Bull. Korean Math. Soc. **42** (2005), No. 2, pp. 327–335

ON A COMPACT AND MINIMAL REAL HYPERSURFACE IN A QUATERNIONIC PROJECTIVE SPACE

YEONG-WU CHOE AND IMSOON JEONG

ABSTRACT. For a compact and orientable minimal real hypersurface M in QP^n, we prove that if the minimum of the sectional curvatures of M is $3/(4n-1)$, then M is isometric to the geodesic minimal hypersphere $M^Q_{0,n-1}$.

1. Introduction

Let QP^n be a quaternionic projective space of real dimension $4n$, $n \geq 2$, with the Fubini-Study metric G of constant Q-sectional curvature 4 and let M be a connected $(4n\text{-}1)$-dimensional real hypersurface of QP^n.

Let N be a local unit normal vector field to M. We denote by $\{J_i\}_{i=1,2,3}$ is a local basis of the quaternionic Kähler structure of QP^n. Then $U_i = -J_iN, i = 1,2,3$ are tangent to M, which will be called structure vectors [10].

Now we put $f_i(X) = g(X, U_i)$, for arbitrary $X \in TM, i = 1,2,3$, where TM is the tangent bundle of M and g denotes the Riemannian metric induced from the metric G.

Now, let us consider the following conditions that the second fundamental tensor A of M in QP^n may satisfy

$$(1.1) \qquad (\nabla_X A)Y = \sum_{i=1}^{3}\{g(X, \phi_i Y)U_i - f_i(Y)\phi_i X\},$$

$$(1.2) \qquad g((A\phi_i - \phi_i A)X, Y) = 0,$$

for any $i = 1,2,3$ and any tangent vector fields X and Y to M.

Received February 3, 2004.

2000 Mathematics Subject Classification: 53C25, 53C42.

Key words and phrases: minimal real hypersurface, quaternionic projective space, quaternionic space form.

Pak[10] investigated the above conditions and showed that they are equivalent to each other. Moreover he used the condition 1.1 to find a lower bound of $\|\nabla A\|$ for real hypersurfaces in QP^n. In fact, it was shown that $\|\nabla A\|^2 \geq 24(n-1)$ for any hypersurfaces and the equality holds if and only if the condition 1.1 holds. In this case it was also known that M is locally congruent to a real hypersurface of type A_1 or type A_2, which means a tube of radius r over $QP^k(1 \leq k \leq n-1)$ in the notion of Berndt[1], and Martínez and Pérez[8].

Now the purpose of this paper is to give another new characterization of a minimal real hypersurface in QP^n by using Lemmas, to be stated in Section 3, which is a quaternionic version of result of Kon[5].

Now we prepare the following theorem [5] without proof in order to compare with our result :

THEOREM 1.1. *Let M be a compact orientable real minimal hypersurface of CP^n. If the sectional curvature K of M satisfies $K \geq 1/(2n-1)$, then M is the geodesic minimal hypersphere $M^c_{0,n-1}$.*

2. Preliminaries

A *quaternionic Kähler manifold* is a Riemannian manifold $(\tilde{M}, G)$ on which there exists a 3-dimensional vector bundle $\tilde{V}$ of tensors of type $(1,1)$ with a local basis of almost Hermitian structures $\{J_i\}_{i=1,2,3}$ satisfying the following conditions :

1. $J_i{}^2 = -id$, $i = 1,2,3$, $J_iJ_j = -J_jJ_i = J_k$, where id denotes the identity endomorphism on $T\tilde{M}$ and (i,j,k) is a cyclic permutation of $(1,2,3)$.
2. If $\tilde{\nabla}$ denotes the Riemannian connection on $\tilde{M}$, then there exist three local 1-forms $P_i, i = 1,2,3$ on $\tilde{M}$ such that
$$\tilde{\nabla}_X J_i = P_k(X)J_j - P_j(X)J_k$$
for all vector field X on $\tilde{M}$, where (i,j,k) is a cyclic permutation of $(1,2,3)$.

Let $Q(X)$ be the 4-dimensional subspace spanned by vectors X, J_1X, J_2X and J_3X for any $X \in T_p\tilde{M}, p \in \tilde{M}$. If the sectional curvature of any section for $Q(X)$ depends only on X, we call it *Q-sectional curvature.* A *quaternionic space form* of Q-sectional curvature c is a connected quaternionic Kähler manifold with constant Q-sectional curvature c.

The standard models of quaternionic space forms are the quaternionic projective space $QP^n(c)(c > 0)$, the quaternionic space $Q^n(c = 0)$ and the quaternionic hyperbolic space $QH^n(c)(c < 0)$ ([1]).

The curvature tensor $\tilde{R}$ of $QP^n(c), n \geq 2$, is given by

$$\tilde{R}(X,Y)Z = \frac{c}{4}[G(Y,Z)X - G(X,Z)Y + \sum_{k=1}^{3}\{G(J_kY,Z)J_kX - G(J_kX,Z)J_kY - 2G(J_kX,Y)J_kZ\}]$$

for any vector fields X, Y and Z on $QP^n(c)$([2]).

From now on we denote by QP^n the quaternionic projective space of constant Q-sectional curvature 4.

Let M be a connected $(4n-1)$-dimensional real hypersurface of QP^n and let N be a local unit normal vector field to M. The Riemannian connection $\tilde{\nabla}$ in QP^n and ∇ in M are related by the following formulas for arbitrary vector fields X and Y tangent to M:

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX,Y)N \tag{2.1}$$

and

$$\tilde{\nabla}_X N = -AX, \tag{2.2}$$

where A is the second fundamental tensor of M in QP^n. The mean curvature h of M is defined by $h = \frac{1}{4n-1} TrA$.

If $h = 0$, then M is said to be *minimal*. Eigenvectors of the second fundamental tensor A are called *principal curvature vectors* and called the corresponding eigenvalues *principal curvatures.* We put

$$J_iX = \phi_iX + f_i(X)N, \quad J_iN = -U_i, \quad i = 1,2,3 \tag{2.3}$$

for any vector field X tangent to M, where ϕ_iX is the tangential parts of J_iX, ϕ_i are tensors of type (1,1) and f_i are 1-forms for $i = 1,2,3$.

As $J_i{}^2 = -id, i = 1,2,3$, id denoting the identity endomorphism on TQP^n, we get

$$\phi_i^2X = -X + f_i(X)U_i, \quad f_i(\phi_iX) = 0, \quad \phi_iU_i = 0, \quad i = 1,2,3 \tag{2.4}$$

for any vector field X tangent to M. As $J_iJ_j = -J_jJ_i = J_k$, (i,j,k) being a cyclic permutation of $(1,2,3)$, we obtain

$$f_i(U_i) = 1, \quad f_i(U_j) = f_i(U_k) = 0, \tag{2.5}$$

$$\phi_iX = \phi_j\phi_kX - f_k(X)U_j = -\phi_k\phi_jX + f_j(X)U_k \tag{2.6}$$

and

$$f_i(X) = f_j(\phi_kX) = -f_k(\phi_jX), \tag{2.7}$$

for any vector field X tangent to M.

It is also easy to see that for any X,Y tangent to M,

$$(2.8)\quad g(\phi_i X, Y)+g(X,\phi_i Y)=0,\quad g(\phi_i X,\phi_i Y)=g(X,Y)-f_i(X)f_i(Y)$$

and

$$(2.9)\qquad \phi_i U_j = -\phi_j U_i = U_k,$$

where (i,j,k) is a cyclic permutation of $(1,2,3)$.

The covariant derivatives of $J_i, i=1,2,3,$ are given by

$$\tilde{\nabla}_X J_i = P_k(X)J_j - P_j(X)J_k$$

for any $X \in TQP^n$,where P_i, $i = 1,2,3,$ are local 1-forms on QP^n. Then from (2.1) and (2.2) we obtain

$$(2.10)\qquad \nabla_X U_i = -P_j(X)U_k + P_k(X)U_j + \phi_i AX$$

and

$$(2.11)\quad (\nabla_X\phi_i)Y = -P_j(X)\phi_k Y + P_k(X)\phi_j Y + f_i(Y)AX - g(AX,Y)U_i$$

for any vector fields X, Y tangent to M, where (i,j,k) is a cyclic permutation of $(1,2,3)$.

Since ϕ_i is skew-symmetric and A is symmetric, (2.10) implies that

$$(2.12)\qquad \operatorname{div} U_i = \sum_{a=1}^{4n-1} g(\nabla_a U_i, e_a) = -P_j(U_k) + P_k(U_j),$$

where (i,j,k) is a cyclic permutation of $(1,2,3)$.

From the expression of the curvature tensor of $QP^n, n\geq 2$, the equations of Gauss and Codazzi are respectively given by

$$\begin{aligned}(2.13)\ R(X,Y)Z \ = \ & g(Y,Z)X - g(X,Z)Y + \sum_{i=1}^{3}\{g(\phi_i Y,Z)\phi_i X \\ & - g(\phi_i X,Z)\phi_i Y - 2g(\phi_i X,Y)\phi_i Z\} \\ & + g(AY,Z)AX - g(AX,Z)AY\end{aligned}$$

and

$$\begin{aligned}(2.14)\quad (\nabla_X A)Y - (\nabla_Y A)X \ = \ & \sum_{i=1}^{3}\{f_i(X)\phi_i Y - f_i(Y)\phi_i X \\ & + 2g(X,\phi_i Y)U_i\}\end{aligned}$$

for any X, Y, Z tangent to M, where R denotes the curvature tensor of M ([8]).

We now put

$$T := \nabla_{U_i}U_i + \nabla_{U_j}U_j + \nabla_{U_k}U_k + (\operatorname{div} U_i)U_i + (\operatorname{div} U_j)U_j + (\operatorname{div} U_k)U_k$$

and take an orthonormal basis $\{e_a\}_{a=1,\dots,4n-1}$ of tangent vectors to M such that

$$\begin{aligned} e_n &:= \phi_i e_1 \ ,\dots,\ e_{2(n-1)} := \phi_i e_{n-1}, \\ e_{2n-1} &:= \phi_j e_1 \ ,\dots,\ e_{3(n-1)} := \phi_j e_{n-1}, \\ e_{3n-2} &:= \phi_k e_1 \ ,\dots,\ e_{4(n-1)} := \phi_k e_{n-1}, \\ e_{4n-3} &:= U_i \ ,\ e_{4n-2} := U_j \ ,\ e_{4n-1} := U_k. \end{aligned}$$

Then it follows from (2.10) and (2.12) that

$$T = \phi_i A U_i + \phi_j A U_j + \phi_k A U_k \tag{2.15}$$

We note that T is a global vector field defined on M. For later use we compute $div(T) = \sum_{i=1}^{4n-1} g(\nabla_{e_a} T, e_a)$. Differentiating (2.15) covariantly and using (2.4), (2.6), (2.9)–(2.11), and (2.14), we have

$$\begin{aligned} \mathrm{div}(T) = (TrA)&(\sum_{i=1}^{3} g(AU_i, U_i)) - \sum_{i=1}^{3} g(A^2U_i, U_i) + \sum_{i=1}^{3} Tr(A\phi_i)^2 \\ &- \sum_{l=1}^{n-1} \{ g((\nabla_{e_l}A)\phi_i e_l - (\nabla_{\phi_i e_l}A)e_l + (\nabla_{\phi_j e_l}A)\phi_k e_l \\ &- (\nabla_{\phi_k e_l}A)\phi_j e_l, U_i) + g((\nabla_{e_l}A)\phi_j e_l - (\nabla_{\phi_j e_l}A)e_l \\ &+ (\nabla_{\phi_k e_l}A)\phi_i e_l - (\nabla_{\phi_i e_l}A)\phi_k e_l, U_j) + g((\nabla_{e_l}A)\phi_k e_l \\ &- (\nabla_{\phi_k e_l}A)e_l + (\nabla_{\phi_i e_l}A)\phi_j e_l - (\nabla_{\phi_j e_l}A)\phi_i e_l, U_k)\} \\ &- g((\nabla_{U_j}A)U_k - (\nabla_{U_k}A)U_j, U_i) - g((\nabla_{U_k}A)U_i \\ &- (\nabla_{U_i}A)U_k, U_j) - g((\nabla_{U_i}A)U_j - (\nabla_{U_j}A)U_i, U_k), \end{aligned}$$

or equivalently

$$\begin{aligned} \mathrm{div}(T) = (TrA)&(\sum_{i=1}^{3} g(AU_i, U_i)) - \sum_{i=1}^{3} g(A^2U_i, U_i) \\ &+ \sum_{i=1}^{3} Tr(A\phi_i)^2 + 12(n-1) \end{aligned}$$

Moreover we should explain model subspaces which will appear in our Theorem 3.3. We consider the Hopf fibration $\tilde{\pi}$:

$$S^3 \longrightarrow S^{4n+3} \xrightarrow{\ \tilde{\pi}\ } QP^n,$$

where S^k denotes the Euclidean sphere of curvature 1. In S^{4n+3} we have the family of generalized Clifford surfaces whose spheres lie in quaternionic subspaces(cf. [7]):

$$M_{4p+3,4q+3} := S^{4p+3}\left(\sqrt{\frac{4p+3}{2(2n+1)}}\right) \times S^{4q+3}\left(\sqrt{\frac{4q+3}{2(2n+1)}}\right),$$

where $p+q=n-1$. Then we have a fibration π :

$$S^3 \longrightarrow M_{4p+3,4q+3} \xrightarrow{\ \pi\ } M^Q_{p,q},$$

compatible with $\tilde{\pi}$. In the special case $p=0$, $M^Q_{0,n-1}$ is called the geodesic minimal hypersphere of QP^n, and is a homogeneous, positively curved manifold diffeomorphic to the sphere (for details, see [1, 7, 10]).

3. Main results

In order to prove our theorem, we need the following result.

LEMMA 3.1. *Let M be a minimal real hypersurface of QP^n. Then*

$$\begin{aligned} g(\nabla^2 A, A) \;&=\; \sum_{a,b} g((R(e_b,e_a)A)e_b, Ae_a) - 9\,TrA^2 \\ &\quad + \frac{3}{2}\sum_i \|[\phi_i, A]\|^2, \end{aligned} \tag{3.1}$$

where $[\phi_i, A]$ denotes $\phi_i A - A\phi_i$.

Proof. Let $\{e_a\}$ be an orthonormal frame for M. Then (2.14) implies

$$\sum_a (\nabla_{e_a} A)e_a = 0. \tag{3.2}$$

Thus, from (2.10), (2.11), (2.14), and (3.2) we obtain

$$\begin{aligned} &g(\nabla^2 A, A) \\ &= \sum_{a,b} g((\nabla_{e_b}\nabla_{e_b} A)e_a, Ae_a) \\ &= \sum_{a,b} g((R(e_b,e_a)A)e_b - \sum_i \{g(\nabla_{e_b}U_i, e_a)\phi_i e_b \\ &\quad + f_i(e_a)(\nabla_{e_b}\phi_i)e_b - g(\nabla_{e_b}U_i, e_b)\phi_i e_a - f_i(e_b)(\nabla_{e_b}\phi_i)e_a \\ &\quad + 2g(e_a, (\nabla_{e_b}\phi_i)e_b)U_i + 2g(e_a, \phi_i e_b)\nabla_{e_b}U_i\}, Ae_a) \end{aligned} \tag{3.3}$$

$$= \sum_{a,b} g((R(e_b,e_a)A)e_b, Ae_a) - 3\sum_i g(A^2U_i,U_i) + 3\sum_i Tr(A\phi_i)^2.$$

Since $Tr(A\phi_i)^2 = -TrA^2 + g(A^2U_i,U_i) + \frac{1}{2}\|[\phi_i,A]\|^2$, we obtain

$$(3.4)\quad -3\sum_i g(A^2U_i,U_i)+3\sum_i Tr(A\phi_i)^2 = -9\,TrA^2+\frac{3}{2}\sum_i \|[\phi_i,A]\|^2.$$

Substituting (3.4) into (3.3), we have our assertion. □

LEMMA 3.2. *Let M be a compact and orientable minimal real hypersurface in QP^n. If the minimum of the sectional curvatures of M is $3/(4n-1)$, then $\|\nabla A\|^2 = 24(n-1)$ and $g((A\phi_i - \phi_i A)X,Y) = 0$, $i = 1,2,3$.*

Proof. We choose an orthonormal frame $\{e_a\}$ of M such that

$$Ae_a = \lambda_a e_a, \quad a = 1,2,\cdots,4n-1.$$

We denote by K_{ab} the sectional curvature of M spanned by e_a and e_b. Then we have

$$\begin{aligned} &\sum_{a,b} g((R(e_a,e_b)A)e_a, Ae_b) \\ =\ &\sum_{a,b}\{g(R(e_a,e_b)Ae_a,Ae_b) - g(AR(e_a,e_b)e_a,Ae_b\} \\ =\ &\frac{1}{2}\sum_{a,b}(\lambda_a-\lambda_b)^2K_{ab} \\ \geq\ &\frac{3}{2(4n-1)}\sum_{a,b}(\lambda_a-\lambda_b)^2 = 3\,TrA^2. \end{aligned}$$

Consequently, we see

$$(3.5)\qquad 3\,TrA^2 - \sum_{a,b} g((R(e_a,e_b)A)e_a,Ae_b) \leq 0.$$

Since we have $\frac{1}{2}\triangle\, TrA^2 = \|\nabla A\|^2 + g(\nabla^2A,A)$, we obtain

$$(3.6)\qquad \int_M \|\nabla A\|^2 * 1 = -\int_M g(\nabla^2A,A) * 1.$$

From Lemma 3.1, (3.6) and (2.16) we have

$$\begin{aligned} 0 &\le \int_M [\, \|\nabla A\|^2 - 24(n-1) + \frac{1}{2}\sum_i \| \, [\phi_i, A]\|^2 \,] * 1 \\ &= \int [\, 9TrA^2 - \sum_{a,b} g((R(e_a, e_b)A)e_a, Ae_b) - 24(n-1) \\ &\qquad - \sum_i \| \, [\phi_i, A] \, \|^2 \,] * 1 \\ &= \int [\, 3TrA^2 - \sum_{a,b} g((R(e_a, e_b)A)e_a, Ae_b) \,] * 1. \end{aligned}$$

From this and (3.5) we complete the proof. □

Combining Lemma 3.2 and the result of Kwon and Pak[6], we see that M is $M^Q_{p,q}$.

On the other hand if $p, q \ge 1$, then the sectional curvature K of $M^Q_{p,q}$ takes values 0 for some plane section [10]. But the sectional curvature K of $M^Q_{0,n-1}$ satisfies $K \ge 3/(4n-1)$.

Consequently, M is the geodesic minimal hypersphere $M^Q_{0,n-1}$.

THEOREM 3.3. *Let M be a compact and orientable minimal real hypersurface in QP^n, If the minimum of the sectional curvatures of M is $3/(4n-1)$, then M is isometric to the geodesic minimal hypersphere $M^Q_{0,n-1}$.*

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Yeong-Wu Choe and Imsoon Jeong, Department of Mathematics, College of Sciences, Catholic University of Daegu, Gyeongsan 713-702, Korea
E-mail: ywchoe@cu.ac.kr
jis3557174@hanmail.net