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RANKS OF k-HYPERGRAPHS

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Abstract. We define the incidence matrices of oriented and nonoriented k-hypergraphs, respectively. We discuss the ranks of some circulant matrices and show that the rank of the incidence matrices of oriented and nonoriented k-hypergraphs H are n under a certain condition on the k-edge set or k-arc set of H.

1. Hypergraphs

A *hypergraph* $H = (V, E)$ is a pair of sets: one is the vertex set $V = \{v_1, \ldots, v_n\}$ and the other is the set $E = \{e_1, \ldots, e_m\}$ of so-called hyperedges which are subsets of V. A hyperedge is called a k-edge if it is a k-subset of V. We call a hypergraph having only k-edges a *k-hypergraph.* Notice that a 2-hypergraph is a usual graph. For a k-hypergraph H, the incidence matrix of H is defined as a $(0, 1)$-matrix of size $n \times m$ such that the ij-entry is 1 if vertex v_i is contained in k-edge e_j and 0 otherwise.

An *orientation* of a k-edge is a linear arrangement of the vertices of the edge. If a k-edge is given an orientation, then it is called a *k-arc.* If all of the hyperedges of a k-hypergraph are k-arcs, then the k-hypergraph is said *oriented.* Given an oriented k-hypergraph $\vec{H} = (V, \vec{E})$ with $|V| = n$ and $|\vec{E}| = m$, we define the incidence matrix

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$M = M(\vec{H}) = [\,m_{ij}\,]$ of $\vec{H}$ as an $n \times m$ $(0, 1, -1)$-matrix by

$$m_{ij} = \begin{cases} 1 & \text{if vertex } v_i \text{ is contained in } k\text{-arc } e_j, \\ & \quad \text{but not as the last element,} \\ -1 & \text{if } v_i \text{ is contained in } e_j \text{ as the last element,} \\ 0 & \text{if } v_i \text{ is not contained in } e_j. \end{cases}$$

For each vertex v of a k-hypergraph H, we define the *degree* of v as the number of k-edges of H containing v. A k-hypergraph is said *regular* if all the vertices have the same degree. We say an oriented k-hypergraph $\vec{H}$ is *regular* if the degrees of the vertices are all equal and the number of k-arcs containing any given vertex as the last element is a constant. From the fact that

$$k|E| = \sum_{v \in V} deg(v),$$

we easily see that for $\vec{H}$ (and H, also) to be regular it necessarily holds that $|V| = n$ divides $k|E| = km$ and so the common degree of the vertices of $\vec{H}$ (and H) is $d = \frac{km}{n}$. Each column sum of M of $\vec{H}$ is $k-2$ and each row sum of M is $\frac{(k-2)m}{n}$.

When an oriented k-hypergraph $\vec{H}$ contains all possible k-edges given some orientation, it is called a *k-hypertournament*. Notice that 2-hypertournaments are usual tournaments. The set $\vec{E}$ of k-arcs is of size $\binom{n}{k}$.

It is known that the ranks of tournaments on n vertices are n or $n-1$. Here, the rank of a tournament means the rank of the adjacency matrix of the tournament. It is natural to ask what the ranks of k-hypergraphs and oriented k-hypergraphs are. Especially, one may ask what the ranks of regular k-hypertournaments are. However, the adjacency matrices of hypergraphs are nor easy to look at than the incidence matrices of them. So, we rather define the rank of a k-hypergraph as the rank of its corresponding incidence matrix.

It is helpful to look at the rank of some circulant matrices to know the rank of k-hypergraphs. Since circulant matrices $A = [\,a_{ij}\,]$ satisfy $a_{i+k,j+k} = a_{ij}$ for $k = 1, 2, \ldots, n-1$, where the addition in the subscripts is considered in modulo n, they are completely determined by their first row. So, we denote a circulant matrix A by $A = \mathrm{circ}(a_{11}, a_{12}, \ldots, a_{1n})$ with its first row $(a_{11}, a_{12}, \ldots, a_{1n})$.

2. Rank of circulant matrices

In this section, we find the ranks of some circulant $(0, 1)$-matrices. Let P be the permutation matrix given by

$$P = E_{12} + E_{23} + \cdots + E_{n-1\,n} + E_{n1},$$

where E_{ij} is the $n \times n$ *unit matrix* having exactly one 1 as the ij-entry and 0's for the other entries. Since the characteristic polynomial of P is $x^n - 1$, the eigenvalues of P are the nth roots of unity in the complex plane.

Let A be a matrix and $f(x)$ a polynomial. Then, for each eigenvalue ζ of A, $f(\zeta)$ is an eigenvalue of the matrix $f(A)$.

Lemma 2.1. [1] *Let* $A = \mathrm{circ}(1, \ldots, 1, 0, \ldots, 0)$ *be the* $n \times n$ *circulant matrix with the first row consisting of* k *1's and* $n - k$ *0's. Then* rank $A = n$ *if and only if* n *and* k *are relatively prime. In this case,* $\det A = k$.

Proof. The circulant matrix A is written as

$$A = I + P + P^2 + \cdots + P^{k-1}.$$

For each eigenvalue ζ of P, $1 + \zeta + \zeta^2 + \cdots + \zeta^{k-1}$ is an eigenvalue of A. So, the determinant of A is

$$\det A = \prod_{j=1}^{n}(1 + \zeta_j + \zeta_j^2 + \cdots + \zeta_j^{k-1}) = k \prod_{j=1}^{n-1} \frac{(1 - \zeta_j^k)}{(1 - \zeta_j)},$$

where $\zeta_j = \omega^j$ for $j = 1, \ldots, n$, and $\omega = e^{\frac{2\pi i}{n}}$ is the primitive nth root of unity.

Note that $\zeta_j^k = e^{\frac{2\pi i}{n} jk} = 1$ for some $j \neq n$ if and only if jk is a multiple of n, i.e., if and only if $j = \frac{qn}{d}$ for some $q = 1, 2, \ldots, d - 1$, where $d = \gcd(n, k)$. That is, the multiplicity of 0 as an eigenvalue of A is $d - 1$. Hence

$$\text{rank } A = n - d + 1$$

and so $\det A = 0$ if and only if $\gcd(n, k) > 1$.

When $\gcd(n,k) = 1$, the sets $\{\zeta_j^k \,|\, j = 1,2,\ldots,n\}$ and $\{\zeta_j \,|\, j = 1,2,\ldots,n\}$ are the same. Consequently we have $\det A = k$. □

Let $S(n,k)$ be the set of $n \times n$ $(0,1)$-matrices with row sums and column sums equal to k. Then, Lemma 2.1 provides an example of the following fact.

Lemma 2.2. [1] *There exists a matrix* $A \in S(n,k)$ *with* $\det A = k\gcd(n,k)$.

Lemma 2.3. [4] *If* $A \in S(n,k)$, *then* $\bar{A} = J - A \in S(n, n-k)$ *and*

$$k \det \bar{A} = (n-k) \det A,$$

where J *is the all* 1*'s matrix.*

Using Lemmas 2.1 and 2.3, we can easily derive the following result.

Corollary 2.4. *Let* $\gcd(n,k) = 1$ *and let* $A \in S(n,k)$ *be* $A = \operatorname{circ}(1,\ldots,1,0,\ldots,0)$. *Then* $\bar{A} = J - A$ *is written as* $\bar{A} = P^k + P^{k+1} + \cdots + P^{n-1}$, $\bar{A} \in S(n, n-k)$ *and* $\det \bar{A} = n-k$.

Theorem 2.5. *Let* $A_1 = (1,0,1,0,\ldots,0) \in S(n,2)$. *Then* A_1 *is a singular matrix if and only if* n *is a multiple of 4.*

Proof. The matrix A_1 is written as $A_1 = I + P^2$, where P is the permutation matrix defined before. Let ω denote the nth primitive root $e^{2\pi i/n}$ of unity. Then

$$\det A_1 = 2 \prod_{j=1}^{n-1} (1 + \omega^{2j}).$$

So $\det A_1 = 0$ if and only if $\omega^{2j} = -1$ for some $j = 1,2,\ldots,n-1$. The equation

$$\omega^{2j} = e^{i(\frac{4\pi j}{n})} = -1$$

implies that $\frac{4\pi j}{n} = \pi N_j$ for some odd integer N_j and hence $4|n$.

Conversely, suppose that $n = 4a$ for some positive integer a. Then we have $\omega^{2j} = e^{\pi i j/a} = -1$ for $j = a$ and $j = 3a$ so that $\det A_1 = 0$. □

For $l = 1, 2, \ldots, n-2$, let $A_l \in S(n,2)$ be the circulant matrix with the first row $(1,0,\ldots,0,1,0,\ldots,0)$, where the second 1 appears in the $(l+2)$th component of the row vector: $A_l = \mathrm{circ}(1,0,\ldots,0,1,0,\ldots,0)$.

Note that for $l = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$, the matrices A_l and A_{n-2-l} are permutationally equivalent, i.e.,

$$A_l P^{n-1-l} = A_{n-2-l} \qquad \text{or} \qquad A_l = A_{n-2-l} P^{l+1}.$$

So the determinants of A_l and A_{n-2-l} satisfy

$$\det A_l = \det A_{n-2-l} \cdot \det P^{l+1},$$

where

$$\det P^{l+1} = \prod_{j=1}^{n} \omega^{j(l+1)} = \omega^{(l+1)\frac{n(n+1)}{2}} = e^{i(l+1)(n+1)\pi}$$

is -1 if both n and l are even, and 1 otherwise. Hence,

$$\det A_{n-2-l} = \begin{cases} -\det A_l, & \text{if both } n \text{ and } l \text{ are even,} \\ \det A_l, & \text{otherwise.} \end{cases}$$

Theorem 2.6. *Let $A_l = \mathrm{circ}(1,0,\ldots,0,1,0,\ldots,0) \in S(n,2)$, as above, for $l = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$. If A_l is singular, then n is even.*

In fact, when l is even, A_l is singular if and only if n is even. When l is odd, let $l+1 = 2^u v$ for some $u \geq 1$ and an odd $v \geq 1$. Then, A_l is singular if and only if $2^{u+1} \mid n$.

Proof. From Theorem 2.5, it is known that A_1 is singular if and only if $4 | n$. For $l = 2, 3, \ldots, \lfloor \frac{n}{2} \rfloor$, $A_l = I + P^{l+1}$ and

$$\det A_l = 2 \prod_{j=1}^{n-1} (1 + \omega^{(l+1)j}).$$

So $\det A_l = 0$ if and only if for some j,

$$\omega^{(l+1)j} = e^{i(\frac{2\pi(l+1)j}{n})} = -1.$$

That is, $\frac{2\pi(l+1)j}{n} = \pi N_j$ for some odd integer N_j. So $2(l+1)j = nN_j$, i.e., n is even.

Suppose that l is even. If $n = 2a$ for some positive integer a, then for $j = a$, $\frac{2\pi(l+1)j}{n} = (l+1)\pi$ and so $\omega^{(l+1)j} = -1$, i.e., $\det A = 0$. Hence when l is even, A_l is singular if and only if n is even.

Now suppose that l is odd. Write $l+1 = 2^u v$, where $u \geq 1$ is an integer and $v \geq 1$ is an odd integer. If $\det A_l = 0$, then $\frac{2\pi(l+1)j}{n} = \frac{\pi 2^{u+1} vj}{n} = \pi N_j$ for some odd N_j implies that $2^{u+1} | n$. Conversely, if $2^{u+1} | n$, n is written $n = 2^{u+1}b$ for some positive integer b. Then $\frac{2\pi(l+1)j}{n} = \frac{\pi vj}{b}$. So for $j = b, 3b, \ldots, (2^{u+1}-1)b$, we have $\omega^{(l+1)j} = -1$ and hence $\det A_l = 0$. $\square$

Corollary 2.7. *If n is odd, then the $n \times n$ circulant matrix A_l is nonsingular for all $l = 1, 2, \ldots, n-2$.*

Theorem 2.8. *Let $A = \mathrm{circ}(-1, 1, \cdots, 1, 0, \cdots, 0)$, where -1 appears once, 1 appears $k-1$ times and 0 appears $n-k$ times. Then* rank $A = n$ *for all $n \geq 2$ and $3 \leq k \leq n$.*

Proof. The matrix A is written $A = -I + P + P^2 + \cdots + P^{k-1}$. Let $\zeta_j = \omega^j$, where ω is the primitive nth root of unity. Then the determinant of A is

$$\det A = \prod_{j=1}^{n} (-1 + \zeta_j + \zeta_j^2 + \cdots + \zeta_j^{k-1}) = (k-2) \prod_{j=1}^{n-1} (\frac{1-\zeta_j^k}{1-\zeta_j} - 2).$$

So $\det A = 0$ if and only if $\frac{1-\zeta_j^k}{1-\zeta_j} - 2 = 0$ for some $j \in \{1, 2, \ldots, n-1\}$. That is, $\zeta_j \neq 1$ is a common root of the equations

$$1 - z^k - 2(1-z) = 0, \qquad |z| = 1.$$

These together imply that

$$|z - \frac{1}{2}| = |\frac{1}{2} z^k| = \frac{1}{2}.$$

The only complex number z satisfying both $|z - \frac{1}{2}| = \frac{1}{2}$ and $|z| = 1$ is $z = 1$. So $\det A \neq 0$, and hence rank $A = n$. $\square$

3. Rank of k-hypergraphs

Now consider the (non-signed) vertex-edge incidence matrix M of a k-hypergraph H on n vertices and m k-edges, which is an $n \times m$ $(0,1)$-matrix. We define the rank of a k-hypergraph H as the rank of the corresponding incidence matrix M. We want to find a condition for the incidence matrices of k-hypergraphs to be of rank n. If $m < n$, then the rank of M is trivially less than n. We assume $m \geq n$. Then $\operatorname{rank} M \leq n$. If n and k are relatively prime and if the incidence matrix includes n columns $e = (1, 1, \dots, 1, 0, \dots, 0)^T$ and $P^j e$ for $j = 1, \dots, n-1$, then by Lemma 2.1, we see that these columns are linearly independent and hence rank $M = n$.

Theorem 3.1. *Let H be a complete k-hypergraph on n vertices. Then the incidence matrix M of H contains a submatrix of size $n \times n$ whose determinant is $k \gcd(n, k) \neq 0$ and so* rank $M = n$.

Proof. The fact that H is a complete k-hypergraph means that H contains all the possible k-edges on n vertex set. So the incidence matrix M contains all the possible (0,1) n-tuples with k 1's and $n-k$ 0's as its columns and so M is of size $n \times \binom{n}{k}$. By Lemma 2.2, it is immediate that M contains a submatrix $A \in S(n,k)$ with $\det A = k \gcd(n,k)$, and hence $\operatorname{rank} M = n$. □

Note that in the theorem if $\gcd(n, k) = 1$, we can pick n columns, $e = (1, 1, \dots, 1, 0, \dots, 0)^T$ and $P^j e$ for $j = 1, \dots, n-1$, which together make the $n \times n$ submatrix B of M, the transpose B^T of which is the circulant matrix $\operatorname{circ}(1, \dots, 1, 0, \dots, 0)$. So by Lemma 2.1, M contains a submatrix whose determinant is k.

For an $(0,1)$ n-tuple e, there exists a positive integer q such that $P^q e = e$. We define the smallest one among such q's the *order* of e under P. Define e_i and e_j to be equivalent under P if $e_i = P^q e_j$ for some integer q. We can, then, partition the set of all $(0,1)$ n-tuples into t equivalence classes of the equivalence relation P. Let $\{e_1, e_2, \dots, e_t\}$ be the representatives of the equivalence classes of P and let the order of e_j be q_j for $j = 1, 2, \dots, t$. Since some $(0,1)$ n-tuples have orders n, we may assume that $q_j = n$ for $j = 1, 2, \dots, s,\ s \leq n$. Then the $(0,1)$-incidence matrix M of a complete k-hypergraph H can be written, by interchanging the columns if necessary, as follows:

$$M = [\ A_1\ A_2\ \cdots\ A_s\ A_{s+1}\ \cdots\ A_t\ ],$$

where each A_j is an $n \times q_j$ circulant submatrix, whose first column is e_j and xth column is $P^{x-1}e_j$ for $x = 2, \dots, q_j$ and $j = 1, 2, \dots, t$. In particular, A_j is an $n \times n$ circulant matrix for $j = 1, 2, \dots, s$.

Theorem 3.2. *Let $n \geq 2$ and $3 \leq k \leq n-1$. If $n|\binom{n}{k}$, then there exists a regular k-hypertournament matrix M. Furthermore, every regular k-hypertournament matrix M has* $\operatorname{rank} M = n$.

Proof. We here just outline the proof. For the details, refer to [2]. Let H be a complete k-hypergraph on n vertices and $\tilde{M}$ the incidence matrix of H. Since $n|\binom{n}{k}$, $r = \frac{1}{n}\binom{n}{k}$ is an integer. So the (0,1)-matrix $\tilde{M}$ of size $n \times \binom{n}{k}$ can be written as

$$\tilde{M} = [\ B_1\ B_2\ \cdots\ B_s\ B_{s+1}\ \cdots\ B_r\ ],$$

where each B_j is $n \times n$ block submatrix for $j = 1, 2, \dots, r$.

Since the rank of a matrix is not changed with the rearrangement of the columns of the matrix, we may assume that, with circulant matrices A_i's mentioned above,

$$\tilde{M} = [\ B_1\ B_2\ \cdots\ B_s\ B_{s+1}\ \cdots\ B_r\ ] = [\ A_1\ A_2\ \cdots\ A_s\ A_{s+1}\ \cdots\ A_t\ ],$$

where t is the number of the equivalence classes under P and $A_i = B_i$ for $i = 1, 2, \dots, s$, and

$$[\ A_{s+1}\ \cdots\ A_t\ ] = [\ B_{s+1}\ \cdots\ B_r\ ].$$

Also, since all of the A_i's are circulant, we may assume that the diagonal entries of each $n \times n$ submatrix B_j are all 1's. In particular, we may assume that $B_1 = A_1 = \operatorname{circ}(1, \dots, 1, 0, \dots, 0) \in S(n, k)$.

Putting negative signs on these diagonal entries of all the blocks, we obtain a $(0, 1, -1)$ regular k-hypertournament matrix M. The first block B_1 is the matrix described in Theorem 2.8, and it is nonsingular and so $\operatorname{rank} M = n$. $\square$

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