Journal of Natural Sciences
Pai Chai University, Korea
Vol. 14, No. 1 : 1~6, 2004

# Klein bottle위의 주기함수의 Nielsen 수

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# The Nielsen numbers of periods for maps on the Klein bottle

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In this paper, we calculate the Nielsen numbers of periods for maps on the Klein bottle.
**Key Words** : Nielsen number, periodic map, Klein bottle.

## 1. Introduction

The object of this paper is to find the Nielsen numbers of periods for maps on the Klein bottle. In order to state our main results, let us fix some notation and terminology. Let $f: X \to X$ be a continous self-map of a compact connected polyhedron $X$, and $n$ be a natural number. A fixed point of $f$ is a point $x$ in $M$ such that $f(x)=x$. Denote the fixed point set of $f$ by Fix( $f$ ). $x$ is a periodic point of period $n$ if $x$ is a fixed point of $f^n$ but is not fixed by any $f^k$, for $1 \leq k < n$. When a map $g: X \to X$ is homotopic to $f$, we shall write $g \simeq f: X \to X$.

An important invariant is the Nielsen number $N(f)$ of $f$, which is a lower bound for the number of the fixed points among all maps in the homotopy class of $f$ and also a lower bound for the component number of the fixed point set $Fix(f)$. It is known that if M is a compact manifold of dimension greater than two with nonnegative Euler characteristic number, then the Nielsen number $N(f)$ can be realized as the number of the fixed points of some maps homotopic to $f$.

Let $G$ be a connected, simply connected nilpotent Lie group. Let $\pi \subset Aff(G) = G \rtimes Aut(G)$ with $\Gamma = \pi \cap G$ is a lattice of $G$. Then $M = \pi/G$ is an infra-nilmanifold with a finite holonomy group $\Psi = \pi/\Gamma$.

In this paper, let $K$ denote a two dimensional klein bottle and 2-dimensional Klein bottle group $\pi_1(K)$ is denoted by $\pi$. In this case, $G = \mathbb{R}^2$, $\Gamma = \mathbb{Z}^2$, and holonomy group $\Psi = \pi/\mathbb{Z}^2$ is $\mathbb{Z}_2$.

Let $f: K \to K$ be any continous map on Klein bottle $K$ with the holonomy group $Z_2$. Since Klein bottle is an infra-nilmanifold, we can apply the averaging formula which was proved by K. B. Lee and J. B. Lee.

## 2. Prelimnaries

In 1911, Bieberbach proved that any automorphism of a crystallographic group is conjugation by an element of $Aff(\mathbb{R}) = \mathbb{R} \rtimes GL(n, \mathbb{R})$. This was generalized to almost crystallographic group. In 1995, K. B. Lee generalized this result to all homomorphisms from isomorphisms (See [7]). Topologically, this implies that every continous map on infra-nilmanifold is homotopic to a map induced by an affine endomorphism on the Lie group level. Because a flat manifold is an infra-nilmanifold, it can be stated as every endomorphism of flat manifolds is semi-conjugate to an affine endomorphism. We can restate K. B. Lee's results in Klein bottle group case as follows:

**Theorem A.** Let $\pi, \pi' \subset Aff(\mathbb{R})$ be two Klein bottle groups. Then for any homomorphism $\theta : \pi \to \pi'$, there exists $g = (d, D) \in aff(\mathbb{R}) = G \rtimes End(\mathbb{R})$ such that $\theta(\alpha) \cdot g = g \cdot \alpha$ for all $\alpha \in \pi$.

**Corollary B**. Let $K=\pi/\mathbb{R}$ be a Klein bottle, and $h: K\to K$ be any map. Then $h$ is homotopic to a map induced from an affine endormorphism $\mathbb{R}\to\mathbb{R}$.

## 3. The Nielsen numbers of periods for maps on the Klein bottle

In the Klein bottle case, $G=\mathbb{R}^2$, $\Gamma=Z^2$, and holonomy group $\Psi=\pi/Z^2$ is $Z_2$. So there is a short exact sequence $1\to Z^2\to\pi\to Z_2\to 1$. More precisely, the klein bottle group $\pi\subset\mathbb{R}^2\rtimes O(2)\subset Aff(\mathbb{R}^2)$ is generated by

$$\langle(e_1,I),(e_2,I),(a,A)\rangle,$$

where $e_1=\begin{pmatrix}1\\0\end{pmatrix}$, $e_1=\begin{pmatrix}0\\1\end{pmatrix}$, $a=\begin{pmatrix}1/2\\0\end{pmatrix}$, and $A=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. Because $(a,A)^2=(e_1,I)$. we may think that a Klein bottle group $\pi$ is generate by $(e_2,I)$, and $(a,A)$. For any continuous map $f: K\to K$, $f_*$ denotes the homomorphism induced on the fundamental group. Such an endomorphism is well-defined up to an inner automorphism. Any continuous map $f: K\to K$ lifts to some continuous map on Torus $f: T^2\to T^2$ by double covering. And $T^2$ is covered by $\mathbb{R}^2$. By Theorem A, for a homomorphism $f_*: \pi\to\pi$, there exist $g=(d,D)\in\mathbb{R}^2\rtimes gl(2,\mathbb{R})$ such that

$$f_*(\alpha)\cdot g=g\cdot\alpha$$

for all $\alpha\in\pi_1(K)$. Moreover, $g=(d,D)$ covers a unique algebraic endomorphism $f_{(d,D)}: K\to K$ with $f_{(d,D)}\simeq f$.

**Theorem C.** Let $f: K\to K$ be a continuous map and let $f_*: \pi\to\pi$ denote the induced homomorphism on the fundamental group. There exist only two types of $g=(d,D)$ which satisfy the semi-conjugate condition $f_*(\alpha)\cdot g=g\cdot\alpha$, for all $\alpha\in\pi$. The first type is $d=\begin{pmatrix}d_1\\d_2\end{pmatrix}$, $D=\begin{pmatrix}2l+1&0\\0&m\end{pmatrix}$, where $d_1\in\mathbb{R}$, $d_2\in Z/2$, and $l,m\in Z$. The second type is

$d=\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$, $D=\begin{pmatrix} 2l & 0 \\ 2m & 0 \end{pmatrix}$, where $r_1, r_2 \in \mathbb{R}$ and $l, m \in \mathbb{Z}$.

Proof. By theorem A, there exists $g=(d, D) \in \mathbb{R}^2 \rtimes gl(2, \mathbb{R})$ such that $f_*(\alpha) \cdot g = g \cdot \alpha$ for all $\alpha \in \pi$. Let $\alpha=(a, A)=\left(\begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$, $t_1=(e_1, I)$ and $t_2=(e_2, I)$. Since $(a, A)^2=(e_1, I)$, we may think $\pi$ is generated by $t_2$ and $\alpha$.

Since all endomorphisms of $\pi$ must map $[\pi, \pi]$ into itself and $[\pi, \pi]=<t_2^{-2}>=<t_2^{2}>$, $f_*(t_2^2)=t_2^{2m}$ for some $m \in \mathbb{Z}$.

Let $f_*(t_2) = \alpha^{\varepsilon_1} t_1^{\varepsilon_2} t_2^{\varepsilon_3}$. Then

$$f_*(t_2^2) = \alpha^{\varepsilon_1} t_1^{\varepsilon_2} t_2^{\varepsilon_3} \alpha^{\varepsilon_1} t_1^{\varepsilon_2} t_2^{\varepsilon_3}.$$

$$= \begin{cases} \alpha^{2\varepsilon_1} t_1^{2\varepsilon_2} = t_1^{2\varepsilon_2+1} & \text{if } \varepsilon_1=1 \\ t_1^{2\varepsilon_2} t_2^{2\varepsilon_3} & \text{if } \varepsilon_1=0 \end{cases}.$$

Hence $\varepsilon_1 \neq 1$ and $\varepsilon_1=0=\varepsilon_2$. That is $f_*(t_2) = t_2^{\varepsilon_3}$. And there are two possibilities for $f_*(\alpha)$. *i. e.* either $f_*(\alpha)=\left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, I\right)$ or

$f_*(\alpha)=\left(\begin{pmatrix} z_1+1/2 \\ z_2 \end{pmatrix}, A\right)$ where $z_1, z_2 \in \mathbb{Z}$

Case 1. $f_*(t_2) = t_2^{\varepsilon_3}$ and $f_*(\alpha)=\left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, I\right)$. Set $(d, D)=\left(\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \begin{pmatrix} s & t \\ u & v \end{pmatrix}\right)$, where $d_1, d_2 \in \mathbb{Z}$ and $s, t, u, v$ are all integers.

From the semi-conjugate condition $f_*(t_2) \cdot g = g \cdot t_2$ and $f_*(\alpha) \cdot g = g \cdot \alpha$, we obtain $d_1 \in \vec{r}$, $d_2 \in \vec{z}/2$, $t=u=0$, $v \in \vec{z}$ and $s$ is odd integers.

Case 2. $f_*(t_2) = t_2^{\varepsilon_3}$ and $f_*(\alpha)=\left(\begin{pmatrix} z_1+1/2 \\ z_2 \end{pmatrix}, A\right)$. From the semi-conjugate condition $f_*(t_2) \cdot g = g \cdot t_2$ and $f_*(\alpha) \cdot g = g \cdot \alpha$, we obtain $d_1$ and $d_2$ are all real numbers, both $s$ and $u$ are even integers, and $t=v=0$.

According to the following Theorem, we may ignore the translation part $d$ to calculate the Nielsen number.

**Theorem D.** [6] Let $f: M \to M$ be any continuous map on an infra-nilmanifold $M$ with the holonomy group $\Psi$. Then

$$L(f) = \frac{1}{|\Psi|} \sum_{A \in \Psi} \frac{det(A_* - f_*)}{det A_*},$$

$$N(f) = \frac{1}{|\Psi|} \sum_{A \in \Psi} | det(A_* - f_*)|$$

In particular, $|L(f)| \le N(f)$.

Since Klein bottle is infra-nilmanifold, We can apply the above formula to calculate the Nielsen number.

**Theorem E.** Let $f: K \to K$ be any continous map with the holonomy group $Z_2$ such that $f \simeq f_{(d,D)}$. Then

(1) $N(f^k) = \frac{1}{2}(|1-m^k| + |1-m^k|)|(2l+1)^k - 1|$

$$\text{if } D = \begin{pmatrix} 2l+1 & 0 \\ 0 & m \end{pmatrix}, \text{ where } l, m \in Z.$$

(2) $N(f^k) = |1-(2l)^k|$ if $D = \begin{pmatrix} 2l & 0 \\ 2m & 0 \end{pmatrix}$, where $l, m \in Z$.

Proof.

(1) If $D = \begin{pmatrix} 2l+1 & 0 \\ 0 & m \end{pmatrix}$, then $D^k = \begin{pmatrix} (2l+1)^k & 0 \\ 0 & m^k \end{pmatrix}$

$$M(f^k) = \frac{1}{2}\left\{\left|det\begin{pmatrix} 1-(2l+1)^k & 0 \\ 0 & 1-m^k \end{pmatrix}\right| + \left|det\begin{pmatrix} 1-(2l+1)^k & 0 \\ 0 & -1-m^k \end{pmatrix}\right|\right\}$$

$$= \frac{1}{2}|((2l+1)^k((m^k-1)+(m^k+1))|$$

$$= |m^k||(2l+1)^k - 1|$$

(2) If $D = \begin{pmatrix} 2l & 0 \\ 2m & 0 \end{pmatrix}$, then $D^k = \begin{pmatrix} (2l)^k & 0 \\ (2m)(2l)^{k-1} & 0 \end{pmatrix}$

$$M(f^k) = \frac{1}{2}(|(1-(2l)^k)1| + |(1-(2l)^k)(-1)|)$$

$$= \frac{1}{2}(|1-(2l)^k| + |1-(2l)^k| = |1-(2l)^k|$$

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