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Numerical Solution For Fredholm Integral Equation With Hilbert Kernel

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Abstract: Here, the Fredholm integral equation with Hilbert kernel is solved numerically , using two different methods . Also the error ,in each case , is estimated .

1. Introduction

Many problems in mathematical physics [1,2], contact problem [3,4]and mixed boundary value problems in the theory of elasticity [5,6], are transformed into an integral equations or an integro-differential equations.

Since closed form solutions of these problems are generally not available, much attention has been focused on numerical methods [7,8,9].

In this work, we present two methods to obtain numerical solutions for Fredholm singular integral equations of the second kind . In section two, we discuss the Toeplitz matrix method and applied this method to a Fredholm integral equation of the second kind with Hilbert kernel . In section three, we discuss the second method ,as one of the famous method ,the product Nystrom method , and we applied this method to obtain another numerical solution of the same Fredholm

Integral equation of the second kind with Hilbert kernel. The error estimate ,in each method ,is calculated.

2. The Toeplitz matrix method

In this section , we present the Toeplitz matrix method [10,11], to obtain numerical solution for Fredholm integral equation of the second kind and applied this method to a Fredholm integral equation of the second kind with Hilbert kernel.

The idea of this method is to obtain a system of 2N+1 linear algebraic equations , where 2N+1 is the number of discretization points used . The coefficients matrix is expressed as sum of two matrices , one of them is the Toeplitz matrix and the other is a matrix with zero elements except the first and last columns.

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Consider the integral equation

$$\mu\,\phi(x)\;\;\lambda\int_{-a}^{a}k(x,y)\,\phi(y)\,dy=f(x) \tag{2.1}$$

where μ is a constant$\neq 0$,
determines the kind of the integral equation, λ is constant,,
may be complex ,that has many physical meaning , the known function k(x,y) is called the kernel of integral equation ,which has a singular term ,f(x) is a known continuous function and is unknownfunction ,which will be defined .
In order to guarantee the existence of a unique solution of equation (2.1), we assume through this work the following conditions:

1- k(x,y) $\in$ C([-a,a] x[-a,a]) , and satisfies the discontinuity conditions

$$\left\{\int_{-a}^{a}\int_{-a}^{a}k^{2}\,(x,y)dx\,dy\right\}^{1/2}=c<\infty \qquad c=const.$$

2- The free f(x) term is continuous with its derivatives and belong to C[-a,a] .
3-the unknown function $\phi(x)$, which is called the potential function ,satisfies the Lipschitz condition i.e.

$$|\phi(x_1)-\phi(x_2)|\le L\,|x_1-x_2| \qquad ,\text{L is Lipschitz constant}$$

Also its norm is defined as

$$\left\{\int_{-1}^{1}|\phi(x)|^{2}\,dx\right\}^{1/2}\le A\,\|\phi\|_2$$

where $\|\;\|_2$ denotes the L_2 norm and A is a constant .
The continuity and normality of the integral operator

$$K\phi=\int_{-a}^{a}k(x,y)\phi(y)\,dy \tag{2.2}$$

can be easily ,proved . i.e. From the discontinuity condition , we have

$$\|K\|\le c \tag{2.3}$$

The integral term in equation (2.1) can be written as

$$\int_{-a}^{a}k(x,y)\,\phi(y)\,dy=\sum_{n=-N}^{N-1}\int_{nh}^{nh+h}k(x,y)\,\phi(y)dy \quad \left(h=\tfrac{a}{N}\right) \tag{2.4}$$

We approximate the integral in the right hand side of equation (2.4)by

$$\int_{nh}^{nh+h} k(x,y)\ \phi\ (y)\ dy = A_n(x)\ \phi\ (nh) + B_n(x)\ \phi\ (nh+h) + R \qquad (2.5)$$

where $A_n(x)$ and $B_n(x)$ are two arbitrary functions to be determined and R is the estimate error which depends on x and on the way that the coefficients $A_n(x)$, $B_n(x)$ are chosen. Putting $\phi\ (y)$ =1,y in equation (2.5) yields a set of two equations in terms of the two functions $A_n(x)$ and $B_n(x)$. For choosing the values of $\phi\ (y)$, the error R ,in this case , must vanish.

We can , clearly solve the result set of two equations for $A_n(x)$ and $B_n(x)$, to obtain

$$A\ (x) = \tfrac{1}{h}\left[(nh+h)I(x) - J(x)\right], \qquad (2.6)$$

$$B_n(x) = \tfrac{1}{h}\left[J(x) - nhI(x)\right] \qquad (2.7)$$

where

$$I(x) = \int_{nh}^{nh+h} k(x,y)\ dy \quad ,$$

and

$$J(x) = \int_{nh}^{nh+h} y\, k(x,y)\, dy \quad .$$

Hence, equation (2.4) takes the form

$$\int_{-a}^{a} k(x,y)\ \phi\ (y)\, dy = \sum_{n=-N}^{N} D_n(x)\phi\ (nh) \qquad (2.8)$$

where

$$D_n(x) = \begin{cases} A_n(x) & n = -N \\ A_n(x) + B_n(x) & N < n < N \\ B_n\ (x) & n = N\ . \end{cases} \qquad (2.9)$$

If we put x = mh , then we get the following system of linear algebraic equations

$$\mu\Phi_m - \lambda \sum_{n=-N}^{N} D_{mn}\ \Phi_n = F_m \qquad -N \le m \le N \qquad (2.10)$$

Here, we use the following notations

$$\phi(x)=\phi(mh)=\Phi_m\ ,\qquad D_n(x)=D_n(mh)=D_{nm}$$
$$\text{and}\quad f(x)=f(mh)=F_m\ . \tag{2.11}$$

where we define D_{mn} as

$$D_{mn}=\begin{cases} A_{-N}(mh) & n=-N \\ A_n(mh)+B_{n-1}(mh) & -N<n<N \\ B_{N-1}(mh) & n=N\ . \end{cases} \tag{2.12}$$

The matrix D_{mn} may be written as

$$D_{mn}=G_{mn}-E_{mn} \tag{2.13}$$

where

$$G_{mn}=A_n(mh)+B_{n-1}(mh)\qquad -N\le m,n\le N \tag{2.14}$$

is a Toeplitz matrix of order 2N+1
and

$$E_{mn}=\begin{cases} B_{-N-1}(mh) & n=-N \\ 0 & -N<n<N \\ A_N(mh) & n=N\ . \end{cases} \tag{2.15}$$

represents a matrix of order 2N+1 whose elements are zeros except the first and last columns .
However, the solution of the system of equation (2.10) can be obtained in the form

$$\Psi_m=\left[\mu\, I-\lambda\,\left(G_{mn}-E_{mn}\right)\right]^{-1}F_m \tag{2.16}$$

where I is the identity matrix of order 2N+1 .
The values of equation(2.16) will be calculated under the condition

$$\left\|\mu\, I-\lambda\,\left(G_{mn}-E_{mn}\right)\right\|\neq 0$$

The error term R is determined from equation (2.5) by letting $\phi(y)=y^2$, to get

$$R=\left|\int_{nh}^{nh+h}y^2\ k(x,y)\ dy-\left(A_n(x)(nh)^2+B_n(x)\ (nh+h)^2\right)\right|=O(h^3) \tag{2.17}$$

The method is said to be convergent of order r in $[-a,a]$ if and only if for N sufficiently large there exists a constant $D>0$ independent of N such that

$$\|\phi(x) - \phi_n(x)\| \le D\, N^{-r} \tag{2.18}$$

under the assumption of the discontinuity kernel and equation (2.3) ,equation (2.8) , we have

$$\sup_N \left\| \sum_{n=-N}^{N} D_{nm}\ \Phi_n \right\| \le c\|\Phi\| \tag{2.19}$$

Now , we apply this method to solve the integral equation (2.1) when $\mu = \lambda = 1$, and the kernel takes the form of Hilbert kernel .

Consider the singular integral equation with Hilbert kernel of the form

$$\mu\phi(x) - \lambda \int_{-\pi}^{\pi} \cot\frac{y-x}{2}\ \phi(y)\,dy = f(x) \tag{2.20}$$

under the conditions

$$\phi(-\pi) = \phi(\pi) = 0 \tag{2.21}$$

With the aid of Bernoulli numbers (see[12],pp.1076), and the integral formula [12, pp. 192]

$$\int x^p \cot x\,dx = \sum_{s=0}^{\infty} \frac{(-1)^s\ 2^{2s}\ B_{2s}}{(p+2s)\ (2s)\,!}\ x^{p+2s} \quad (p \ge 1, |x| < \pi\,) \tag{2.22}$$

where B_{2s} are Bernoulli numbers , we obtain

$$\begin{aligned} A_n(x) = \frac{2}{h}\Big\{ &(nh+h-x)\ln\left|\sin\frac{nh+h-x}{2}\right| - (nh-x) \\ &- \sum_{s=0}^{\infty} \frac{(-1)^s\ B_{2s}}{(1+2s)\ (2s)!}\Big[(nh+h-x)^{1+2s} - (nh-x)^{1+2s}\Big]\Big\} \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} B_n(x) = \tfrac{2}{h}\Big\{ &-(nh+h-x)\ln\left|\sin\tfrac{nh+h-x}{2}\right| + (nh-x)\ln\left|\sin\tfrac{nh-x}{2}\right| \\ &+ \sum_{s=0}^{\infty} \tfrac{(-1)^s\ B_{2s}}{(1+2s)\ (2s)!}\Big[(nh+h-x^{1+2s}) - (nh-x)^{1+2s}\Big]\Big\} \end{aligned} \tag{2.24}$$

Therefore ,the elements of the Toeplitz matrix G_{mn} are given by

$$G_{m,n} = 2(n-m+1)\ln\left|\sin\frac{h(n-m+1)}{2}\right| - 4(n-m)\ln\left|\sin\frac{h(n-m)}{2}\right|$$

$$+(n-m-1)\ln\left|\sin\frac{h(n-m-1}{2}\right| - 4\sum_{s=0}^{\infty}\frac{(-1)^s h^{2s} B_{2s}}{2(1+2s)(2s)!}\times$$

$$\left[(n-m+1)^{1+2s} - 2(n-m)^{1+2s} + (n-m-1)^{1+2s}\right],$$

$$(-N+1 \le m,n \le N-1)$$

The two conditions $\phi(-\pi) = \phi(\pi) = 0$ reduce the Toeplitz matrix elements G_{mn} and E_{mn} to 2N-1 order .

3.The product Nystrom method:

In this section , we present the product Nystrom method [13,14] to obtain the solution of the integral eqution (2.1), when the kernel k(x,y) is singular within the range of integration . We can often factor out the singularity in k(x,y) by writing $k(x,y) = p(x,y)\bar{k}(x,y)$ where p and k are respectively "badly behaved" and "well behaved" functions of their arguments .

Using Nystrom method ,the integral equation (2.1) will be transformed into the following system of linear algebraic equations.

$$\mu\phi(x_i) - \lambda\sum_{j=0}^{N} w_{ij}\bar{k}(y_i,y_j)\phi(y_j) = f(x_i) \qquad (3.1)$$

$$\left(x_i = y_i = a + ih,\ h = \tfrac{2a}{N},\ i = 0,1,....N\right)$$

where

$$w_{i,0} = \beta_1(y_i) \quad , \ w_{i,2j+1} = 2\gamma_{j+1}(y_i)\,.$$

$$w_{i,2j} = \alpha_j(y_i) + \beta_{j+1}(y_i)\,,\ w_{i,N} = \alpha_{N/2}(y_i)\,. \qquad (3.2)$$

Also

$$\alpha_j(y_i) = \frac{1}{2h^2}\int_{y_{2j-2}}^{y_{2j}} p(y_i,y)(y-y_{2j-2})(y-y_{2j-1})\,dy,$$

where

$$\beta_j(y_i) = \frac{1}{2h^2}\int_{y_{2j-2}}^{y_{2j}} p(y_i,y)(y_{2j-1}-y)(y_{2j}-y)\,dy$$

and

$$\gamma_j(y_i) = \frac{1}{2h^2}\int_{y_{2j-2}}^{y_{2j}} p(y_i,y)(y-y_{2j-2})(y_{2j}-y)\,dy \qquad (3.3)$$

The linear system of equation (3.1) can be written in the form

$$\mu\Phi - \lambda\ W\Phi = F \tag{3.4}$$

which has the solution

$$\Phi = \left[\mu\ I - \lambda W\right]^{-1} F \quad , \ \left\|\mu I - \lambda\ W\right\| \neq 0 \tag{3.5}$$

where I is the identity matrix . Let us define the linear operator

$$A_N : C[-1,1] \to C[-1,1]$$

$$A_N \phi_i = \sum_{j=0}^{N} w_{ij} \bar{k}(y_i, y_j)\phi(y_j) \tag{3.6}$$

Here, for every bounded function ϕ , we get

$$\sup_N \left\|A_N\ \phi_i\right\| \leq C^* \left\|\phi\right\| \quad , \quad C^* \ \ is\ a\ const \tag{3.7}$$

The error, can be calculated from the following formula

$$E_N = \left| \int_a^a p(x,y)\bar{k}(x,y)\phi(y)\ dy - \sum_{j=0}^{N} w_{ij}\ \bar{k}(y_i, y_j)\phi\ (y_i) \right| \tag{3.8}$$

Now, we go to apply Nystrom method ,when the integral equation takes the form of equation (2.20) .For this, we introduce the change of variable $y = y_{2j-2} + \xi\ h \quad , 0 \leq \xi \leq 2$, Thus the system (3.3) becomes

$$\alpha_j(y_i) = \tfrac{h}{2} \int_0^2 \xi(\xi\ - 1)\cot\tfrac{h(\xi - i + 2j - 2)}{2} d\xi\ ,$$

$$\beta_j(y_i) = \tfrac{h}{2} \int_0^2 (1 - \xi\ )(2 - \xi\ )\cot\tfrac{h(\xi - i + 2j - 2)}{2}\ d\xi\ , \tag{3.9}$$

and

$$\gamma_j(y_i) = \tfrac{h}{2} \int_0^2 \xi\ (2 - \xi\ )\cot\tfrac{h(\xi - i + 2j - 2)}{2}\ d\xi$$

Integrating (3.9) with the aid of (2.22) , we obtain

$$w_{i,2j+1} = 2z(2-z)\left[\ln\left|\sin\frac{h(2-z)}{2}\right| - \ln\left|\sin\frac{hz}{2}\right|\right] + \\ 4\sum_{s=0}^{\infty} \frac{(-1)^s\ h^{2s}\ B_{2s}}{(2s)!}\left[\frac{(2-z)\ (1-z)\ + z^{1+2s}(1-z)}{1+2s} - \frac{(2-z)^{2+2s} - z^{2+2s}}{2(2+2s)}\right] \tag{3.10}$$

and

$$w_{i,2j} = 6(z-2)\ln\left|\sin\frac{h(2-z)}{2}\right| + (z-3)(z-4)\ln\left|\sin\frac{h(4-z)}{2}\right| - z(z-1)\ln\left|\sin\frac{hz}{2}\right| +$$
$$\sum_{s=0}^{\infty}\frac{(-1)^s\, h^{2s}}{(2s)!}B_{2s}\left[\frac{(4-z)^{2+2s}-z^{2+2s}}{2(1+s)}+\frac{6(2-z)^{1+2s}+(2z-1)z^{1+2s}+(2z-7)(4-z)^{1+2s}}{1+2s}\right],$$
$$\left(z=i-2j+2;1\le i,j\le N-1\right). \qquad (3.11)$$

The conditions $\phi(\pi)=\phi(-\pi)=0$, avoid the use of the two points $x=\pm\pi$ and avoids the calculation of $w_{i,0}$ and $w_{i,N}$.

4. Numerical results

Table 1. show the values of the approach solution $\phi_n^{(T)}$ and the error $R^{(T)}$ the interior points by using the Toeplitz matrix method with n=20, $\mu=\lambda=1$.
Also show the approximate solution $\phi_n^{(N)}$ and the error $R^{(N)}$ at the interior points by using the product Nystrom method . The exact solution is $\phi(x)=\sin x$,
See Figs.(1-4).

5. Conclusion :

From the above results and discussion , we deduce the following :

(1) In the contact problem in the theory of elasticity the conditions of equation (2.21) can be removed to the equivalent condition

$$\int_{-\pi}^{\pi}\phi(y)\,dy = p<\infty \qquad (\text{ P is a constant})$$

which is called the pressure condition . In this case the integral equation (2.20) under the pressure condition ,represents the contact problem of a strip occupying the region $0\le y\le h,$ that made of material satisfies the stress and strain relations [15]. Assume a rigid rectangular stamp of width 2π is impressed into the boundary of a strip y=h by contact force p whose eccentricity of application is e. assume the frictional force in the contact area between the stamp and the strip are small ,so that it can be neglected . Also, we assume the width 2π of the area of contact is independent of the magnitude of the force applied. This kind of contact problem leads us to the integral equation (2.20) under the pressure condition [16].

(2) The integral equation

$$\phi(x)-\lambda\int_{-\pi}^{\pi}\cot\left(\frac{y-x}{2}\right)\phi(y)\,dy=f(x) \tag{5.1}$$

subject to $\phi(-\pi)=\phi(\pi)=0$, transformation $x=2u-\pi$, $y=2v-\pi$, becomes

$$\psi(u)-\lambda\int_{0}^{\pi}\cot(v-u)\psi(v)\,dv=g(u) \tag{5.2}$$

$$(\phi(x)=\phi(2u-\pi)=\psi(u),\qquad f(x)=f(2u-\pi)=g(u))$$

under the conditions

$$\psi(0)=\psi(\pi)=0 \tag{5.3}$$

The equation(5.2) subject to the conditions (5.3) has appeared in both combined infrared gaseous radiation and molecular condition , where $\psi(u)$ represents the unknown temperature and λ is a constant composed of several physical properties [2].

(3) From the spectral relations of the integral equation with logarithmic kernel one has [1]

$$-\int_{-a}^{a}\frac{\ln|x-y|\,T_n\left(\frac{y}{a}\right)dy}{\sqrt{a^2-y^2}}=\begin{cases}\pi\ln\frac{2}{a} & n=0\\ \frac{\pi}{n}\,T_n\left(\frac{x}{a}\right) & n\ge 1,(|x|<a)\end{cases} \tag{5.4}$$

, where $T_n(.)$ is the Chebyshev polynomials of the first kind of order n.

Let in (5.4) n=2m , $\frac{x}{a}=\frac{\sin\frac{\xi}{2}}{\sin\frac{\alpha}{2}}$, $\frac{y}{a}=\frac{\sin\frac{\eta}{2}}{\sin\frac{\alpha}{2}}$, we have

$$\int_{-\alpha}^{\alpha}\ln\frac{1}{2\left|\sin\frac{\xi-\eta}{2}\right|}\,\frac{T_{2m}\left(\frac{\sin\frac{\eta}{2}}{\sin\frac{\alpha}{2}}\right)\cos\left(\frac{\eta}{2}\right)d\eta}{\sqrt{2(\cos\eta-\cos\alpha)}}=\mu_{2m}\,T_{2m}\left(\frac{\sin\frac{\xi}{2}}{\sin\frac{\alpha}{2}}\right) \tag{5.5}$$

$$(-\alpha<\xi,\eta<\alpha\ ,\ m=0,1,2,....$$

where

$$\mu_{2m} = \begin{cases} -\pi \quad n\left(\sin\frac{\alpha}{2}\right) & m = 0 \\ \frac{\pi}{2m} & m \geq 1 \end{cases}$$

Differentiating (5.5) with respect to ξ and using the famous relation [1].

$$T_n^{'}(x) = nU_{n-1}(x)$$

where $U_n(.)$ is the Chebyshev polynomials of the second kind of order n, we obtain

$$\int_{-\alpha}^{\alpha} \cot\left(\frac{\xi-\eta}{2}\right) \frac{T_{2m}\left(\frac{\sin\frac{\eta}{2}}{\sin\frac{\alpha}{2}}\right) \cos\left(\frac{\eta}{2}\right) \sin\left(\frac{\alpha}{2}\right) \ d\eta}{\sqrt{2(\cos\eta - \cos\alpha)}} = \mu_{2m}\left(\cos\frac{\xi}{2}\right) U_{2m-1}\left(\frac{\sin\frac{\xi}{2}}{\sin\frac{\alpha}{2}}\right) \qquad (5.6)$$

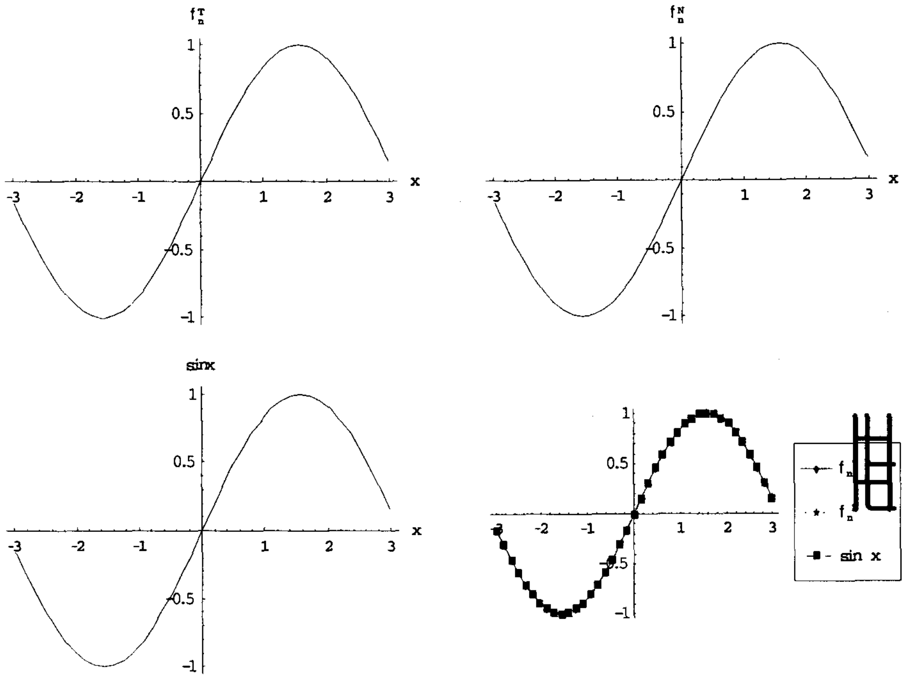

Table1:

X	$\phi_n^{(T)}$	$R^{(T)}$	$\phi_n^{(N)}$	$R^{(N)}$
-.298451E+01	-.157566E+00	.113183E-02	-.157904E+00	.146956E-02
-.282743E+01	-.306774E+00	.224258E-02	.-.307179E+00	.183761E-02
-.267035E+01	-.456815E+00	.282412E-02	-.454282E+00	.291620E-03
-.251327E+01	-.587984E+00	.198949E-03	-.587120E+00	.665654E-03
-.235619E+01	-.710086E+00	.297899E-02	-.706959E+00	.147790E-03
-.219911E+01	-.810573E+00	.155627E-02	-.808990E+00	.265871E-04
-.204204E+01	-.894161E+00	.315436E-02	-.890639E+00	.367293E-03
-.188496E+01	-.953450E+00	.239302E-02	-.951288E+00	.231250E-03
-.172788E+01	-.990983E+00	.329504E-02	-.987275E+00	.413104E-03
-.157080E+01	-.100286E+01	.285655E+01	-.100020E+01	.197142E-03
-.141372E+01	-.991010E+00	.332117E-02	.987354E+00	.334045E-03
-.125664E+01	-.954070E+00	.301384E-02	-.951009E+00	.476253E-04
-.109956E+01	-.894194E+00	.318768E-02	-.890833E+00	.173579E-03
-.942478E+00	-.811927E+00	.291043E-02	-.808591E+00	.425900E-03
-.785398E+00	-.709991E+00	.288390E-02	-.707135E+00	.285526E-04
-.628319E+00	-.590376E+00	.259122E-02	-.586921E+00	.864526E-03
-.471239E+00	-.456418E+00	.242748E-02	-.454227E+00	.236237E-03
-.314159E+00	-.311125E+00	.210820E-02	-.307721E+00	.129617E-02
-.157080E+00	-.158292E+00	.185783E-02	-.156852E+00	.417934E-03
.000000E+00	-.152180E-02	.152180E-02	.166220E-02	.166120E-02
.157080E+00	.155205E+00	.122941E-02	.155886E+00	.548042E-03
.314159E+00	.308119E+00	.898460E-03	.310933E+00	.191594E-02
.471239E+00	.453386E+00	.604871E-03	.453382E+00	.608263E-03
.628319E+00	.587479E+00	.306284E-03	.589811E+00	.202547E-02
.785398E+00	.707059E+00	.481999E-04	.706518E+00	.588612E-03
.942478E+00	.809207E+00	.190301E-03	.810993E+00	.197563E-02
.109956E+01	.891388E+00	.381596E-03	.890519E+00	.487828E-03
.125664E+01	..951592E+00	.535616E-03	.952825E+00	.176875E-02
.141372E+01	.988324E+00	.635698E-03	.987375E+00	.313036E-03
.157080E+01	.100069E+01	.686574E-03	.100142E+01	.142411E-02
.172788E+01	.988367E+00	.678469E-03	.987610E+00	.784910E-04
.188496E+01	.951670E+00	.613645E-03	.952033E+00	.976132E-03
.204204E+01	.891492E+00	.485934E-03	.891203E+00	.196925E-03
.219911E+01	.809312E+00	.294613E-03	.809489E+00	.472495E-03
.235619E+01	.707138E+00	.308381E-04	.707602E+00	.495305E-03
.251327E+01	.587465E+00	.320492E-03	.587762E+00	.234950E-04
.267035E+01	.453182E+00	.808603E-03	.454816E+00	.825687E-03
.282743E+01	.307399E+00	.161841E-02	.308643E+00	.373926E-03
.298451E+01	.152392E+00	.404226E-02	.157947E+00	.151280E-02

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