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INVERSION FORMULA FOR C-REGULARIZED SEMIGROUPS

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Abstract. In this paper, we establish an inversion formula for exponentially bounded C-regularized semigroup.

1. Introduction

This paper is concerned with the study of inversion formula for C-semigroups. The C-regularized semigroup theory has been introduced by Da Prato [2], and Davies and Pang [3]. This is a generalization of strongly continuous semigroups that may be applied to an abstract Cauchy problem on a Banach space X

$$\frac{d}{dt}u(t) = Au(t), \quad u(0) = x.$$

Let $A : D(A) \subset X \to X$ be a closed linear operator. If A generates a strongly continuous semigroup, then the abstract Cauchy problem has the unique mild solution for all x in X. To generate a strongly continuous semigroup, A must be densely defined and has a nonempty resolvent set. However, operators with empty resolvent set may occur in the abstract Cauchy problem, e. g., Petrovsky correct systems of partial differential equations [4]. Since the generator of C-regularized semigroup may have an empty resolvent set, C-regularized semigroup theory can be applied very efficiently to the abstract Cauchy problem for A with an empty resolvent set.

Throughout this paper X is a Banach space, all operators are linear and M, ω are constants. By $B(X)$, we denote the space of all bounded

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linear operators from X to $X$and C is an injective operator in $B(X)$. For an operator A, we will write $D(A)$ and $R(A)$ for the domain and the range of A, respectively.

2. Inversion formula

First, we recall the definition and basic facts about C-regularized semigroups and generators (see [4]).

Definition. The strongly continuous family $\{T(t) : t \geq 0\} \subset B(X)$ is called a C-regularized semigroups if it satisfies $S(0) = C$ and $T(t)T(s) = CT(t+s)$ for all $t,\ s \geq 0$.

The generator A of $\{T(t) : t \geq 0\}$ is defined by

$$Ax = C^{-1}\left(\lim_{h\to 0}\frac{1}{h}(T(h)x - Cx)\right)$$

with

$$D(A) = \{x \in X : \lim_{h\to 0}\frac{1}{h}(T(h)x - Cx) \text{ exists and is in } R(C)\}.$$

The complex number λ is in $\rho_C(A)$, the C-resolvent set of A, if $\lambda - A$ is injective and $R(C) \subset R(\lambda - A)$.

Lemma 2.1. *Let A be the generator of a C-regularized semigroup $\{T(t) : t \geq 0\}$ satisfying $||T(t)|| \leq Me^{\omega t}$ for all $t \geq 0$. Then $(\omega,\ \infty) \subset \rho_C(A)$ and for $\lambda > \omega$ $R(C) \subset R((\lambda - A))$ and*

$$(\lambda - A)^{-1}C = \int_0^\infty e^{-\lambda t}T(t)dt.$$

The C-resolvent $(\lambda - A)^{-1}C$ is the Laplace transform of $\{T(t) : t \geq 0\}$. Thus we want to have $T(t)$ from the C-resolvent by the inverse Laplace transform. For a C_0 semigroup $\{S(t) : t \geq 0\}$, the Phragmén inversion formula is known (see Theorem 5.1 in [5] and cf. Phragmén Doetsch inversion in [1]).

$$\int_0^t S(s)xds = \lim_{n\to\infty}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j!}e^{tjn}R(jn, A)x$$

for all x in X, where $R(jn, A)$ is the resolvent of the generator A of $\{S(t) : t \geq 0\}$.

In the Phragmén inversion formula, we have the representation of integral of the semigroup. Our main result is to have a representation of the semigroup itself. The idea comes from the differentiation of the Phragmén inversion formula.

Theorem 2.2. *Let A be the generator of a C-regularized semigroup $\{T(t) : t \geq 0\}$ satisfying $||T(t)|| \leq Me^{\omega t}$ for all $t \geq 0$. Let $R(\lambda) = (\lambda - A)^{-1}C$ for $\lambda > \omega$. Then*

$$T(t)x = \lim_{n\to\infty} ne^{\omega t} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} e^{n(j+1)t} R((j+1)n + \omega)x$$

for all $x \in X$ and $t > 0$.

Proof. First we assume that $\{T(t) : t \geq 0\}$ is bounded, that is, $\omega = 0$. Let $t > 0$.

Note that $\int_0^\infty ne^{n(t-s)}e^{-e^{n(t-s)}}ds = \int_{e^{nt}}^0 -e^{-u}du = 1 - e^{-e^{nt}}$. So we have

$$\lim_{n\to\infty} \int_0^\infty ne^{n(t-s)}e^{-e^{n(t-s)}}ds = 1.$$

By the continuity of $T(s)x$, given $\varepsilon > 0$ there exists $\delta > 0$ such that $|s - t| < \delta$ implies $\|T(s)x - T(t)x\| < \varepsilon$. Thus we have

$$\begin{aligned}
&\| \int_0^\infty ne^{n(t-s)}e^{-e^{n(t-s)}}(T(s)x - T(t)x)ds\| \\
&= \int_0^{t-\delta} ne^{n(t-s)}e^{-e^{n(t-s)}}\|T(s)x - T(t)x\|ds \\
&+ \int_{t-\delta}^{t+\delta} ne^{n(t-s)}e^{-e^{n(t-s)}}\|T(s)x - T(t)x\|ds \\
&+ \int_{t+\delta}^{\infty} ne^{n(t-s)}e^{-e^{n(t-s)}}\|T(s)x - T(t)x\|ds \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Since $\|T(t)\| \leq M$, we have

$$\begin{aligned}
I_1 &\leq 2M\|x\| \int_0^{t-\delta} ne^{n(t-s)}e^{-e^{n(t-s)}}ds \\
&= 2M\|x\| \left[e^{-e^{n(t-s)}}\right]_0^{t-\delta} \\
&= 2M\|x\|(e^{-e^{n\delta}} - e^{-e^{nt}}) \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

and

$$
\begin{aligned}
I_3 &\le 2M\|x\| \int_{t+\delta}^{\infty} n e^{n(t-s)} e^{-e^{n(t-s)}} ds \\
&= 2M\|x\|(1 - e^{-e^{-n\delta}}) \;\to\; 0 \quad \text{as} \;\; n \to \infty.
\end{aligned}
$$

By the continuity of $T(s)x$, we have

$$
\begin{aligned}
I_2 &\le \varepsilon \int_{t-\delta}^{t+\delta} n e^{n(t-s)} e^{-e^{n(t-s)}} ds \\
&\le \varepsilon \int_{0}^{\infty} n e^{n(t-s)} e^{-e^{n(t-s)}} ds \\
&= \varepsilon(1 - e^{-e^{nt}}) \;\to\; 0 \quad \text{as} \;\; n \to \infty.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
T(t)x &= \lim_{n\to} \int_0^{\infty} n e^{n(t-s)} e^{-e^{n(t-s)}} T(t)x ds \\
&= \lim_{n\to\infty} \int_0^{\infty} n e^{n(t-s)} e^{-e^{n(t-s)}} T(s)x ds \\
&= \lim_{n\to\infty} \int_0^{\infty} n e^{n(t-s)} (\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} e^{nj(t-s)}) T(s)x ds \\
&= \lim_{n\to\infty} \int_0^{\infty} \sum_{j=0}^{\infty} n \frac{(-1)^j}{j!} e^{n(j+1)(t-s)} T(s)x ds \\
&= \lim_{n\to\infty} n \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} e^{n(j+1)t} \int_0^{\infty} e^{-n(j+1)s} T(s)x ds \\
&= \lim_{n\to\infty} n \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} e^{n(j+1)t} R(n(j+1))x.
\end{aligned}
$$

Suppose that $\|T(t)\| \le Me^{\omega t}$. Let $S(t) = e^{-\omega t}T(t)$. Then $\{S(t) : t \ge 0\}$ is a bounded C-regularized semigroup with the generator $A - \omega$. So we have

$$
e^{-\omega t}T(t)x = \lim_{n\to\infty} n \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} e^{n(j+1)t} R(n(j+1)+\omega)x.
$$

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