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OSCILLATION CRITERIA FOR SECOND-ORDER NONLINEAR DIFFERENCE EQUATIONS WITH "SUMMATION SMALL" COEFFICIENT

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ABSTRACT. We consider the second-order nonlinear difference equation

$$\Delta(a_n h(x_{n+1})\Delta x_n) + p_{n+1} f(x_{n+1}) = 0,\ n \geq n_0, \tag{1}$$

where $\{a_n\}, \{p_n\}$ are sequences of integers with $a_n > 0, \{p_n\}$ is a real sequence without any restriction on its sign. h and f are real-valued functions. We obtain some necessary conditions for (1) existing nonoscillatory solutions and sufficient conditions for (1) being oscillatory.

1. Introduction

We are mainly concerned with oscillation of solutions of the following second-order nonlinear difference equation:

$$\Delta(a_n h(x_{n+1})\Delta x_n) + p_{n+1} f(x_{n+1}) = 0,\ n \geq n_0, \tag{1.1}$$

where Δ is the forward difference equation, $\{a_n\}$ is an eventually positive real sequence, $\{p_n\}$ is a real sequence without any restriction on its sign. $h \in C(R, [c, \infty))$ here $c > 0$, $f \in C(R, R)$ and $xf(x) > 0$ for $x \neq 0$. We assume that the following conditions are satisfied:

(c_1) $\sum\limits_{s=n_0}^{\infty} 1/a_s = \infty$, for all $n_0 \geq 0$.

(c_2) $f(x) - f(y) = F(x,y)(x-y)$, for all $x, y \neq 0$, where F is a nonnegative function, $\inf\limits_{x \neq 0} F(x,y)/h(x) \geq \varepsilon$, ε is a positive constant.

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(c_3) f/h is a monotonically nondecreasing function.

A number of dynamical behaviors of solutions of second-order difference equations are possible. Our concern is motivated by several papers, especially those by Li Wantong[2], Zhang Zhenguo and Zhang Jinlian[8], Thandapani et.al.[3], Wong and Agarwal[4, 5], as well as Zhang and Chen[7]. In [8], the authors obtain oscillation criteria for equation

$$\Delta(a_n \Delta x_n) + p_n x_{g_n} = 0. \tag{1.2}$$

But in (1.2), $p_n \geq 0$, $p_n \not\equiv 0$. In [3], the authors obtain oscillation criteria for a special case of (1.1) ($h(x) = 1$, $a_n = 1$)

$$\Delta^2 y_n + q_{n+1} f(y_{n+1}) = 0, \quad n = 0, 1, \cdots . \tag{1.3}$$

In this paper, we weakened the condition that p_n has the designed sign and use discrete inequalities to offer sufficient conditions for (1.1) is oscillatory and some necessary conditions for (1.1) existing nonoscillatory solution, which extends some results in [1, 3, 4, 5, 7, 8]. Our technique is an extension of the methods employed in the works of Zhang and Chen[7] and Zhang Zhenguo et.al.[8]. The main results in this paper are discrete analogues of the corresponding results for the continuous version by Yan[6].

By a solution of (1.1), we mean a nontrivial sequence $\{x_n\}$ satisfying (1.1) for $n \geq n_0$. A solution $\{x_n\}$ is said to be oscillatory if it is neither eventually positive nor negative, and nonoscillatory otherwise.

2. Several lemmas

LEMMA 2.1. *Suppose $\{x_n\}$ is a positive (negative) solution of* (1.1) *for $n \in N_{n_0}^{\alpha}$, where $N_{n_0}^{\alpha} = \{n_0, n_0+1, \cdots, \alpha\}$, α can be infinite. Assume that there exists an integer $n_1 \in N_{n_0}^{\alpha}$ and $m > 0$ such that*

$$-w_{n_0} + \sum_{s=n_0}^{n-1} p_{s+1} + \sum_{s=n_0}^{n_1-1} \frac{w_{s+1}\Delta(f(x_{s+1}))}{f(x_{s+1})} \geq m, \quad \text{for all } n \in N_{n_1}^{\alpha}, \tag{2.1}$$

where

$$w_n = \frac{a_n h(x_{n+1}) \Delta x_n}{f(x_{n+1})}. \tag{2.2}$$

Then

$$a_n h(x_{n+1}) \Delta x_n \leq (\geq) - m f(x_{n_1}), \quad n \in N_{n_1}^{\alpha}. \tag{2.3}$$

Proof. Define w_n as (2.2), then

$$\Delta w_n = -p_{n+1} - \frac{w_{n+1}\Delta(f(x_{n+1}))}{f(x_{n+1})}. \tag{2.4}$$

Summing (2.4) from n_0 to $n-1$, where $n \in N_{n_1}^{\alpha}$ and using (2.1), we find

$$\begin{aligned} -w_{n_0} + \sum_{s=n_0}^{n-1} p_{s+1} + \sum_{s=n_0}^{n_1-1} \frac{w_{s+1}\Delta(f(x_{s+1}))}{f(x_{s+1})} \\ + \sum_{s=n_1}^{n-1} \frac{w_{s+1}\Delta(f(x_{s+1}))}{f(x_{s+1})} = -w_n, \end{aligned} \tag{2.5}$$

then

$$-w_n \geq m + \sum_{s=n_1}^{n-1} \frac{a_{s+1}h(x_{s+2})\Delta x_{s+1}\Delta(f(x_{s+1}))}{f(x_{s+1})f(x_{s+2})} > 0, \tag{2.6}$$

which follows from (c_2). Hence, if $\{x_n\}$ be a positive solution of (1.1), then $\Delta x_n < 0$, for $n \in N_{n_1}^{\alpha}$. Set $-w_n = v_n > 0$, and (2.6) becomes

$$v_n \geq m - \sum_{s=n_1}^{n-1} \frac{v_{s+1}\Delta(f(x_{s+1}))}{f(x_{s+1})}, \quad \text{for } n \in N_{n_1}^{\alpha}.$$

We consider the corresponding equation

$$u_n = m - \sum_{s=n_1}^{n-1} \frac{u_{s+1}\Delta(f(x_{s+1}))}{f(x_{s+1})}, \quad \text{for } n \in N_{n_1}^{\alpha}.$$

We follow the same arguments in the proofs of Lemma 2.1 [7] and conclude our proof. □

COROLLARY 2.1. *Let $\{x_n\}$ is a positive solution of* (1.1). *If*

$$\lim_{n\to\infty} \inf \sum_{s=n_1}^{n-1} p_{s+1} > -\infty,$$

then

$$\sum_{s=n_1}^{\infty} \frac{w_{s+1}\Delta(f(x_{s+1}))}{f(x_{s+1})} < \infty. \tag{2.7}$$

Proof. Otherwise, then

$$\sum_{s=n_1}^{\infty} \frac{w_{s+1}\Delta(f(x_{s+1}))}{f(x_{s+1})} = \infty,$$

hence there exists $n_1^* \geq n_1$ such that (2.1) holds. Hence, by Lemma 2.1

$$ca_n \Delta x_n \leq a_n h(x_{n+1})\Delta x_n \leq -mf(x_{n_1^*}), \text{ for } n \geq n_1^*, \tag{2.8}$$

where $m > 0$ is a constant. (2.8) and (c_1) imply that x_n is negative eventually, which is a contradiction. The proof is complete. □

By Corollary 2.1, it is easy to see that the following result is true.

COROLLARY 2.2. *Assume that* $\sum\limits_{s=n_1}^{\infty} p_{s+1} = \infty$. *Then every solution of* (1.1) *is oscillatory.*

We now consider the case that $\lim\limits_{n\to\infty} \sum\limits_{s=n_1}^{n-1} p_{s+1}$ exists.

LEMMA 2.2. *Suppose that* (c_1)–(c_3) *and*

$$\lim_{|x|\to\infty} |f(x)| = \infty \tag{2.9}$$

hold. If $\{x_n\}$ *is a nonoscillatory solution of* (1.1), *then*

$$\lim_{n\to\infty} w_n = 0 \tag{2.10}$$

and

$$w_n = \sum_{s=n}^{\infty} p_{s+1} + \sum_{s=n}^{\infty} \frac{w_{s+1}\Delta(f(x_{s+1}))}{f(x_{s+1})} \geq P_n + \frac{\varepsilon}{2}\sum_{s=n}^{\infty}\frac{w_{s+1}^2}{a_{s+1}}, \tag{2.11}$$

here $P_n = \sum\limits_{s=n}^{\infty} p_{s+1}$.

Proof. Let $\{x_n\}$ be a nonoscillatory of (1.1). Without loss of generality, we assume $x_n > 0$ for $n \geq n_0$. From (2.5) we have

$$\begin{aligned} w_n &= w_{n_0} - \sum_{s=n_0}^{n-1} p_{s+1} - \sum_{s=n_0}^{n-1} \frac{w_{s+1}\Delta(f(x_{s+1}))}{f(x_{s+1})} \\ &= w_{n_0} - \sum_{s=n_0}^{\infty} p_{s+1} - \sum_{s=n_0}^{\infty} \frac{w_{s+1}\Delta(f(x_{s+1}))}{f(x_{s+1})} \\ &\quad + \sum_{s=n}^{\infty} p_{s+1} + \sum_{s=n}^{\infty} \frac{w_{s+1}\Delta(f(x_{s+1}))}{f(x_{s+1})} \\ &= \alpha + P_n + \sum_{s=n}^{\infty} \frac{w_{s+1}\Delta(f(x_{s+1}))}{f(x_{s+1})}, \end{aligned} \tag{2.12}$$

where

$$\alpha = w_{n_0} - \sum_{s=n_0}^{\infty} p_{s+1} - \sum_{s=n_0}^{\infty} \frac{w_{s+1}\Delta(f(x_{s+1}))}{f(x_{s+1})}.$$

We claim $\alpha = 0$. If $\alpha < 0$, we choose n_2 so large that

$$\left|\sum_{s=n_2}^{n-1} p_{s+1}\right| < -\frac{\alpha}{4}, \quad n \geq n_2,$$

and

$$\sum_{s=n_2}^{\infty} \frac{w_{s+1}\Delta(f(x_{s+1}))}{f(x_{s+1})} < -\frac{\alpha}{4}.$$

If we take $n_0 = n_1 = n_2$ in Lemma 2.1, then all the assumptions of Lemma 2.1 hold and so

$$ca_n\Delta x_n \leq a_n h(x_{n+1})\Delta x_n \leq -mf(x_{n_2}), \quad \text{for } n \geq n_2,$$

so

$$\Delta x_n \leq -\frac{M}{a_n}, \quad \text{for } n \geq n_2,$$

here $M = mf(x_{n_2})/c > 0$, which in view of (c_1) contradicts the positivity of $\{x_n\}$.

If $\alpha > 0$, from (2.12) we have $\lim_{n\longrightarrow\infty} w_n = \alpha > 0$, which implies that $\Delta x_n > 0$ eventually. So there exists $n_1 \geq n_0$ such that

$$w_n \geq \frac{\alpha}{2}, \quad n \geq n_1. \tag{2.13}$$

Define

$$r(t) = f(x_{n+1}) + (t-n-1)\Delta(f(x_{n+1})), \quad n+1 \le t \le n+2.$$

It is easy to see that $r'(t) = \Delta(f(x_{n+1}))$ and $f(x_{n+1}) \le r(t) \le f(x_{n+2})$ for $n+1 \le t \le n+2$. Hence

$$\begin{aligned}\frac{\Delta(f(x_{n+1}))}{f(x_{n+1})} &= \int_{n+1}^{n+2} \frac{\Delta(f(x_{n+1}))}{f(x_{n+1})} dt \\ &= \int_{n+1}^{n+2} \frac{r'(t)}{f(x_{n+1})} dt \ge \int_{n+1}^{n+2} \frac{r'(t)}{r(t)} dt.\end{aligned}$$

From (2.13) we obtain

$$\begin{aligned}\infty > \sum_{s=n_1}^{\infty} \frac{w_{s+1}\Delta(f(x_{s+1}))}{f(x_{s+1})} &\ge \frac{\alpha}{2} \sum_{s=n_1}^{\infty} \frac{\Delta(f(x_{s+1}))}{f(x_{s+1})} \\ &\ge \frac{\alpha}{2} \sum_{s=n_1}^{\infty} \int_{s+1}^{s+2} \frac{r'(t)}{r(t)} dt \\ &= \frac{\alpha}{2} \lim_{n\longrightarrow\infty} \ln\left(\frac{r(n)}{r(n_1+1)}\right).\end{aligned}$$

Hence $\ln r(t) < \infty$, which implies that $f(x_n) < +\infty$ as $n \longrightarrow \infty$. From (2.9) we know $\{x_n\}$ is bounded. On the other hand from (2.13) and (c_3), we have

$$a_n \Delta x_n \ge \frac{\alpha}{2} \frac{f(x_{n+1})}{h(x_{n+1})} \ge \frac{\alpha f(x_{n_1+1})}{2h(x_{n_1+1})}, \quad n \ge n_1.$$

From (c_1) it follows that $\lim_{n\longrightarrow\infty} x_n = \infty$, which contradicts the boundedness of $\{x_n\}$. So

$$\begin{aligned} w_n &= P_n + \sum_{s=n}^{\infty} \frac{w_{s+1}\Delta(f(x_{s+1}))}{f(x_{s+1})} \\ &= P_n + \sum_{s=n}^{\infty} \frac{w_{s+1}^2 F(x_{s+2}, x_{s+1}) f(x_{s+2})}{a_{s+1} h(x_{s+2}) f(x_{s+1})} \qquad (2.14) \\ &\ge P_n + \varepsilon \sum_{s=n}^{\infty} \frac{w_{s+1}^2 f(x_{s+2})}{a_{s+1} f(x_{s+1})}.\end{aligned}$$

In the following we will discuss the two cases:

(i) $\Delta x_n > 0$, then $f(x_{n+2}) \geq f(x_{n+1})$, so $f(x_{n+2})/f(x_{n+1}) \geq 1 > 1/2$, then from (2.14) we know (2.11) holds.

(ii) $\Delta x_n < 0$, then $\{x_n\}$ is a monotonically decreasing positive sequence. If $\lim_{n\to\infty} x_n = l > 0$, then there exists a sufficient large n_1 such that for $n \geq n_1$ we have $3f(l)/4 \leq f(x_{n+2}) \leq f(x_{n+1}) \leq 3f(l)/2$, so $f(x_{n+2})/f(x_{n+1}) \geq 1/2$. If $l = 0$, then there exists a sufficient large n_1 such that for $n \geq n_1$ we have $\varepsilon/4 \leq f(x_{n+2}) \leq f(x_{n+1}) \leq \varepsilon/2$, ε is an arbitrarily small constant, so $f(x_{n+2})/f(x_{n+1}) \geq 1/2$. Then from (2.14) we have (2.11) holds. □

3. Main results

For studying the oscillatory properties of (1.1), we construct the following sequence for each m for which $\alpha_m(n)$ is defined.

$$\alpha_0(n) = P_n = \sum_{s=n}^{\infty} p_{s+1}, \quad \alpha_1(n) = \frac{\varepsilon}{2} \sum_{s=n}^{\infty} \frac{\alpha_0^2(s+1)_+}{a_{s+1}},$$

$$\alpha_{m+1}(n) = \sum_{s=n}^{\infty} \frac{[\alpha_0(s+1) + \frac{\varepsilon}{2}\alpha_m(s+1)]_+^2}{a_{s+1}}, \quad m = 1, 2, \cdots. \tag{3.1}$$

Here $\beta(s)_+$ and $[\beta(s)]_+$ are defined as $\frac{1}{2}[\beta(s) + |\beta(s)|]$.

It is easy to see that $\alpha_m(n) \leq \alpha_{m+1}(n)$ and $\lim_{n\to\infty} \alpha_m(n) = 0$.

THEOREM 3.1. *If* (1.1) *has a nonoscillatory solution, then all* $\alpha_m(n)$, $m = 1, 2, \cdots$ *in* (3.1) *are defined and*

$$\lim_{m\to\infty} \alpha_m(n) = \alpha(n) < \infty. \tag{3.2}$$

Proof. Assume that $\{x_n\}$ is a nonoscillatory solution of (1.1), then there exists a positive integer $n_1 \geq n_0$ such that $x_n \neq 0$ for $n \geq n_1$. From Lemma 2.2 we know $w_n \geq P_n = \alpha_0(n)$, so $w_n^2 \geq \alpha_0^2(n)_+$, from which we have

$$\alpha_1(n) = \sum_{s=n}^{\infty} \frac{\alpha_0^2(s+1)_+}{a_{s+1}} \leq \sum_{s=n}^{\infty} \frac{w_{s+1}^2}{a_{s+1}} < \infty. \tag{3.3}$$

From (2.11) and (3.3) we obtain

$$w_n \geq \alpha_0(n) + \frac{\varepsilon}{2}\alpha_1(n).$$

So $w_n^2 \geq [\alpha_0(n) + \frac{\varepsilon}{2}\alpha_1(n)]_+^2$. From Lemma 2.2 we get

$$\begin{aligned}\alpha_1(n) \leq \alpha_2(n) &= \sum_{s=n}^{\infty} \frac{[\alpha_0(s+1) + \frac{\varepsilon}{2}\alpha_1(s+1)]_+^2}{a_{s+1}} \\ &\leq \sum_{s=n}^{\infty} \frac{w_{s+1}^2}{a_{s+1}} < \infty.\end{aligned}$$

So by mathematical induction we have

$$\alpha_m(n) \leq \sum_{s=n}^{\infty} \frac{w_{s+1}^2}{a_{s+1}}, \quad m = 1, 2, \cdots. \tag{3.4}$$

Therefore, sequence $\{\alpha_m(n)\}$ is bounded. Note that $\{\alpha_m(n)\}$ nondecreasing implies that (3.1) is defined and

$$\lim_{m \longrightarrow \infty} \alpha_m(n) = \alpha(n) < \infty.$$

The proof of Theorem 3.1 is completed. □

From Theorem 3.1 we can easily obtain the sufficient conditions for (1.1) to be oscillatory.

THEOREM 3.2. *Suppose one of the following conditions is satisfied, then* (1.1) *is oscillatory:*

(a) $\alpha_m(n)$ *in* (3.1) *exists for* $m = 1, 2, \cdots, m_0 - 1$, *but* $\alpha_{m_0}(n)$ *doesn't exist, where* $m_0 \geq 1$ *is a positive integer;*

(b) $\alpha_m(n)$ *in* (3.1) *exists, but for every sufficiently large* n_1, *there exists* $n^* \geq n_1$ *such that* $\lim_{m \longrightarrow \infty} \alpha_m(n^*) = \infty$.

REMARK 3.1. In Theorem 3.1 and Theorem 3.2, we have extended the results in [6] to discrete equation (1.1).

EXAMPLE 1. Consider the equation

$$\Delta(\frac{1}{n^2}h(x_{n+1})\Delta x_n) + (\frac{1}{\sqrt{n+2}} - \frac{1}{\sqrt{n+1}})f(x_{n+1}) = 0,$$

where f and h satisfy the necessary conditions. Since

$$\alpha_0(n) = P_n = \sum_{s=n}^{\infty}\left(\frac{1}{\sqrt{s+2}} - \frac{1}{\sqrt{s+1}}\right) = \frac{-1}{\sqrt{n+1}} > -\infty.$$

But

$$\alpha_1(n) = \sum_{s=n}^{\infty} \frac{\alpha_0^2(s+1)}{a_{s+1}} = \sum_{s=n}^{\infty} \frac{(s+1)^2}{s+2} = \infty.$$

So by Theorem 3.2 this equation is oscillatory.

REMARK 3.2. Theorem 3.1, Theorem 3.2 weakened the condition $p_n \geq 0$. So we generalized and improved the results in [7, 8].

THEOREM 3.3. *If* (1.1) *has a nonoscillatory solution, then* $\alpha_m(n)$ *and* $\alpha(n)$ *satisfy the following expression:*

$$\lim_{n\longrightarrow\infty} \sup \alpha_m(n) \prod_{s=n_0}^{n-1} \left[1 + \frac{2\varepsilon[P_{s+1}]_+}{a_{s+1}}\right] < \infty, \tag{3.5}$$

and

$$\lim_{n\longrightarrow\infty} \alpha(n) \prod_{s=n_0}^{n-1} \left[1 + \frac{2\varepsilon[P_{s+1}]_+}{a_{s+1}}\right] < \infty. \tag{3.6}$$

Proof. Suppose there exists a sufficiently large n_1 such that, for $n \geq n_1$, we have $x_n > 0$ and, similar to the case (i) of the proof of Lemma 2.1, $f(x_{n+2})/f(x_{n+1}) \geq \frac{1}{2}$. From Lemma 2.2, we know

$$w_n = P_n + u_n,$$

where $u_n = \sum_{s=n}^{\infty} \frac{w_{s+1}\Delta(f(x_{s+1}))}{f(x_{n+1})}$. Then

$$\begin{aligned} -\Delta u_n &= \frac{w_{n+1}\Delta(f(x_{n+1}))}{f(x_{n+1})} \geq \frac{\varepsilon}{2}\frac{w_{n+1}^2}{a_{n+1}} \\ &= \frac{\varepsilon}{2}\frac{(P_{n+1}+u_{n+1})^2}{a_{n+1}} \geq \frac{2\varepsilon u_{n+1}[P_{n+1}]_+}{a_{n+1}}, \end{aligned}$$

that is

$$u_n - u_{n+1} \geq \frac{2\varepsilon u_{n+1}[P_{n+1}]_+}{a_{n+1}},$$

thus

$$\frac{u_{n+1}}{u_n} \le \left[1 + \frac{2\varepsilon[P_{n+1}]_+}{a_{n+1}}\right]^{-1}.$$

Forming the product of both sides of the above inequality from n_0 to $n-1$, we have

$$u_n \le u_{n_0} \prod_{s=n_0}^{n-1} \left[1 + \frac{2\varepsilon[P_{s+1}]_+}{a_{s+1}}\right]^{-1}. \tag{3.7}$$

On the other hand, we have

$$u_n \ge \frac{\varepsilon}{2} \sum_{s=n}^{\infty} \frac{w_{s+1}^2}{a_{s+1}} \ge \frac{\varepsilon}{2} \sum_{s=n}^{\infty} \frac{[P_{s+1}]_+^2}{a_{s+1}} = \frac{\varepsilon}{2}\alpha_1(n).$$

So

$$w_n \ge P_n + \frac{\varepsilon}{2}\alpha_1(n).$$

We then have

$$u_n \ge \frac{\varepsilon}{2} \sum_{s=n}^{\infty} \frac{w_{s+1}^2}{a_{s+1}} \ge \frac{\varepsilon}{2} \sum_{s=n}^{\infty} \frac{[P_{s+1} + \frac{\varepsilon}{2}\alpha_1(s+1)]_+^2}{a_{s+1}} = \frac{\varepsilon}{2}\alpha_2(n).$$

Using mathematical induction, we have

$$u_n \ge \frac{\varepsilon}{2}\alpha_m(n), \quad m = 1, 2, \cdots.$$

So from (3.7) we have

$$\alpha_m(n) \prod_{s=n_0}^{n-1} \left[1 + \frac{2\varepsilon[P_{s+1}]_+}{a_{s+1}}\right] \le \frac{2}{\varepsilon}u_{n_0}, \quad m = 1, 2, \cdots. \tag{3.8}$$

From Theorem 3.1 we know $\{\alpha_m(n)\}_{m=0}^{\infty}$ is convergent. According to (3.8) we have

$$\lim_{m \longrightarrow \infty} \alpha_m(n) \prod_{s=n_0}^{n-1} \left[1+\frac{2\varepsilon[P_{s+1}]_+}{a_{s+1}}\right] = \alpha(n) \prod_{s=n_0}^{n-1} \left[1+\frac{2\varepsilon[P_{s+1}]_+}{a_{s+1}}\right] \le \frac{2}{\varepsilon}u_{n_0}.$$

Therefore we have

$$\lim_{n\longrightarrow\infty} \sup \alpha_m(n) \prod_{s=n_0}^{n-1} \left[1 + \frac{2\varepsilon[P_{s+1}]_+}{a_{s+1}}\right] < \infty,$$

and

$$\lim_{n\longrightarrow\infty} \alpha(n) \prod_{s=n_0}^{n-1} \left[1 + \frac{2\varepsilon[P_{s+1}]_+}{a_{s+1}}\right] < \infty.$$

The proof is completed. □

From Theorem 3.3, the following Theorem is easily obtained.

THEOREM 3.4. *Suppose conditions* (c_1)–(c_3) *hold and one of the following conditions is satisfied, then* (1.1) *is oscillatory.*

(a) *There exists a positive integer* m_0 *such that*

$$\lim_{n\longrightarrow\infty} \alpha_{m_0}(n) \prod_{s=n_0}^{n-1} \left[1 + \frac{2\varepsilon[P_{s+1}]_+}{a_{s+1}}\right] = \infty; \tag{3.9}$$

or

(b)

$$\lim_{n\longrightarrow\infty} \alpha(n) \prod_{s=n_0}^{n-1} \left[1 + \frac{2\varepsilon[P_{s+1}]_+}{a_{s+1}}\right] = \infty. \tag{3.10}$$

THEOREM 3.5. *If conditions* (c_1)–(c_3) *hold and*

$$\lim_{n\longrightarrow\infty} \sum_{s=n_1}^{n-1} \prod_{k=n_0}^{s-1} \left[1 + \frac{2\varepsilon[P_{k+1}]_+}{a_{k+1}}\right]^{-1} < \infty, \tag{3.11}$$

and there exists a positive integer m_0 *such that*

$$\lim_{n\longrightarrow\infty} \sum_{s=n_1}^{n-1} \alpha_{m_0}(s) = \infty. \tag{3.12}$$

Then every solution of (1.1) *is oscillatory.*

REMARK 3.3. In Theorem 3.3, Theorem 3.4 and Theorem 3.5 we have extended the results in [6] to discrete equation. If $h(x) \equiv 1$, $f(x) \equiv x$, $p_n \geq 0$ and not eventually equal to zero, then the above theorems can be reduced to the corresponding results in [8]. If $h(x) = 1$, $a_n = 1$, $p_n \geq 0$, then (1.1) reduces to (1.3). So the results in this paper generalized and improved the corresponding results in [3, 8].

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