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ON CLOSED CONVEX HULLS AND THEIR EXTREME POINTS

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Abstract. In this paper, the new subclass denoted by $S_p(\alpha,\beta,\xi,\gamma)$ of p-valent holomorphic functions has been introduced and investigate the several properties of the class $S_p(\alpha,\beta,\xi,\gamma)$. In particular we have obtained integral representation for mappings in the class $S_p(\alpha,\beta,\xi,\gamma)$ and determined closed convex hulls and their extreme points of the class $S_p(\alpha,\beta,\xi,\gamma)$.

1. Introduction

Let S_p (p a fixed integer greater than zero) denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \tag{1.1}$$

which are holomorphic in the unit disc $E = \{z : |z| < 1\}$. A function f in S_p satisfying the inequality

$$Re\left(z\frac{f'(z)}{f(z)}\right) > \alpha$$

for $z \in E$ and $0 \le \alpha < p$, is known as p-valent starlike function of order α. We denote it by $S_p(\alpha)$.

Now we define a subclass $S_p(\alpha,\beta,\xi,\gamma)$ of S_p as follows.

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Definition.. A function f in S_p belongs to the class $S_p(\alpha, \beta, \xi, \gamma)$ if f satisfies the condition.

$$\left| \frac{z\frac{f'(z)}{f(z)} - p}{2\xi\left(z\frac{f'(z)}{f(z)} - \alpha\right) - \gamma\left(z\frac{f'(z)}{f(z)} - p\right)} \right| < \beta$$

where $0 \le \alpha < p, \quad 0 < \beta \le 1, \quad 1/2 < \xi \le 1, 0 < \gamma \le 1.$

By giving specific values to $\alpha, \beta, \xi, \gamma$ we obtain several subclasses studied by various authors Brickman ([1]), Lakshminarsimhan([2]), Brickman, Hallenbeck and MacGregor ([3]), Juneja and Mogra([4]).

2. Closed Convex Hulls and Their Extreme Points

To be able to identify the closed-convex hulls and the extreme points of closed convex hulls of $S_p(\alpha, \beta, \xi, \gamma)$, we need the following lemma.

Lemma. *Let $G(z) = p + c_1 z + c_2 z^2 + \cdots$ be holomorphic in E and satisfy the condition*

$$\left| \frac{G(z) - p}{2\xi(G(z) - \alpha) - \gamma(G(z) - p)} \right| < \beta$$

for $z \in E$ if and only if

$$G(z) = \frac{p - \beta z\theta(z)(\gamma p - 2\xi\alpha)}{1 + \beta z\theta(z)(2\xi - \gamma)}$$

where $0 \le \alpha < p, \quad 0 < \beta \le 1, \quad 1/2 < \xi \le 1, \quad 0 < \gamma \le 1$ and $\theta(z)$ is a holomorphic function such that $|\theta(z)| \le 1$.

Proof. Suppose that

$$\left| \frac{G(z) - p}{2\xi(G(z) - \alpha) - \gamma(G(z) - p)} \right| < \beta$$

for $z \in E$, where $0 \le \alpha < p, \quad 0 < \beta \le 1, \quad 1/2 < \xi \le 1, \quad 0 < \gamma \le 1.$ Let us consider the function

$$g(z) = \frac{G(z) - p}{2\xi(G(z) - \alpha) - \gamma(G(z) - p)}. \tag{2.1}$$

We observe that $g(0) = 0$ and $|g(z)| < \beta$. By Schwartz's Lemma we have $|g(z)| \leq |z|$. Thus we get

$$g(z) = \beta z\theta(z) \tag{2.2}$$

where $\theta(z)$ is holomorphic in E and $|\theta(z)| \leq 1$.

From (2.1) and (2.2) we have

$$\beta z\theta(z) = \frac{G(z) - p}{2\xi(G(z) - \alpha) - \gamma(G(z) - p)}.$$

Therefore

$$G(z) = \frac{p - \beta z\theta(z)(\gamma p - 2\xi\alpha)}{1 + \beta z\theta(z)(2\xi - \gamma)}$$

where $\theta(z)$ is holomorphic in E and $|\theta(z)| \leq 1$.

Conversely, suppose that $G(z)$ having above representation, is holomorphic in E. Then

$$G(z) = \frac{p - \beta z\theta(z)(\gamma p - 2\xi\alpha)}{1 + \beta z\theta(z)(2\xi - \gamma)}$$

where $0 \leq \alpha < p$, $0 < \beta \leq 1$, $1/2 < \xi \leq 1$, $0 < \gamma \leq 1$ and $\theta(z)$ is holomorphic in E and $|\theta(z)| \leq 1$. Thus

$$\beta z\theta(z) = \frac{G(z) - p}{2\xi(G(z) - \alpha) - \gamma(G(z) - p)}.$$

Since $|\theta(z)| \leq 1$ and $|z| < 1$, we have $|\beta z\theta(z)| < \beta$. Therefore

$$\left|\frac{G(z) - p}{2\xi(G(z) - \alpha) - \gamma(G(z) - p)}\right| < \beta.$$

□

THEOREM 1. *Let $f(z)$ be holomorphic in E. Then $f(z) \in S_p(\alpha, \beta, \xi, \gamma)$ if and only if*

$$f(z) = \exp \int_a^z \frac{p - \beta t\theta(t)(\gamma p - 2\xi\alpha)}{(1 + \beta t\theta(t)(2\xi - \gamma))t} dt$$

where $a \in E \setminus 0$. and $\theta(t)$ is holomorphic in E and $|\theta(t)| \leq 1$.

Proof. Let $f(z) \in S_p(\alpha, \beta, \xi, \gamma)$. Then

$$\left| \frac{z\frac{f'(z)}{f(z)} - p}{2\xi\left(z\frac{f'(z)}{f(z)} - \alpha\right) - \gamma\left(z\frac{f'(z)}{f(z)} - p\right)} \right| < \beta$$

By Lemma

$$z\frac{f'(z)}{f(z)} = \frac{p - \beta z\theta(z)(\gamma p - 2\xi\alpha)}{1 + \beta z\theta(z)(2\xi - \gamma)}$$

or

$$\frac{f'(z)}{f(z)} = \frac{p - \beta z\theta(z)(\gamma p - 2\xi\alpha)}{(1 + \beta z\theta(z)(2\xi - \gamma))z} \tag{2.3}$$

where $|\theta(z)| \leq 1$. Integrating both sides of (2.3), we have

$$\log f(z) = \int_a^z \frac{f'(t)}{f(t)} dt = \int_a^z \frac{p - \beta t\theta(t)(\gamma p - 2\xi\alpha)}{(1 + \beta t\theta(t)(2\xi - \gamma))t} dt$$

where $a \in E \setminus 0$. Hence

$$f(z) = \exp \int_a^z \frac{p - \beta t\theta(t)(\gamma p - 2\xi\alpha)}{(1 + \beta t\theta(t)(2\xi - \gamma))t} dt.$$

Conversely, assume that

$$f(z) = \exp \int_a^z \frac{p - \beta t\theta(t)(\gamma p - 2\xi\alpha)}{(1 + \beta t\theta(t)(2\xi - \gamma))t} dt$$

where $0 \leq \alpha < p,\ 0 < \beta \leq 1,\ 1/2 < \xi \leq 1,\ 0 < \gamma \leq 1$ and $a \in E \setminus 0$. Taking log and differentiating both sides

$$\frac{f'(z)}{f(z)} = \frac{p - \beta z\theta(z)(\gamma p - 2\xi\alpha)}{(1 + \beta z\theta(z)(2\xi - \gamma))z}$$

or

$$\frac{zf'(z)}{f(z)} = \frac{p - \beta z\theta(z)(\gamma p - 2\xi\alpha)}{(1 + \beta z\theta(z)(2\xi - \gamma))}.$$

By Lemma,

$$\left| \frac{z\frac{f'(z)}{f(z)} - p}{2\xi\left(z\frac{f'(z)}{f(z)} - \alpha\right) - \gamma\left(z\frac{f'(z)}{f(z)} - p\right)} \right| < \beta.$$

Hence $f(z) \in S_p(\alpha, \beta, \xi, \gamma)$. □

Remark.

1. $S_p(\alpha, 1, 1, 1) = S_p^*(\alpha)$ (H. Silverman Class).
2. $S_p(\alpha, \beta, \frac{1+\alpha}{2}, 1) = S_p^*(\beta, \frac{1+\alpha}{2})$ (T. V. Lakshminarsimhan Class([2])).
3. $S_p(\alpha, 1, \beta, 1) = S_p^*(\alpha, 1, \beta)$ (Juneja Mogra Class ([4])).

Theorem 2. *Suppose that $f(z) \in S_p(\alpha, \beta, \xi, \gamma)$. Then f has the following integral representation*

$$f(z) = z^p \exp\left(\frac{-2\xi(\gamma p - \alpha)}{2\xi - \gamma} \int_X \log(1 + \beta x z(2\xi - \gamma)) d\mu(x)\right)$$

where X is the unit circle and $\int_X d\mu(x) = 1$.

Proof. As $f(z) \in S_p(\alpha, \beta, \xi, \gamma)$ we have

$$\left| \frac{z\frac{f'(z)}{f(z)} - p}{2\xi\left(z\frac{f'(z)}{f(z)} - \alpha\right) - \gamma\left(z\frac{f'(z)}{f(z)} - p\right)} \right| < \beta$$

where $0 \le \alpha < p,\ 0 < \beta \le 1,\ 1/2 < \xi \le 1,\ 0 < \gamma \le 1$. By routine calculation, we have

$$z\frac{f'(z)}{f(z)} = \int_X \frac{p - \beta x z(\gamma p - 2\xi\alpha)}{1 + \beta x z(2\xi - \gamma)} d\mu(x),$$

where X is the unit circle and $\int_X d\mu(x) = 1$. This can be further transcribed as,

$$\log\left(\frac{f(z)}{z^p}\right) = \frac{-2\xi(\gamma p - \alpha)}{2\xi - \gamma} \int_X \log(1 + \beta x z(2\xi - \gamma)) d\mu(x).$$

This is equivalent to

$$f(z) = z^p \exp\left(\frac{-2\xi(\gamma p - \alpha)}{2\xi - \gamma} \int_X \log(1 + \beta x z(2\xi - \gamma)) d\mu(x)\right).$$

□

We now state a very general result that is originally due to Brickman, Hallenbeck. Our findings heavily depend on this very useful result. This result is popularly described as “Geometric Mean Theorem”.

Theorem 3. *Let X be the unit circle, $\mathcal{P}$ the set of probability measures on X and $p > 0$. Then given $\mu \in \mathcal{P}$, there exists $\gamma \in \mathcal{P}$ such that*

$$\exp\left(\int_X -p\log(1-xz)d\mu(x)\right) = \int_X (1-xz)^{-p} d\gamma(x).$$

We shall put to use the integral representation of the members of $S_p(\alpha, \beta,$
$\xi, \gamma)$ along with the "Geometric Mean Theorem" to determine the set of extreme points of the closed convex hull of $S_p(\alpha, \beta, \xi, \gamma)$.

Theorem 4. *Let X be the unit circle, $\mathcal{P}$ the set of probability measures on X and $\mathcal{F}$ the set of functions f_μ on E, defined by*

$$f_\mu(z) = \int_X z^p \left(1 + \beta xz(2\xi - \gamma)\right)^{-\frac{2\xi(\gamma p - \alpha)}{2\xi - \gamma}} d\mu(x)$$

where $\mu \in \mathcal{P}$and $0 \le \alpha < p, 0 < \beta \le 1, \frac{1}{2} < \xi \le 1, 0 < \gamma \le 1$. Then

$$\mathcal{F} = \text{CLCO } S_p(\alpha, \beta, \xi, \gamma)$$

and the map $\mu \to f_\mu$ is one to one and each function

$$z \to z^p(1 + \beta xz(2\xi - \gamma))^{-\frac{2\xi(\gamma p - \alpha)}{2\xi - \gamma}}$$

is in the Ext. CLCO $S_p(\alpha, \beta, \xi, \gamma)$.

Proof. By theorem 2 and 3, clearly if $f \in S_p(\alpha, \beta, \xi, \gamma)$, then $f \in \mathcal{F}$ is compact and convex. Hence $\mathcal{F}$ is the closed convex hull of $\mathcal{P}$. Thus CLCO$S_p(\alpha, \beta, \xi, \gamma) \subseteq \mathcal{F}$. Again as each kernel function

$$z \to z^p(1 + \beta xz(2\xi - \gamma))^{-\frac{2\xi(\gamma p - \alpha)}{(2\xi - \gamma)}}$$

belongs to $S_p(\alpha, \beta, \xi, \gamma)$. Thus $\mathcal{F} \subseteq$ CLCO $S_p(\alpha, \beta, \xi, \gamma)$. Therefore $\mathcal{F}$ = CLCO $S_p(\alpha, \beta, \xi, \gamma)$.

If $f_{\mu_1} = f_{\mu_2}$ for $\mu_1 \in \mathcal{P}$ and $\mu_2 \in \mathcal{P}$, then

$$\int_X \frac{z}{(1+\beta xz(2\xi-\gamma))^{-\frac{2\xi(\gamma p-\alpha)}{(2\xi-\gamma)}}} d\mu_1(x) = \int_X \frac{z}{(1+\beta xz(2\xi-\gamma))^{-\frac{2\xi(\gamma p-\alpha)}{(2\xi-\gamma)}}} d\mu_2(x)$$

for $z \in E$. It follows that

$$\int_X x^n d\mu_1(x) = \int_X x^n d\mu_2(x)$$

for $n = 0, 1, 2, \cdots$. Therefore we conclude that $\mu_1 = \mu_2$. Hence the map $\mu \to f_\mu$ is one-to-one and we get the desired conclusion about the extreme points. □

We now state several particular cases of our result. The very fact our result subsumes several particular cases many of which seem to be new stands in testimony of the effectiveness and utility of the new family of $S_p(\alpha, \beta, \xi, \gamma)$ that we have introduced.

Corollary 1. *Let X be the unit circle, $\mathcal{P}$ the set of probability measures on X and $\mathcal{F}$ the set of functions f_μ on E defined by*

$$f_\mu(z) = \int_X z(1+xz)^{-2} d\mu(x)$$

where $\mu \in \mathcal{P}$. Then $\mathcal{F} = \text{CLCO } S(0,1,1,1) = \text{CLCO } S^*$ and the map $\mu \to f_\mu$ is one-to-one.

This corollary is due to ([1], p=1). By giving specific values say $\alpha = 0, \beta = \xi = \gamma = 1$ in Theorem 4, we get the family S^* of univalent starlike functions of order α.

Corollary 2. *Let X be the unit circle, $\mathcal{P}$ the set of probability measures on X and $\mathcal{F}$ the set of functions f_μ on E defined by*

$$f_\mu(z) = \int_X z(1+\alpha\beta xz)^{-\frac{1+\alpha}{\alpha}} d\mu(x)$$

where $\mu \in \mathcal{P}, 0 < \alpha < 1, 0 \leq \beta < 1$. *Then*

$$\mathcal{F} = \text{CLCO } S(\alpha, \beta, \frac{1+\alpha}{2}, 1) = \text{CLCO } S^*(\beta, \frac{1+\alpha}{2}).$$

Further the map $\mu \to f_\mu$ *is one-to-one and each function*

$$z \to z(1 + \alpha\beta xz)^{-\frac{1+\alpha}{\alpha}}$$

is in the Ext. CLCO $S(\alpha, \beta, \frac{1+\alpha}{2}, 1)$.

This corollary is due to ([2], p=1). By giving specific values say $\alpha = \alpha, \beta = \beta, \xi = \frac{1+\alpha}{2}, \gamma = 1$ in Theorem 4, we get the family $S^*(\beta, \frac{1+\alpha}{2})$ of univalent starlike functions of order α.

Corollary 3. *Let* X *be the unit circle,* $\mathcal{P}$ *the set of probability measures on* X *and* $\mathcal{F}$ *the set of functions* f_μ *on* E *defined by,*

$$f_\mu(z) = \int_X z(1 + xz)^{-2(1-\alpha)} d\mu(x)$$

where $\mu \in \mathcal{P},\ 0 \leq \alpha < 1$. *Then*

$$\mathcal{F} = \text{CLCO } S(\alpha, 1, 1, 1) = \text{CLCO } S^*(\alpha).$$

Further the map $\mu \to f_\mu$ *is one-to-one and each function*

$$z \to z(1 + xz)^{-2(1-\alpha)}$$

is in the Ext. CLCO $S(\alpha, 1, 1, 1)$.

This corollary is due to ([3], p=1). By giving specific values say $\alpha = \alpha, \beta = \xi = \gamma = 1$ in Theorem 4, we get the family $S^*(\alpha)$ of univalent starlike functions of order α.

Corollary 4. *Let* X *be the unit circle,* $\mathcal{P}$ *the set of probability measures on* X *and* $\mathcal{F}$ *the set of functions* f_μ *on* E *defined by*

$$f_\mu(z) = \int_X z(1 + xz(2\beta - 1))^{-\frac{2\beta(1-\alpha)}{2\beta-1}}$$

where $\mu \in \mathcal{P}, \frac{1}{2} < \beta \leq 1, 0 \leq \alpha < \frac{1}{2}\beta$. *Then*

$$\mathcal{F} = \text{CLCO } S(\alpha, 1, \beta, 1) = \text{CLCO } S^*(\alpha, \beta).$$

Further the map $\mu \to f_\mu$ *is one-to-one and each function*

$$z \to z\,(1 + xz(2\beta - 1))^{-\frac{2\beta(1-\alpha)}{2\beta-1}}$$

is in the Ext. CLCO $S(\alpha, 1, \beta, 1)$.

This corollary is due to ([4], p=1). By giving specific values say $\alpha = \alpha, \beta = 1, \xi = \beta, \gamma = 1$ in Theorem, we get the family $S^*(\alpha, \beta)$ of univalent starlike functions of order α.

References

1. Brickman L., MacGregor T. H. and Wilken D. R., *Convex hulls of some classical families of univalent functions*, Trans. Amer. Math. Soc. 156 (1971), 91-107.
2. Lakshminarsinhan T. V., *On subclasses of functions starlike in the unit disk*, Jour. Indian Math. Soc. 41 (1977), 233-243.
3. Brickman L., Hallenbeck D., MacGregor T. H and Wilken D. R., *Convex hulls and extreme points of families of starlike convex mappings*, Trans. Amer. Math. Soc. 185 (1973), 413-428.
4. Juneja O. P. and Mogra M. L., *Radii of convexity for certain classes of univalent analytic functions*, Pacif. Jour. Math. 78 (1978), 359-368.

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