Kangweon-Kyungki Math. Jour. **12** (2004), No. 2, pp. 171–175

ON THE LIFTING PROPERTIES OF HOMOMORPHISMS OF FLOWS

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Abstract. The purpose of this paper is to investigate some lifting properties of homomorphisms of flows. It is shown that an almost one to one extension of a minimal proximal flow is proximal. It is also shown that a distal extension of a pointwise almost periodic flow is pointwise almost periodic.

1. Introduction

Let (X, T) be a flow with compact Hausdorff phase space X. The flow is *minimal* if every orbit is dense. If (X, T) and (Y, T) are flows, a *homomorphism* is a continuous equivariant map $\varphi : X \to Y$, $\varphi(xt) = \varphi(x)t$ $(x \in X,\ t \in T)$. If φ is onto, φ is said to be an *epimorphism*. In this case, (X, T) is said to be an *extension* of (Y, T). The enveloping semigroup $E(X)$ of a flow is a kind of compactification of the acting group. A pair of points (x, y), $x, y \in X$ is *proximal* if $xp = yp$ for some $p \in E(X)$. We write $P(X, T)$ for the proximal relation in X. If x and y are not proximal, they are said to be *distal*, and the flow (X, T) is called *distal* if there are no non-trivial proximal pairs. We say that a homomorphism $\varphi : (X, T) \to (Y, T)$ is *proximal* (*distal*) if whenever $x_1, x_2 \in \varphi^{-1}(y)$ then x_1 and x_2 are proximal (distal). A homomorphism $\varphi : (X, T) \to (Y, T)$ is *almost one to one* if there exists a point $y_0 \in Y$ such that $\varphi^{-1}(\{y_0\})$ is a singleton. We say that X is a *proximal*, *distal*, and *almost one to one extension* of Y provided that there exists a proximal, distal, and almost one to one homomorphism of

Received July 26, 2004.

2000 Mathematics Subject Classification: 54H20.

Key words and phrases: proximal extension, distal extension, almost one to one extension, minimal, pointwise almost periodic.

The present research has been conducted by the Research Grant of Kwangwoon University in 2003.

(X,T) onto (Y,T), respectively. Let (X,T) be a flow, and let $x \in X$. We say that x is an *almost periodic point* if for every neighborhood U of x, there is a syndetic subset A of T such that $xA \subset U$. We also say that x is *of characteristic 0* if $D(x) = \overline{xT}$, where $D(x) = \{y \in X \mid x_i t_i \to y$ for some $x_i \to x$ and $t_i \in T\}$. Note that x is an almost periodic point if and only if $\overline{xT}$ is a compact minimal subset of X.

Let $\varphi : (X,T) \to (Y,T)$ be a homomorphism. Then which dynamical properties of (Y,T) lift to (X,T)? In general very little can be said. However, if we start with a bi-flow (H,X,T) and $Y = X/H$, then it is possible to lift 'information' from (Y,T) to (X,T) (see Proposition 3.1 and Proposition 3.2).

In this paper we investigate some lifting properties of homomorphisms of flows.

2. General lifting properties

PROPOSITION 2.1. *[1] A distal extension of a distal flow is distal.*

PROPOSITION 2.2. *A proximal extension of a proximal flow is proximal.*

Proof. Let $\varphi : (X,T) \to (Y,T)$ be a proximal epimorphism, let (Y,T) be proximal and let $x_1, x_2 \in X$. Then $(\varphi(x_1), \varphi(x_2)) \in P_Y$, which implies $\varphi(x_1)q = \varphi(x_2)q$, for some $q \in E(Y)$. Hence there exists an element $p \in E(X)$ such that $\psi(p) = q$, where $\psi : E(X) \to E(Y)$ is the unique epimorphism induced by φ, and $\varphi(x_1 p) = \varphi(x_2 p)$. Since φ is proximal, we have $(x_1 p, x_2 p) \in P(X,T)$. Then there exists an element $r \in E(X)$ such that $(x_1 p)r = (x_2 p)r$. Therefore $x_1(pr) = x_2(pr)$ and $pr \in E(X)$. This follows that $(x_1, x_2) \in P(X,T)$. □

PROPOSITION 2.3. *An almost one to one extension of a minimal proximal flow is proximal.*

Proof. Let $\varphi : (X,T) \to (Y,T)$ be an almost one to one epimorphism and let (Y,T) be minimal proximal. Suppose that there exists a point $y_0 \in Y$ such that $\varphi^{-1}(\{y_0\}) = \{x_0\}$. Now let $x_1, x_2 \in X$ and let $\varphi(x_1) = y_1, \varphi(x_2) = y_2$. Since $(y_1, y_2) \in P(Y,T)$, there exists an element $p \in$

$E(Y)$ such that $y_1p = y_2p$. But since Y is minimal, there exists an element $q \in E(Y)$ such that $y_0 = (y_1p)q = y_1(pq)$. Then there exists an element $r \in E(X)$ such that $\psi(r) = pq$, where $\psi : E(X) \to E(Y)$ is the unique epimorphism induced by φ. Hence $\varphi(x_1r) = \varphi(x_1)\psi(r) = y_1pq = y_2pq = \varphi(x_2)\psi(r) = \varphi(x_2r)$. It follows that $x_1r = x_2r$, which means that $(x_1, x_2) \in P(X,T)$.

Note that if $\varphi : (X,T) \to (Y,T)$ is a homomorphism and Y is minimal, then φ is an epimorphism. Also note that a proximal and distal homomorphism is one to one. The proof of the following proposition is immediate.

PROPOSITION 2.4. *A proximal and distal extension of a minimal flow is minimal.*

LEMMA 2.5 (2). *Let $\varphi : (X,T) \to (Y,T)$ be an epimorphism and let y be an almost periodic point of (Y,T). Then there exists an almost periodic point x of (X,T) with $\varphi(x) = y$.*

PROPOSITION 2.6. *A distal extension of a pointwise almost periodic flow is pointwise almost periodic.*

Proof. Let $\varphi : (X,T) \to (Y,T)$ be a distal epimorphism, let (Y,T) be pointwise almost periodic and let $x \in X$. Then there exists an almost periodic point x_0 of (X,T) with $\varphi(x_0) = \varphi(x)$ by Lemma 2.5. Since φ is a distal homomorphism, it follows that x_0 and x are distal. Since (Y,T) is pointwise almost periodic, we have $\tilde{\varphi}\ (P(X,T)) = P(Y,T)$, where $\tilde{\varphi} : X \times X \to Y \times Y$ the map induced by φ (see Proposition 5.22.3 in [2]). Because $\tilde{\varphi}\ (x, x_0) = (\varphi(x), \varphi(x_0))$, we have that $(x, x_0) \in P(X,T)$. Therefore $x = x_0$.

PROPOSITION 2.7. *Let (X,T) be a proximal extension of a pointwise almost periodic flow. Then the following holds.*

For each $x \in X$, there exists an element $p \in E(X)$ such that xp is an almost periodic point.

Proof. Let $\varphi : (X,T) \to (Y,T)$ be a proximal epimorphism, let (Y,T) be pointwise almost periodic and let $x \in X$. Then there exists an almost periodic point x_0 of (X,T) with $\varphi(x_0) = \varphi(x)$. Therefore x_0 and x are proximal. Hence there exists an element $p \in E(X)$ such that $xp = x_0 p$. Since $xp \in \overline{x_0 T}$ and $\overline{x_0 T}$ is compact minimal, we have that xp is an almost periodic point.

Proposition 2.8. *Let $\varphi : (X,T) \to (Y,T)$ be a monomorphism (or one to one homomorphism). If (Y,T) is of characteristic 0, so is (X,T).*

Proof. Let $x \in X$ and let $x_0 \in D(x)$. Then there exist nets $\{x_i\}$ in X and $\{t_i\}$ in T such that $x_i \to x$ and $x_0 = \lim x_i t_i$. Since $\varphi(x_i) \to \varphi(x)$, it follows that $\varphi(x_0) \in D(\varphi(x))$. Since $D(\varphi(x)) = \overline{\varphi(x)T} = \varphi(\overline{xT})$ and φ is one to one, we have $x_0 \in \overline{xT}$. Therefore $D(x) \subset \overline{xT}$, and hence (X,T) is of characteristic 0.

Proposition 2.9. *Let $\varphi : (X,T) \to (Y,T)$ be a distal monomorphism. Then (X,T) is a distal flow of characteristic 0 if and only if $(\varphi(X),T)$ is a distal flow of characteristic 0.*

Proof. This follows from Corollary 5.7 [2], Proposition 2.1, Proposition 2.8, and Corollary 2.10 [5].

3. Some lifting properties in bi-flows

A bi-flow (H,X,T) involves two flows (H,X) and (X,T). We use $E(X)$ to designate the enveloping semigroup of (X,T).

Note that in general X/H is compact but it need not be Hausdorff. However if H is compact Hausdorff, then X/H is Hausdorff.

Proposition 3.1 (2). *Let (H,X,T) be a bi-flow such that X/H is Hausdorff and $(X/H,T)$ is pointwise almost periodic. Then (X,T) is pointwise almost periodic.*

Proposition 3.2 (2). *Let (H,X,T) be a bi-flow such that X/H is Hausdorff and $(X/H,T)$ is distal. Then (X,T) is also distal.*

Proposition 3.3. *Let (H,X,T) be a bi-flow such that X/H is Hausdorff and $(X/H,T)$ is minimal. Then (X,T) is a disjoint union of minimal sets.*

Proof. Let $x_0 \in X$, $\varphi : (X,T) \to (X/H,T)$ be the canonical map, and let $x \in X$. Since X/H is minimal, we have $\varphi(x) \in \varphi(\overline{x_0 T})$. Therefore $x \in H\overline{x_0 T}$, and hence $X = \cup\{\overline{hx_0 T} : h \in H\}$. Since $(X/H,T)$ is minimal, we have it is pointwise almost periodic. By Proposition 3.1 (X,T) is also pointwise almost periodic. Hence $\{\overline{hx_0 T} : h \in H\}$ is a partition of X consisting of compact minimal sets.

Proposition 3.4. *Let (H,X,T) be a bi-flow such that X/H is Hausdorff and $(X/H,T)$ is minimal. If $\varphi : (X,T) \to (X/H,T)$ is a proximal homomorphism, then (X,T) is also minimal.*

Proof. Let $\varphi : (X,T) \to (X/H,T)$ be a proximal homomorphism, and let $(X/H,T)$ be Hausdorff minimal. Then (X,T) contains a unique minimal set (see Lemma 1.1 of Chapter 2 in [3]). By Proposition 3.3 (X,T) is minimal.

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