RECURRENCE RELATION FOR QUOTIENTS OF THE POWER DISTRIBUTION BY RECORD VALUES

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ABSTRACT. In this paper we establish some recurrence relations satisfied by quotient moments of upper record values from the power distribution. Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with a common continuous distribution function(cdf) $F(x)$ and probability density function(pdf) $f(x)$. Let $Y_n = max\{X_1, X_2, \cdots, X_n\}$ for $n \geq 1$. We say X_j is an upper record value of $\{X_n, n \geq 1\}$, if $Y_j > Y_{j-1}, j > 1$. The indices at which the upper record values occur are given by the record times $\{u(n)\}, n \geq 1$, where $u(n) = \min\{j | j > u(n-1), X_j > X_{u(n-1)}, n \geq 2\}$ and $u(1) = 1$. Suppose $X \in POW(0, 1, \theta)$ then

$$E\left(\frac{X_{u(m)}^r}{X_{u(n)}^{s+1}}\right) = \frac{\theta}{s} E\left(\frac{X_{u(m)}^r}{X_{u(n-1)}^s}\right) + \frac{(s-\theta)}{s} E\left(\frac{X_{u(m)}^r}{X_{u(n)}^s}\right) \text{ and}$$

$$E\left(\frac{X_{u(m)}^{r+1}}{X_{u(n)}^s}\right)$$
$$= \frac{\theta}{(r+1)} \left[E\left(\frac{X_{u(m-1)}^{r+1}}{X_{u(n+1)}^s}\right) - E\left(\frac{X_{u(m)}^{r+1}}{X_{u(n-1)}^s}\right) + \frac{(r+1)}{\theta} E\left(\frac{X_{u(m)}^r}{X_{u(n)}^s}\right) \right].$$

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with a common cdf $F(x)$ and pdf $f(x)$. Sup-

pose $Y_n = \max\{X_1, X_2, \cdots, X_n\}$ for $n \geq 1$. We say X_j is an upper record value of this sequence if $Y_j > Y_{j-1}, j > 1$. We define the record times $u(n)$ by $u(1) = 1$ and

$$u(n) = \min\{\, j \mid j > u(n-1)\,,\,\, X_j > X_{u(n-1)}\,,\,\, n \geq 2\,\}.$$

The record times of the sequence $\{X_n, n \geq 1\}$ are random variables and are the same as those for the sequence $\{F(X_n), n \geq 1\}$. We know that the distribution of $u(n)$ does not depend on $F(x)$. Hence, the distribution of $u(n)$ can be determined by considering the uniform distribution $F(x) = x$. We will call the random variable $X \in POW(0, 1, \theta)$ if the corresponding probability cumulative function $F(x)$ of x is of the form

$$(1) \qquad F(x) = \begin{cases} 1 - (1-x)^{\theta} & \text{if } 0 < x < 1 \text{ and } \theta > 0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

Some characterizations of the power distribution are known. But mainly some recurrence relations satisfied by the single and product moments of record values. Such results have been established by Ahsanullah[1]. Balakrishnan, Ahsanullah and Chan[4, 5], and Balakrishnan and Ahsanullah[2, 3] for the extreme value, exponential, Pareto and generalized extreme value distributions. And Dallas[6] characterizes the power and Pareto distributions using the conditional expectations.

In this paper, we will give some recurrence relations satisfied by the quotient moments of upper record values from the power distribution.

2. Main Results

THEOREM 2.1. *For* $1 \leq m \leq n-2$, $r = 0, 1, 2, \cdots$ *and* $s = 1, 2, \cdots$,

$$E\left(\frac{X_{u(m)}^r}{X_{u(n)}^{s+1}}\right) = \frac{\theta}{s} E\left(\frac{X_{u(m)}^r}{X_{u(n-1)}^s}\right) + \frac{(s-\theta)}{s} E\left(\frac{X_{u(m)}^r}{X_{u(n)}^s}\right).$$

Proof. First of all, we have that for the power distribution in (1), $(1-x)f(x) = \theta[1 - F(x)]$. The joint pdf of $X_{u(m)}$ and $X_{u(n)}$ is

$$f_{m,n}(x, y) = \frac{1}{\Gamma(m)\Gamma(n-m)} R^{m-1}(x)r(x)[R(y) - R(x)]^{n-m-1} f(y).$$

Let us consider for $1 \le m \le n-2$, $r = 0, 1, 2, \cdots$ and $s = 1, 2, \cdots$,

$$E\left(\frac{X_{u(m)}^r}{X_{u(n)}^{s+1}} - \frac{X_{u(m)}^r}{X_{u(n)}^s}\right)$$

$$= \int\int_{0<x<y<1} \left(\frac{x^r}{y^{s+1}} - \frac{x^r}{y^s}\right) f_{m,n}(x, y)dydx$$

$$= \frac{1}{\Gamma(m)\Gamma(n-m)} \times$$

$$\int\int_{0<x<y<1} \frac{x^r}{y^{s+1}}(1-y)R^{m-1}(x)r(x)[R(y) - R(x)]^{n-m-1}f(y)dydx$$

$$= \frac{1}{\Gamma(m)\Gamma(n-m)} \times$$

$$\int_0^1 x^r R^{m-1}(x)r(x) \left(\int_x^1 \frac{1}{y^{s+1}}(1-y)[R(y) - R(x)]^{n-m-1}f(y)dy\right) dx$$

$$= \frac{1}{\Gamma(m)\Gamma(n-m)} \times$$

$$\int_0^1 x^r R^{m-1}(x)r(x) \left(\int_x^1 \frac{1}{y^{s+1}}[R(y) - R(x)]^{n-m-1}\theta[1 - F(y)]dy\right) dx.$$

Using integrating by parts treating $\dfrac{\theta}{y^{s+1}}$ for integration and $[R(y) - R(x)]^{n-m-1}[1 - F(y)]$ for differentiation on the second integration, we

get

$$\int_x^1 \frac{1}{y^{s+1}} [R(y) - R(x)]^{n-m-1} \theta[1 - F(y)] dy$$

$$= \left[-\frac{\theta}{s} \frac{1}{y^s} [R(y) - R(x)]^{n-m-1} [1 - F(y)] \right]_x^1$$

$$+ \frac{(n-m-1)\ \theta}{s} \int_x^1 \frac{1}{y^s} [R(y) - R(x)]^{n-m-2} f(y) dy$$

$$- \frac{\theta}{s} \int_x^1 \frac{1}{y^s} [R(y) - R(x)]^{n-m-1} f(y) dy.$$

Then we have

$$E\left(\frac{X_{u(m)}^r}{X_{u(n)}^{s+1}} - \frac{X_{u(m)}^r}{X_{u(n)}^s} \right) = \frac{\theta}{s\ \Gamma(m)\Gamma(n-m-1)} \times$$

$$\int \int_{0<x<y<1} \frac{x^r}{y^s} R^{m-1}(x) r(x) [R(y) - R(x)]^{n-m-2} f(y) dy dx$$

$$- \frac{\theta}{s\ \Gamma(m)\Gamma(n-m)} \times$$

$$\int \int_{0<x<y<1} \frac{x^r}{y^s} R^{m-1}(x) r(x) [R(y) - R(x)]^{n-m-1} f(y) dy dx$$

$$= \frac{\theta}{s} \int \int_{0<x<y<1} \frac{x^r}{y^s} f_{m,n-1}(x,y) dy dx$$

$$- \frac{\theta}{s} \int \int_{0<x<y<1} \frac{x^r}{y^s} f_{m,n}(x,y) dy dx$$

$$= \frac{\theta}{s} E\left(\frac{X_{u(m)}^r}{X_{u(n-1)}^s} \right) - \frac{\theta}{s} E\left(\frac{X_{u(m)}^r}{X_{u(n)}^s} \right).$$

Hence

$$E\left(\frac{X_{u(m)}^r}{X_{u(n)}^{s+1}} \right) = \frac{\theta}{s} E\left(\frac{X_{u(m)}^r}{X_{u(n-1)}^s} \right) + \frac{(s-\theta)}{s} E\left(\frac{X_{u(m)}^r}{X_{u(n)}^s} \right).$$

This completes the proof. $\qquad\square$

COROLLARY 2.2. *For* $m \geq 1, \quad r = 0, 1, 2, \cdots \quad$ *and* $s = 1, 2, \cdots$,

$$E\left(\frac{X_{u(m)}^r}{X_{u(m+1)}^{s+1}}\right) = \frac{\theta}{s} E(X_{u(m)}^{r-s}) + \frac{(s-\theta)}{s} E\left(\frac{X_{u(m)}^r}{X_{u(m+1)}^s}\right).$$

Proof. Upon substituting $n = m+1$ in Theorem 2.1 and simplifying, then we have

$$E\left(\frac{X_{u(m)}^r}{X_{u(m+1)}^{s+1}}\right) = \frac{\theta}{s} E(X_{u(m)}^{r-s}) + \frac{(s-\theta)}{s} E\left(\frac{X_{u(m)}^r}{X_{u(m+1)}^s}\right).$$

$\square$

THEOREM 2.3. *For* $1 \leq m \leq n-2$ *and* $r, s = 0, 1, 2, \cdots$,

$$E\left(\frac{X_{u(m)}^{r+1}}{X_{u(n)}^s}\right)$$
$$= \frac{\theta}{(r+1)}\left[E\left(\frac{X_{u(m-1)}^{r+1}}{X_{u(n+1)}^s}\right) - E\left(\frac{X_{u(m)}^{r+1}}{X_{u(n-1)}^s}\right) + \frac{(r+1)}{\theta}E\left(\frac{X_{u(m)}^r}{X_{u(n)}^s}\right)\right].$$

Proof. In the same manner as Theorem 2.1, let us consider for $1 \leq m \leq n-2$ and $r, s = 0, 1, 2, \cdots$,

$$E\left(\frac{X_{u(m)}^{r+1}}{X_{u(n)}^s} - \frac{X_{u(m)}^r}{X_{u(n)}^s}\right)$$
$$= \int\int_{0<x<y<1}\left(\frac{x^{r+1}}{y^s} - \frac{x^r}{y^s}\right)f_{m,n}(x,y)dxdy$$
$$= \frac{1}{\Gamma(m)\Gamma(n-m)}\int\int_{0<x<y<1}\frac{x^r}{y^s}(x-1)R^{m-1}(x)r(x)$$
$$\times [R(y) - R(x)]^{n-m-1}f(y)dxdy.$$

Since the power distribution, $r(x) = \dfrac{f(x)}{1 - F(x)} = \dfrac{\theta}{1 - x}$. Upon substituting the above expressions and simplifying the resulting equations, we obtain that

$$E\left(\frac{X_{u(m)}^{r+1}}{X_{u(n)}^s} - \frac{X_{u(m)}^r}{X_{u(n)}^s}\right)$$

$$= a_{mn} \int_0^1 \frac{1}{y^s} f(y) \left(-\theta \int_0^y x^r R^{m-1}(x)[R(y) - R(x)]^{n-m-1} dx\right) dy,$$

where $a_{mn} = \dfrac{1}{\Gamma(m)\Gamma(n-m)}$.

Using integrating by parts treating x^r for integration and $R^{m-1}(x)[R(y) - R(x)]^{n-m-1}$ for differentiation on the second integration, we get

$$\int_0^y x^r R^{m-1}(x)[R(y) - R(x)]^{n-m-1} dx$$

$$= \left[\frac{1}{(r+1)} x^{r+1} \ R^{m-1}(x)[R(y) - R(x)]^{n-m-1}\right]_0^y$$

$$- \frac{(m-1)}{(r+1)} \int_0^y x^{r+1} R^{m-2}(x) r(x)[R(y) - R(x)]^{n-m-1} dx$$

$$+ \frac{(n-m-1)}{(r+1)} \int_0^y x^{r+1} R^{m-1}(x) r(x)[R(y) - R(x)]^{n-m-2} dx.$$

Then we have

$$E\left(\frac{X_{u(m)}^{r+1}}{X_{u(n)}^s} - \frac{X_{u(m)}^r}{X_{u(n)}^s}\right)$$

$$= \frac{\theta}{(r+1) \ \Gamma(m-1)\Gamma(n-m)} \int\int_{0<x<y<1} \frac{x^{r+1}}{y^s} R^{m-2}(x) r(x)$$

$$\times [R(y) - R(x)]^{n-m-1} f(y) dx dy$$

$$- \frac{\theta}{(r+1) \ \Gamma(m)\Gamma(n-m-1)} \int\int_{0<x<y<1} \frac{x^{r+1}}{y^s} R^{m-1}(x) r(x)$$

$$\times [R(y) - R(x)]^{n-m-2} f(y) dx dy$$

$$= \frac{\theta}{(r+1)} \int \int_{0<x<y<1} \frac{x^{r+1}}{y^s} f_{m-1,n+1}(x,y)dxdy$$

$$- \frac{\theta}{(r+1)} \int \int_{0<x<y<1} \frac{x^{r+1}}{y^s} f_{m,n-1}(x,y)dxdy$$

$$= \frac{\theta}{(r+1)} E\left(\frac{X_{u(m-1)}^{r+1}}{X_{u(n+1)}^s}\right) - \frac{\theta}{(r+1)} E\left(\frac{X_{u(m)}^{r+1}}{X_{u(n-1)}^s}\right).$$

Hence

$$E\left(\frac{X_{u(m)}^{r+1}}{X_{u(n)}^s}\right)$$

$$= \frac{\theta}{(r+1)} \left[E\left(\frac{X_{u(m-1)}^{r+1}}{X_{u(n+1)}^s}\right) - E\left(\frac{X_{u(m)}^{r+1}}{X_{u(n-1)}^s}\right) + \frac{(r+1)}{\theta} E\left(\frac{X_{u(m)}^r}{X_{u(n)}^s}\right)\right].$$

This completes the proof. □

COROLLARY 2.4. *For $m \geq 1$ and $r, s = 0, 1, 2, \cdots$,*

$$E\left(\frac{X_{u(m)}^{r+1}}{X_{u(m+1)}^s}\right)$$

$$= \frac{\theta}{(r+1)} \left[E\left(\frac{X_{u(m-1)}^{r+1}}{X_{u(m+2)}^s}\right) - E(X_{u(m)}^{r-s+1}) + \frac{(r+1)}{\theta} E\left(\frac{X_{u(m)}^r}{X_{u(m+1)}^s}\right)\right].$$

Proof. Upon substituting $n = m+1$ in Theorem 2.3 and simplifying, then we have

$$E\left(\frac{X_{u(m)}^{r+1}}{X_{u(m+1)}^s}\right)$$

$$= \frac{\theta}{(r+1)} \left[E\left(\frac{X_{u(m-1)}^{r+1}}{X_{u(m+2)}^s}\right) - E(X_{u(m)}^{r-s+1}) + \frac{(r+1)}{\theta} E\left(\frac{X_{u(m)}^r}{X_{u(m+1)}^s}\right)\right].$$

□

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