ANALYSIS OF A ONE-DIMENSIONAL FIN USING THE ANALYTIC METHOD AND THE FINITE DIFFERENCE METHOD

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ABSTRACT

The straight rectangular fin is analyzed using the one-dimensional analytic method and the finite difference method. For the finite difference method, the numbers of nodes vary from 20 to 100. The relative errors of heat loss and temperature between the analytic method and the finite difference method are represented as a function of Biot Number and dimensionless fin length. One of the results shows that the relative error between the analytic method and the finite difference method decreases as the numbers of nodes for finite difference method increase.

Nomenclature

A_c : horizontal confrontation of differential element [m²]

A_s : surface area of differential element [m²]

b : slop of the upper lateral surface, $0 \leq b \leq \dfrac{1}{L}$

Bi : Biot number over the fin surface (= hl/k), dimensionless

f : ratio of the rate of increase (= $\Delta x / \Delta y$)

h : heat transfer coefficient over the fin surface [W/m². C]

k : thermal conductivity [W/m ℃]

l : one half fin thickness [m]

L' : fin length [m]

L : dimensionless fin length (= L'/l)

Q : heat loss per unit width [W/m]

T : temperature [℃]

T_w : wall temperature [℃]

T_∞ : ambient temperature [℃]

w' : fin width [m]

x' : length direction coordinate

x : dimensionless length direction coordinate (= x'/l)

Δx : the rate of increase of dimensionless length direction

y' : height direction coordinate

y : dimensionless height direction coordinate (= y'/l)

Δy : the rate of increase of dimensionless height direction

θ : dimensionless temperature (= $(T - T_\infty)/(T_w - T_\infty)$)

Θ_w : modified temperature (= $T_w - T_\infty$)

Superscript
′ : dimensional variable

Subscript
w : wall
∞ : surrounding

1. Introduction

The fin is usually used when the convection heat transfer coefficient is low, especially under free convection. In the field of industry the fin is used widely, for instance, electronic accessories, motorcycle, air cooling cylinder of lawn mower, cooling fin attached on refrigerator. Therefore, calculating the heat through the fin surface is very useful and there are many papers about study of the fin. Xia and Jacobi [1] use a numerical solution to conduction within the composite medium comprised of a fin and coating material to conduct a parametric study of the effects of geometry, thermal conductivity of the fin and coating material, and convection coefficient on the temperature profiles. Lee et al [2] solve the two dimensional inverse problem of estimating the unknown heat flux at a pin fin base by the conjugate gradient method. Lin and Jang [3] explain the fin efficiency for the elliptic fin using both mathematical analysis and numerical analysis. Kang and Look [4] optimize a thermally asymmetric annular rectangular fin as a function of bottom to top convection characteristic number ratio and fin volume using two dimensional analytic method. Burmeister [5] analyzes triangular fin performance using heat balance integral method. All these papers used different methods for analysing the fin. Also, some papers compare different methods for the specific shape of the fin. For example, Abrate and Newnham [6] shows the comparision of the fin centerline temperature between the finite element method and the analytic method for the triangular fin. Kim and Kang [7] compare the heat loss from the parabolic fin between the analytic method and the finite differnce method. But in this paper the number of nodes are fixed for the finite difference method.

The purpose of this paper insures the accuracy of the analytic method and the finite difference method by comparing these two methods with the view of heat transfer from the rectangular fin. Especially, for the finite difference method, a number of nodes vary from 20 to 100 and the effect of node numbers on the relative error is shown. The relative errors of heat loss and temperature distribution between the analytic method and the finite difference method are represented as a function of Biot number and dimensionless fin length.

2.1. One-dimensional analytic method

Under steady state conditions, a general form of the energy equation for an extended surface is given by Eq. (1).

$$\frac{d^2 T}{dx'^2} + \left(\frac{1}{A_c} \frac{dA_c}{dx'} \right) \frac{dT}{dx'} - \left(\frac{1}{A_c} \frac{h}{k} \frac{dA_s}{dx'} \right) (T - T_\infty) = 0 \qquad (1)$$

Equation (1) can be written for a rectangular fin with assuming $w' \gg 2l$

$$\frac{d^2T}{dx'^2} - \frac{h}{k}\frac{1}{l}(T - T_\infty) = 0 \tag{2}$$

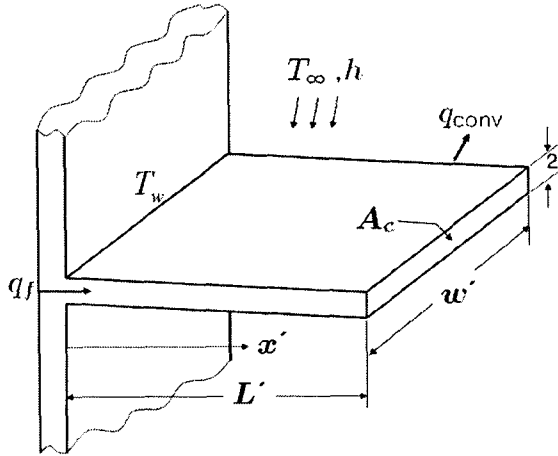Two boundary conditions are needed to solve Eq. (2) and are shown by Eqs. (3) and (4).

Fig. 1 Straight rectangular fin of uniform cross

section

$$x' = 0, \qquad T = T_w \tag{3}$$

$$x' = L', \qquad h(T - T_\infty) = -k\frac{dT}{dx'} \tag{4}$$

Equation (3) represents the constant fin base temperature while Eq. (4) means that the convection heat transfer from the fin tip equals to the heat conduction through the fin tip.

Equation (2) can be written by dimensionless form as Eq. (5).

$$\frac{d^2\Theta}{dx^2} + Bi\Theta = 0 \tag{5}$$

Two boundary conditions (3) and (4) are transformed into dimensionless forms as Eqs. (6) and (7).

$$x = 0, \qquad \Theta = \Theta_w \tag{6}$$

$$x = L, \qquad \frac{d\Theta}{dx} + Bi\Theta = 0 \tag{7}$$

By solving Eq. (5) with the boundary conditions listed as Eqs. (6) through (7), the temperature distribution can be obtained. The result is

$$\Theta = \frac{\cosh\{\sqrt{Bi}(L-x)\} + \sqrt{Bi}\sinh\{\sqrt{Bi}(L-x)\}}{\cosh(\sqrt{Bi}L) + \sqrt{Bi}\sinh(\sqrt{Bi}L)} \qquad (8)$$

Heat transfer from the fin can be obtained by appling temperature distribution equation to Fourier's conduction law.

$$Q = -kA\left(\frac{dT}{dx'}\right)_{x'=0} \qquad (9)$$

$$= 2k\Theta_w\left\{\frac{\sqrt{Bi}\sinh(\sqrt{Bi}L) + \sqrt{Bi}\cosh(\sqrt{Bi}L)}{\cosh(\sqrt{Bi}L) + \sqrt{Bi}\sinh(\sqrt{Bi}L)}\right\} \qquad (10)$$

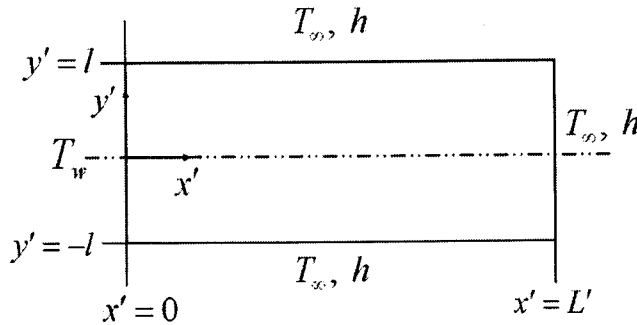## 2.2. One-dimensional finite difference method

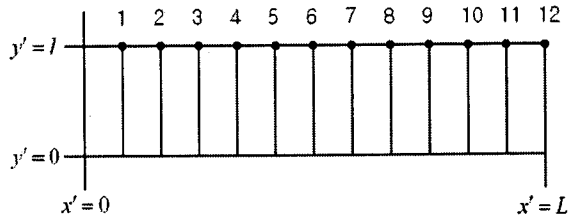Fig. 2 Geometry of a rectangular fin

Fig. 3 Upper half rectangular fin presenting

12 nodes for the finite different method

As shown in Fig. 2, the straight rectangular fin is symmetry so that the upper half fin is divided by 12 nodes for finite difference method. For each of the nodes shown in Fig. 3, the equations are given by Eqs. (11) through (13).

For node 1

$$1 - (2 + fBi\triangle x)\Theta_1 + \Theta_2 = 0 \qquad (11)$$

For node 2 (and a similar form for the points 3 ~ 11)

$$\Theta_1 - (2 + fBi\triangle x)\Theta_2 + \Theta_3 = 0 \qquad (12)$$

For node 12

$$\Theta_{11} - (1 + fBi\frac{\triangle x}{2} + fBi\triangle y)\Theta_{12} = 0 \qquad (13)$$

The convection heat loss Q from the fin can be calculated using Eq. (14).

$$\frac{Q}{2k\Theta_w} = Bi\triangle x\left(\frac{1}{2} + \sum_{i=1}^{11}\Theta_i + \frac{1}{2}\Theta_{12} + \frac{\Theta_{12}}{\triangle x}\right) \qquad (14)$$

3. Results

Fig. 4 shows the relative errors in the convection heat loss between the analytic method and the finite difference method as a function of Biot number for L=6 and N=20, 25, 35 and 50 for the finite difference method. It shows the relative error increases linearly as Biot number increases and that increasing rate decreases as a number of nodes increases. For a given range of Biot number the maximum relative error is less than 1.1% even though N is 20. This means that the finite difference method with just 20 nodes is accurate enough to analyze the short rectangular fin.

Fig. 5 represents the relative error of heat loss between the analytic method and the finite difference method as a function of dimensionless fin length for Bi=0.01 and N=20, 30, 50 and 100. The relative errors increase parabolically as L increases for all values of N. It also shows that the maximum relative error is less than 0.12% even though N=20 and L=20.
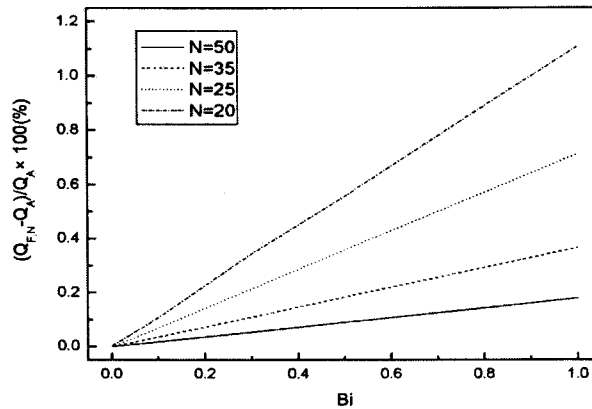

Fig. 4 Relative error of the heat loss versus
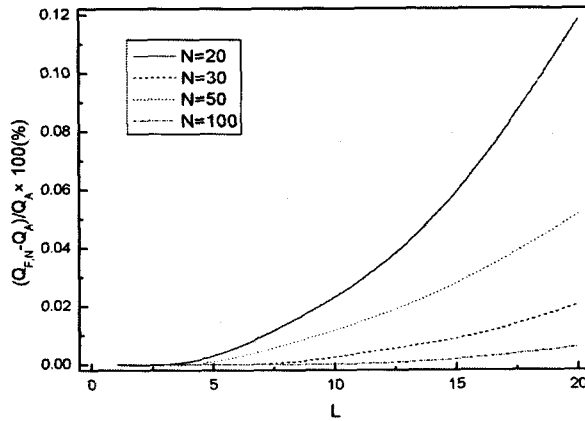
Biot number for L=6

Fig. 5 Relative error of the heat loss versus

dimensionless fin length for Bi=0.01

The relative errors of the temperature at the fin tip for L=6 and N=20, 25, 35 and 50 using the finite difference method as compared to the analytic method versus Biot number are illustrated in Fig. 6. The relative error increases linearly as Biot number increases for all values of N. It can be noted that the relative error decreases from over 2.7% to 0.3% as N increases from 20 to 50 for Bi=1.

Figure 7 illustrates the relative errors of temperature at the fin tip between the finite difference method and the analytic method as a function of dimensionless fin length in case of Bi=0.01, and N=20, 25, 35 and 50. It shows that the relative error seems to be independent on the dimensionless fin length for N=100 and it increase parabolically as L increases for N=20, 30 and 50. Even for the smallest value of N, the relative error remains less than 0.1% until L increase to 20.
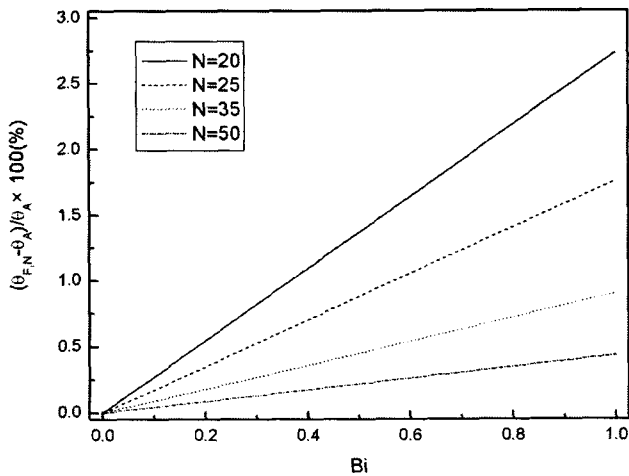

Fig. 6 Relative error of the temperature at the fin
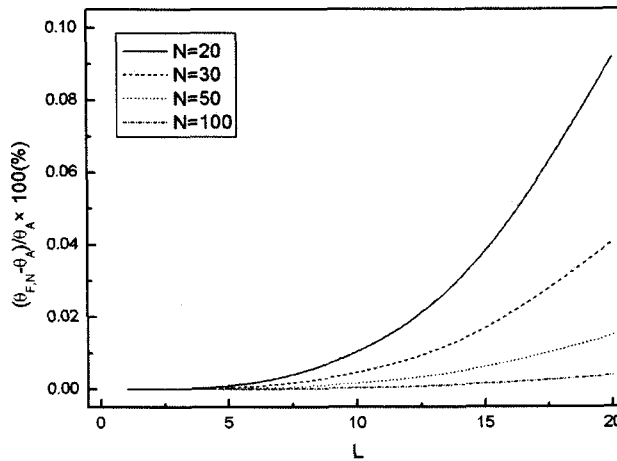
tip versus Biot number for L=6

Fig. 7 Relative error of the temperature at the fin

tip versus the dimensionless fin length

for Bi=0.01

4. Conclusion

The results presented produce the following straightforward conclusions.
(1) The relative error increases linearly as Biot number increases for fixed fin length.
(2) The relative error increases parabolically with the increase of dimensionless fin length for fixed Biot number.
(3) The relative error between the analytic method and the finite difference method decreases as the numbers of nodes for the finite difference method increase.

Reference

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