = \varepsilon A\xi + 3\{(\rho-\lambda)^2 - \frac{c}{4}\}U. \tag{2.35}$$

REMARK 1. We notice here, using (2.33), that $\rho - \lambda \neq 0$ on Ω. □

3. Lemmas

We will continue now, our arguments under the same hypotheses $R_\xi\phi = \phi R_\xi$ and $(*)$ as in section 2.

First of all, we prove

LEMMA 3.1. *Let M be a real hypersurface satisfying $R_\xi\phi = \phi R_\xi$ and $(*)$. Then we have*

$$\alpha(\nabla\rho - \nabla\lambda) = \theta U \tag{3.1}$$

on Ω, where θ is given by

$$\theta = 3(\rho-\lambda)^2 - \frac{3}{4}c. \tag{3.2}$$

Proof. By differentiating (2.35) covariantly and taking the skew-symmetric parts obtained one, we find

$$\begin{aligned} & d\alpha(Y)d(\rho-\lambda)(X) - d\alpha(X)d(\rho-\lambda)(Y) \\ & - 6(\rho-\lambda)\{d(\rho-\lambda)(Y)u(X) - d(\rho-\lambda)(X)u(Y)\} \\ = {} & d\varepsilon(Y)g(A\xi,X) - d\varepsilon(X)g(A\xi,Y) - \frac{c}{2}\varepsilon g(\phi Y,X) \\ & - 2\varepsilon g(A\phi AY,X) + \theta du(Y,X) \end{aligned}$$

by virtue of (1.3), or using (2.35) again,

$$\begin{aligned} & \theta\{u(Y)d\alpha(X) - u(X)d\alpha(Y)\} - \frac{c}{2}\varepsilon\alpha g(\phi Y,X) \\ & - 2\varepsilon\alpha g(A\phi AY,X) + \theta\alpha du(Y,X) \\ = {} & \{\varepsilon d\alpha(Y) - \alpha d\varepsilon(Y) + 6(\rho-\lambda)\varepsilon u(Y)\}g(A\xi,X) \\ & - \{\varepsilon d\alpha(X) - \alpha d\varepsilon(X) + 6(\rho-\lambda)\varepsilon u(X)\}g(A\xi,Y). \end{aligned} \tag{3.3}$$

Putting $Y=\xi$ in (3.3), we get

$$\begin{aligned} & \varepsilon\{d\alpha(X) + 6(\rho-\lambda)u(X)\} - \alpha d\varepsilon(X) \\ = {} & \{\frac{\varepsilon}{\alpha}d\alpha(\xi) - d\varepsilon(\xi)\}g(AX,\xi) + \theta du(\xi,X) \\ & + \{\frac{\theta}{\alpha}d\alpha(\xi) - 2\varepsilon(\rho-\lambda)\}u(X). \end{aligned}$$

Combining this with (3.3), we have

$$\begin{aligned} & \theta\{u(Y)d\alpha(X) - u(X)d\alpha(Y)\} \\ & - \frac{c}{2}\varepsilon\alpha g(\phi Y,X) - 2\varepsilon\alpha g(A\phi AY,X) + \theta\alpha du(Y,X) \\ = {} & \{\theta du(\xi,Y) - 2\varepsilon(\rho-\lambda)u(Y) + \frac{\theta}{\alpha}d\alpha(\xi)u(Y)\}g(A\xi,X) \\ & - \{\theta du(\xi,X) - 2\varepsilon(\rho-\lambda)u(X) + \frac{\theta}{\alpha}d\alpha(\xi)u(X)\}g(A\xi,Y). \end{aligned}$$

If we put $Y=U$ in this and take account of (1.4), (2.4), and (2.24), then we get

$$\begin{aligned} & \{-\mu\theta d\alpha(W) + \theta(\lambda-\alpha)d\alpha(\xi) - 2\varepsilon(\rho-\lambda)\mu^2\}A\xi \\ = {} & \theta\{\mu^2\nabla\alpha - d\alpha(U)U\} + \frac{c}{2}\mu\alpha\varepsilon W - 2\alpha\varepsilon(\rho-\lambda)\mu AW + \theta\alpha TU, \end{aligned} \tag{3.4}$$

where $g(TU, X) = du(U, X)$.

On the other hand, it is, using (2.20), verified that

$$\begin{aligned} TU = &\{(\rho-\lambda)(3\lambda-\alpha-\rho)+\frac{c}{4}+\frac{1}{\alpha}d\alpha(U)\}U \\ &+A\nabla\alpha+(\lambda+\alpha-\rho)\nabla\alpha \\ &-\frac{1}{2}\nabla\beta-\mu\varepsilon W+(\rho-\lambda)d\alpha(\xi)\xi-\{d\alpha(\xi)+\frac{\mu}{\alpha}d\alpha(W)\}A\xi, \end{aligned}$$

where we have used (2.14) and (2.27), or using (2.3), (2.8), and (2.10),

$$\begin{aligned} TU = &\{2(\rho-\lambda)(3\lambda-\alpha-\rho)+\frac{c}{2}+\frac{1}{\alpha}d\alpha(U)\}U+\alpha(\nabla\alpha-\nabla\lambda) \\ &-\{d\alpha(\xi)+\frac{\mu}{\alpha}d\alpha(W)-d\lambda(\xi)\}A\xi. \end{aligned}$$

Substituting this into (3.4), we find

$$\begin{aligned} &\{\theta(\lambda d\alpha(\xi)-\alpha d\lambda(\xi))-2\varepsilon(\rho-\lambda)\mu^2\}A\xi \\ &=\theta\alpha(\lambda\nabla\alpha-\alpha\nabla\lambda)+\frac{c}{2}\mu\alpha\varepsilon W-2\varepsilon\alpha(\rho-\lambda)\mu AW \\ &+\theta\alpha\{2(\rho-\lambda)(3\lambda-\alpha-\rho)+\frac{c}{2}\}U. \end{aligned} \tag{3.5}$$

If we take the inner product (3.5) with W and use (2.5), we find

$$\begin{aligned} &\theta\{\mu\lambda d\alpha(\xi)-\mu\alpha d\lambda(\xi)-\alpha\lambda d\alpha(W)+\alpha^2 d\lambda(W)\} \\ &\qquad\qquad +2\{(\rho-\lambda)^2-\frac{c}{4}\}\epsilon\mu\alpha=0. \end{aligned} \tag{3.6}$$

Since $\epsilon = d(\rho-\lambda)(\xi)$, we, using (2.11) and this, verify that

$$3\theta\{(\lambda-2\alpha)d\alpha(\xi)-2\mu d\alpha(W)\}+\theta\alpha\{8d\rho(\xi)-5d\lambda(\xi)\}=0,$$

which together with (2.10) implies that $\theta d(\rho-\lambda)(\xi)=0$, that is, $\theta\varepsilon=0$ and hence $\varepsilon=0$ because of (3.2). Thus (3.1) is established by virtue of (2.35). □

LEMMA 3.2. *Under the same hypotheses as those in Lemma 3.1, if* $\theta \neq 0$ *and* $d(h-\rho)(\xi)=0$, *then we have*

$$\nabla\alpha=(3\lambda-2\rho)U, \tag{3.7}$$

$$\alpha\nabla\lambda=\{2(\rho-\lambda)(3\lambda-\alpha-\rho)+\lambda(3\lambda-2\rho)+\frac{c}{2}\}U, \tag{3.8}$$

$$\alpha\nabla\rho=(\rho^2-2\rho\alpha+2\alpha\lambda-\frac{c}{4})U. \tag{3.9}$$

Proof. Using (3.1) and (3.2), it is clear that $\alpha\nabla\theta = 6(\rho - \lambda)\theta U$. Thus, differentiating (3.1) covariantly and taking the skew-symmetric part obtained one, we find

$$d\alpha(Y)d(\rho-\lambda)(X) - d\alpha(X)d(\rho-\lambda)(Y) = \theta du(Y,X),$$

which together with (3.1) and $\theta \neq 0$ gives

$$d\alpha(Y)u(X) - d\alpha(X)u(Y) = \alpha du(Y,X). \tag{3.10}$$

Since $\epsilon = 0$ and $\theta \neq 0$, (3.6) can be written as

$$\lambda d\alpha(\xi) - \alpha d\lambda(\xi) = 2\mu d\alpha(W) - 2\alpha(d\rho(\xi) - d\alpha(\xi)), \tag{3.11}$$

where we have used (2.11).

On the other hand, if we take the trace of (1.9) and make use of (1.3), (1.4) and (3.10), then we obtain

$$\alpha d(h-\alpha)(\xi) = \mu d\alpha(W),$$

which together with (3.11) implies that

$$\lambda d\alpha(\xi) - \alpha d\lambda(\xi) = 2\alpha d(h-\rho)(\xi).$$

Thus, (3.5) turns out to be

$$\lambda\nabla\alpha - \alpha\nabla\lambda = 2\{(\rho-\lambda)^2 + (\rho-\lambda)(\alpha-2\lambda) - \frac{c}{4}\}U \tag{3.12}$$

since we have $\epsilon = 0$ and $d(h-\rho)(\xi) = 0$ was assumed.

Using the same method as that used to derive (3.10) from (3.1), we can derive from (3.12) the following:

$$\begin{aligned} &d\lambda(Y)d\alpha(X) - d\lambda(X)d\alpha(Y) \\ &= (\rho-\lambda)\{d(\alpha-2\lambda)(Y)u(X) - d(\alpha-2\lambda)(X)u(Y)\} \\ &\quad + \{(\rho-\lambda)^2 + (\rho-\lambda)(\alpha-2\lambda) - \frac{c}{4}\}du(Y,X), \end{aligned}$$

where we have used (3.1). From (3.10), (3.12) and the last equation, we verify that $\theta d\alpha(W) = 0$ and hence $d\alpha(\xi) = 0$ by virtue of $\theta \neq 0$. Thus putting $Y = \xi$ in (3.10), we have $du(\xi, X) = 0$ for any vector field X. Therefore (2.23) turns out to be $\phi\nabla\alpha = \mu(2\rho - 3\lambda)W$, which shows that $\nabla\alpha = (3\lambda - 2\rho)U$. Thus (3.10) is reduced to $du = 0$. So (2.28) implies that

$$\frac{1}{2}\nabla g(U,U) = \{(\rho-\lambda)(3\lambda-\alpha-\rho) + (\lambda-\alpha)(3\lambda-2\rho) + \frac{c}{4}\}U$$

withe the aid of (3.7). From this and (3.7), it follows, using $g(U,U) = \alpha(\lambda-\alpha)$, that (3.8) is accomplished. Because of (3.1) and (3.8), we see that (3.9) is established. Hence, required formulas are obtained. □

REMARK 2. In the proof of above lemma, we verify that Lemma 3.2 is valid if we replace the assumption $\theta \neq 0$ and $d(h-\rho)(\xi) = 0$ by $du(\xi, X) = 0$ for any vector field X.

LEMMA 3.3. *Let M be a real hypersurface satisfying $R_\xi \phi = \phi R_\xi$ and $(*)$ in $M_n(c)$. If $\nabla\sigma = 0$, then Ω is void.*

REMARK 3. This lemma was proved in the previous paper [8]. But, we give a simple proof of it here.

Proof. Since $\nabla\sigma = 0$ is assumed, we see, making use of (2.3) and (3.1), that

$$(\rho - \lambda)\nabla\alpha + \theta U = 0, \tag{3.13}$$

which implies $d\alpha(\xi) = 0$ and $d\alpha(W) = 0$ by virtue of Remark 1. By differentiating (3.13) and using (3.1), we obtain (3.10) and hence $\theta du(\xi, X) = 0$.

Now, if we suppose that $du(\xi, X) \neq 0$. Then we have $\theta = 0$ and hence $(\rho - \lambda)^2 = \frac{c}{4}$. So (3.13) and Remark 1 tells us that $\nabla\alpha = 0$. Since we have $\nabla\rho = \nabla\lambda$, it is seen that $\nabla\beta = \alpha\nabla\rho$. From these and (2.10) we have $d\rho(\xi) = 0$. Thus, (2.8) turns out to be

$$\alpha\nabla\rho = 2(\rho - \lambda)(2\lambda - \alpha)U \tag{3.14}$$

by virtue of $(\rho - \lambda)^2 = \frac{c}{4}$. Using above arguments, it is, making use of (3.14), verified that $(\rho - \lambda)(2\lambda - \alpha)du(\xi, X) = 0$ and consequently $2\lambda - \alpha = 0$, that is, $2\mu^2 + \alpha^2 = 0$, which produces a contradiction. Thus, we have $du(\xi, X) = 0$ on Ω. By Remark 2, we verify that Lemma 3.2 is valid. Combining (3.13) with (3.2) and (3.7), we have

$$\rho(\rho - \lambda) = \frac{3}{4}c. \tag{3.15}$$

Thus, (3.2) is reduced to

$$\theta = (\rho - \lambda)(2\rho - 3\lambda). \tag{3.16}$$

Differentiation (3.15) gives $(\rho - \lambda)\nabla\rho = -\rho\theta U$ because of (3.1), which together with (3.16) yields

$$\nabla\rho = \rho(3\lambda - 2\rho)U$$

and hence $\nabla\rho = (\rho^2 - \frac{9}{4}c)U$ with the aid of (3.15), or using (3.7)

$$\nabla\rho = \rho\nabla\alpha. \tag{3.17}$$

If we substitute $\nabla\rho = (\rho^2 - \frac{9}{4}c)U$ into (3.9), then we obtain

$$\alpha(\rho^3 - \frac{9}{4}c\rho + \frac{3}{2}c) = \rho^3 - \frac{c}{4}\rho. \tag{3.18}$$

Differentiating this and taking account of (3.17), we find

$$\{9\alpha(\rho^2 - \frac{c}{4}) - 3\rho^2 - \frac{c}{4}\}\nabla\alpha = 0,$$

which connected to (3.18) gives $\nabla\alpha = 0$. Therefore we see, using (3.7) and (3.16), that $\theta = 0$, which together with (3.2) and (3.15) implies that $\lambda = 0$, a contradiction. Hence, Ω is void. □

REMARK 4. We notice here, using (1.6) and (1.13), that the condition (*) with $\sigma = \frac{c}{4}$, that is, $A^2\xi = \rho A\xi + \frac{c}{4}\xi$ if and only if $R_\xi A = AR_\xi$ on Ω. In fact, from (1.13) we have

$$\begin{aligned} &g(R_\xi Y, AX) - g(R_\xi X, AY) \\ &= g(A^2\xi, Y)g(A\xi, X) - g(A^2\xi, X)g(A\xi, Y) \\ &+ \frac{c}{4}\{g(A\xi, Y)\eta(X) - g(A\xi, X)\eta(Y)\}. \end{aligned} \tag{3.19}$$

The if part is immediately true from above equation. So we are going to check the only if part. Since $R_\xi A - AR_\xi = 0$, by putting $X = \xi$ in (3.19), we find

$$\frac{c}{4}A\xi = \beta A\xi - \alpha A^2\xi + \frac{c}{4}\alpha\xi. \tag{3.20}$$

Substituting this into (3.19), we obtain

$$\begin{aligned} &\mu\{w(Y)g(A^2\xi, X) - w(X)g(A^2\xi, Y)\} \\ &= \beta\{\eta(X)g(A\xi, Y) - \eta(Y)g(A\xi, X)\}, \end{aligned}$$

where we have used (1.7). Replacing X by $A\xi$, we have

$$\mu^2 A^2\xi = (\gamma - \beta\alpha)A\xi + (\beta^2 - \alpha\gamma)\xi$$

by virtue of $\mu^2 = \beta - \alpha^2$. If we put $\mu^2\rho = \gamma - \beta\alpha$, then a function ρ is defined on Ω. Hence, it follows that

$$A^2\xi = \rho A\xi + (\beta - \rho\alpha)\xi.$$

From this and (3.20) we see that $\beta = \rho\alpha + \frac{c}{4}$ because of $\mu \neq 0$.

4. Real hypersurfaces satisfying

Let M be a real hypersurface of a complex space form $M_n(c), c \neq 0$. Suppose that the Ricci tensor S of type (1,1) and the Jacobi operator R_ξ with respect to the structure vector ξ commute to each other, that is, $R_\xi S = SR_\xi$. Then we have

$$\begin{aligned} &g(A^3\xi, Y)g(A\xi, X) - g(A^3\xi, X)g(A\xi, Y) \\ &= g(A^2\xi, Y)g(hA\xi - \frac{c}{4}\xi, X) - g(A^2\xi, X)g(hA\xi - \frac{c}{4}\xi, Y) \\ &+ \frac{c}{4}h\{g(A\xi, Y)\eta(X) - g(A\xi, X)\eta(Y)\}, \end{aligned}$$

where we have used (1.6) and (1.13), which shows that

$$\alpha A^3\xi = (\alpha h - \frac{c}{4})A^2\xi + (\gamma - \beta h + \frac{c}{4})A\xi + \frac{c}{4}(\beta - h\alpha)\xi. \tag{4.1}$$

Combining above two equations, we, using (1.7), that

$$\begin{aligned} &\mu\{g(A^2\xi, Y)w(X) - g(A^2\xi, X)w(Y)\} \\ &= \beta\{\eta(Y)g(A\xi, X) - \eta(X)g(A\xi, Y)\}. \end{aligned}$$

Putting $Y = A\xi$ in this, we find

$$\mu^2 g(A^2\xi, X) = \mu\gamma w(X) - \beta\alpha g(A\xi, X) + \beta^2\eta(X).$$

Thus, it follows that

$$\mu^2 A^2\xi = (\gamma - \beta\alpha)A\xi + (\beta^2 - \alpha\gamma)\xi$$

and consequently

$$A^2\xi = \rho A\xi + (\beta - \rho\alpha)\xi, \tag{4.2}$$

where we have put $\mu^2\rho = \gamma - \beta\alpha$ and $\mu^2(\beta - \rho\alpha) = \beta^2 - \alpha\gamma$ on Ω. Thus the condition $(*)$ stated in section 2 is established.

From (4.2) we have

$$A^3\xi = (\rho^2 - \beta - \rho\alpha)A\xi + \rho(\beta - \rho\alpha)\xi.$$

Comparing this with (4.1), we find

$$\mu(h-\rho)(\beta-\rho\alpha-\frac{c}{4})=0. \tag{4.3}$$

Let Ω_0 be a set of points such that $\mu(p)(h(p)-\rho(p))\neq 0$ at $p\in M$. Then we have $\beta-\rho\alpha=\frac{c}{4}$ on Ω_0. Thus, by Lemma 3.3 we see that ξ is a principal curvature vector. Hence we have $h=\rho$ on Ω. From this fact and (4.2), the equation (1.6) turns out to be

$$S\xi=g(S\xi,\xi)\xi, \tag{4.4}$$

where we have put $g(S\xi,\xi)=\frac{c}{2}(n-1)-(\beta-h\alpha)$.

If $g(S\xi,\xi)$ =const., then we conclude that ξ is a principal curvature vector by virtue of Lemma 3.3. Hence (2.1) implies that $A\phi=\phi A$ if $\alpha\neq 0$.

Thus, by Theorem O-MR, we have

THEOREM 4.1. *Let M be a real hypersurface in a nonflat complex space form $M_n(c)$. If it satisfies $R_\xi\phi=\phi R_\xi$, $R_\xi S=SR_\xi$ and $g(S\xi,\xi)=$* const., *then M is locally congruent to be of type* A *provided that $\eta(A\xi)\neq 0$.*

REMARK 5. It is proved in [8] that a real hypersurface satisfying $R_\xi\phi=\phi R_\xi$ and at the same time $S\xi=\tau\xi$ for some constant τ in a complex space form $M_n(c)$ is a Hopf hypersurface in $M_n(c)$.

According to Theorem 4.1 and Remark 4, we have

COROLLARY 4.2. *Let M be a real hypersurface in a nonflat complex space form. If it satisfies $R_\xi\phi=\phi R_\xi$ and at the same time $R_\xi A=AR_\xi$, then M is locally congruent to be of type* A *provided that $\eta(A\xi)\neq 0$.*

Now, we prove

THEOREM 4.3. *Let M be a real hypersurface with constant mean curvature in a nonflat complex space form. If it satisfies $R_\xi\phi=\phi R_\xi$ and at the same time satisfies $R_\xi S=SR_\xi$, then M is locally congruent to be of type* A *provided that $\eta(A\xi)\neq 0$.*

Proof. By Lemma 3.3 and (4.3), we may only discuss the case where $h=\rho$ on Ω. The mean curvature of M being constant, (3.1) becomes

$$\alpha\nabla\lambda=-\theta U. \tag{4.5}$$

This shows that $d\lambda(\xi) = 0$ and $d\lambda(W) = 0$. Thus, we verify, using (2.10) and (2.11), that $\{(2\lambda - \alpha)^2 + 4\mu^2\}d\alpha(\xi) = 0$ since we have $\nabla\rho = 0$. So we have $d\alpha(\xi) = 0$. Hence, the same method as that used in Lemma 3.2, it is, making use of (4.5), seen that $\theta du(\xi, X) = 0$ for any vector field X.

If we assume $du(\xi, X) \neq 0$ on Ω, then we have $\theta = 0$ and hence $\nabla\lambda = 0$. So, by definition we have $\nabla\beta = \lambda\nabla\alpha$, which together with (2.3) and $d\alpha(\xi) = 0$ gives $d\sigma(\xi) = 0$. We also have $d\alpha(W) = 0$ because of (2.11). From these facts (2.8) and (2.17) are reduced respectively to

$$\begin{aligned} A\nabla\alpha &= (\rho - \frac{1}{2}\lambda)\nabla\alpha + (\rho - \lambda)(2\lambda - \alpha)U, \\ (\alpha - \frac{1}{2}\lambda)A\nabla\alpha &+ (\frac{1}{2}\lambda\rho + \frac{1}{2}\alpha\lambda - \alpha\rho)\nabla\alpha \\ &= (\rho - \lambda)(3\alpha\lambda - 2\rho\alpha - 2\lambda^2)U, \end{aligned} \tag{4.6}$$

where we have used $(\rho - \lambda)^2 = \frac{c}{4}$. Combing the last two equations, it follows that

$$\lambda^2\nabla\alpha = 2(\rho - \lambda)(\alpha\lambda - 2\lambda^2 + 2\alpha^2 - 4\rho\alpha)U.$$

In the same way as above, we have from this

$$(\alpha\lambda - 2\lambda^2 + 2\alpha^2 - 4\rho\alpha)du(\xi, X) = 0,$$

which shows that $\alpha\lambda - 2\lambda^2 + 2\alpha^2 - 4\rho\alpha = 0$. From this we see that $\nabla\alpha = 0$ because of (3.2) and $\nabla\lambda = 0$. Therefore (4.6) implies that $2\lambda - \alpha = 0$, a contradiction. Thus, it follows that $du(\xi, X) = 0$ on Ω. So we have $\nabla\alpha = (3\lambda - 2\rho)$ by virtue of (2.23). By Remark 2, we see that (3.8) and (3.9) are valid. Since the mean curvature of M being constant, that is, $\nabla\rho = 0$, (3.9) means that $\rho^2 - 2\alpha\rho + 2\alpha\lambda - \frac{c}{4} = 0$, which implies that $(\rho - \lambda)\nabla\alpha = \alpha\nabla\lambda$. From this, (3.2), (3.7) and (4.5) we have $\rho(\rho - \lambda) = \frac{3}{4}c$, which enables us to obtain $\nabla\lambda = 0$ and hence $\theta = 0$ by virtue of (4.5). Thus, (3.8) gives $2(\rho - \lambda)(2\lambda - \alpha) + \lambda(3\lambda - 2\rho) = 0$, which tells us that $\nabla\alpha = 0$ because ρ and λ are both constant. By (3.7) we have $2\lambda - \alpha = 0$, a contradiction. Therefore we conclude that Ω is void. So by (2.2) we have $A\phi = \phi A$ if $\alpha \neq 0$. Owing to Theorem O-MR, we arrive at the conclusion. □

References

[1] J. Berndt, *Real hypersurfaces with constant principal curvatures in a complex hyperbolic space*, J. Reine Agnew. Math. **395** (1989), 132–141.

[2] T. E. Cecil and P. J. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc. **269** (1982), 481–499.

[3] J. T. Cho and U.-H. Ki, *Real hypersurfaces of a complex projective space in terms of the Jacobi operators*, Acta Math. Hungar. **80** (1998), 155–167.

[4] ———, *Jacobi operators on real hypersurfaces of a complex projective space*, Tsukuba J. Math. **22** (1998), 145–156.

[5] U.-H. Ki, *Cyclic-parallel real hypersurfaces of a complex space form*, Tsukuba J. Math. **12** (1988), 259–268.

[6] U.-H. Ki, H.-J. Kim, and A.-A. Lee, *The Jacobi operator of real hypersurfaces in a complex space form*, Comm. Korean Math. Soc. **13** (1998), 545–560.

[7] U.-H. Ki, S.-J. Kim, and S.-B. Lee, *Some characterizations of a real hypersurfaces of type A*, Kyungpook Math. J. **31** (1991), 73–82.

[8] U.-H. Ki, A. A. Lee, and S.-B. Lee, *On real hypersurfaces of a complex space form in terms of Jacobi operators*, Comm. Korean Math. Soc. **13** (1998), 317–336.

[9] U.-H. Ki and H. Song, *Jacobi operators on a semi-invariant submanifold of codimension 3 in a complex projective space*, Nihonkai Math. J. **14** (2003), 1–16.

[10] U.-H. Ki and Y. J. Suh, *On real hypersurfaces of a complex space form*, Math. J. Okayama Univ. **32** (1990), 207–221.

[11] M. Kimura, *Real hypersurfaces and complex submanifolds in complex projective space*, Trans. Amer. Math. Soc. **296** (1986), 137–149.

[12] M. Kimura and S. Maeda, *Lie derivatives on real hypersurfaces in a complex projective space*, Czechoslovak Math. J. **45** (1995), 135–148.

[13] S. Montiel, *Real hypersurfaces of a complex hyperbolic space*, J. Math. Soc. Japan **37** (1985), 515–535.

[14] S. Montiel and A. Romero, *On some real hypersurfaces of a complex hyperbolic space*, Geometriae Dedicata **20** (1986), 245–261.

[15] R. Niebergall and P. J. Ryan, *Real hypersurfaces in complex space forms, in Tight and Taut submanifolds*, Cambridge Univ. Press (1998(T.E. Cecil and S.S. Chern, eds.)), 233–305.

[16] M. Okumura, *On some real hypersurfaces of a complex projective space*, Trans. Amer. Math. Soc. **212** (1975), 355–364.

[17] R. Takagi, *On homogeneous real hypersurfaces in a complex projective space*, Osaka J. Math. **10** (1973), 495–506.

[18] ———, *Real hypersurfaces in a complex projective space with constant principal curvatures I, II*, J. Math. Soc. Japan **27** (1975), 43–53, 507–516.

[19] K. Yano and M. Kon, *CR submanifolds of Kaehlerian and Sasakian manifolds*, Birkhäuser, 1983.

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