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MULTIFRACTAL ANALYSIS OF A CODING SPACE OF THE CANTOR SET

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Abstract. We study Hausdorff and packing dimensions of subsets of a coding space with an ultra metric from a multifractal spectrum induced by a self-similar measure on a Cantor set using a function satisfying a Hölder condition.

1. Introduction

Recently we obtained some results([1, 4]) of relationship between spectral classes of a self-similar Cantor set([1, 3, 4, 8, 9]) using distribution sets([1, 4]) and their set-theoretical relationship of subsets in spectral classes. We also found some relationship([5]) between subsets of a Cantor set and their corresponding subsets of a coding space. Nowadays most of the fractals have been dealt in the Euclidean space for the discoveries of their Hausdorff and packing dimensions([8]) in the Euclidean space are essential to the scientific progress. However it is also fruitful to consider a non-Euclidean metric space for dimensions can be related to a non-Euclidean metric. We consider such an example as a coding space with an ultra metric. Recently we([5]) studied a relationship between subsets in a coding space with an ultra metric and subsets in a Cantor set with the Euclidean metric. Combining the results([1, 4, 5]), we get some information of multifractal analysis of a coding space of a Cantor set. We note that the bridge to connect the two subsets which are in a Cantor set and in a coding space is a natural code function([2]).

In this paper using the relationship([1, 3, 4]) between spectral classes of a self-similar Cantor set and their corresponding subsets in a coding space, we get the Hausdorff dimensions and packing dimensions of multifractal spectral members of a coding space.

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2. Preliminaries

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{R}$ be the set of real numbers. Let I_ϕ =[0,1]. We can obtain the left subinterval $I_{\tau,1}$ and the right subinterval $I_{\tau,2}$ of I_τ deleting a middle third open subinterval of I_τ inductively for each $\tau \in \{1,2\}^n$ where $n = 0, 1, 2, \cdots$. Let $E_n = \cup_{\tau\in\{1,2\}^n} I_\tau$. Then $\{E_n\}$ is a decreasing sequence of closed sets.

The set $F = \bigcap_{n=0}^\infty E_n$ is called the classical ternary Cantor set. In this case, if $x \in F$ is chosen, we easily see that there corresponds a code $\sigma \in \{1,2\}^{\mathbb{N}}$ such that $\bigcap_{k=0}^\infty I_{\sigma|k} = \{x\}$ (Here $\sigma|k = i_1, i_2, \cdots, i_k$ where $\sigma = i_1, i_2, \cdots, i_k, i_{k+1}, \cdots$).

We assume that $\{1,2\}^{\mathbb{N}}$ is an ultra metric space with the ultra metric ρ satisfying $\rho(\sigma,\sigma) = 0$ and if $\sigma \neq \tau$ then $\rho(\sigma,\tau) = (\frac{1}{2})^k$ where $\sigma = i_1 i_2 \cdots i_k i_{k+1} \cdots$ and $\tau = i_1 i_2 \cdots i_k j_{k+1} \cdots$ where $i_{k+1} \neq j_{k+1}$ for some $k = 0, 1, 2 \cdots$. We call $\{1,2\}^{\mathbb{N}}$ a coding space([7]) with an ultra metric for the Cantor set.

In the coding space we can define a probability measure induced by a natural set function defined on the class of its cylinders. Let $P(\tau \times \{1,2\}^{\mathbb{N}}) = \frac{1}{2^n}$ if $\tau \in \{1,2\}^n$ for each $n = 0, 1, 2, \cdots$. Then the set function P easily extends to a Borel probability measure on the coding space.

We define a natural code function $f : F \longrightarrow \{1,2\}^{\mathbb{N}}$ such that $f(x) = \sigma$ with $\{x\} = \bigcap_{k=0}^\infty I_{\sigma|k}$. Note that f is the one-to-one corresponding. If we define $p(I_{f(x)|n}) = P((f(x)|n) \times \{1,2\}^{\mathbb{N}})$ for all $x \in F$, then p is easily extended to a Borel probability measure on F.

For $x \in F$, we can consider a ternary expansion of x from $\sigma = f(x)$, that is if $\sigma = i_1, i_2, \cdots, i_k, i_{k+1}, \cdots$ then the ternary expansion of x is $0.j_1, j_2, \cdots, j_k, j_{k+1}, \cdots$ where $j_k = 0$ if $i_k = 1$ and $j_k = 2$ if $i_k = 2$. We denote $n_0(x|k)$ the number of times the digit 0 occurs in the first k places of the ternary expansion of x([1]).
For $r \in [0,1]$, we define a distribution set $F(r)$ containing the digit 0 in proportion r by

$$F(r) = \{x \in F : \lim_{k\to\infty} \frac{n_0(x|k)}{k} = r\}.$$

From now on, $\dim_H(E)$ denotes the Hausdorff dimension of $E \subset \mathbb{R}$ and $\dim_p(E)$ denotes the packing dimension of E. In this paper, we assume that $0 \log 0 = 0$ for convenience.

3. Main results

PROPOSITION 1. *Let E be a metric space with a metric ρ. Let $f : F \longrightarrow E$ be a function satisfying a Hölder condition*

$$c_1|x-y|^\alpha \le \rho(f(x), f(y)) \le c_2|x-y|^\alpha$$

for some constants $c_1, c_2 > 0$, $\alpha > 0$ and each $x, y \in F$. Then $\dim_H(f(F)) = \frac{1}{\alpha}\dim_H(F)$ and $\dim_p(f(F)) = \frac{1}{\alpha}\dim_p(F)$.

Proof. $\dim_H(f(F)) = \frac{1}{\alpha}\dim_H(F)$ follows from an easy version of Proposition 2.3 in [8] for a metric space instead of Euclidean space. $\dim_p(f(F)) = \frac{1}{\alpha}\dim_p(F)$ follows from [5] or the similar arguments with the proof of Proposition 2.3 in [8]. □

PROPOSITION 2. *([5]) Let $f : F \longrightarrow \{1,2\}^{\mathbb{N}}$ be a function such that $f(x) = \sigma$ with $\{x\} = \bigcap_{k=0}^{\infty} I_{\sigma|k}$ where $\sigma \in \{1,2\}^{\mathbb{N}}$ and F is the classical Cantor ternary set. Then it satisfies a Hölder condition*

$$|x-y|^{\frac{\log 2}{\log 3}} \le \rho(f(x), f(y)) \le 2|x-y|^{\frac{\log 2}{\log 3}}$$

for each $x, y \in F$.

Proof. Let $x, y \in F$ with $x \ne y$. Then $f(x) = i_1 i_2 \cdots i_k i_{k+1} \cdots$ and $f(y) = i_1 i_2 \cdots i_k j_{k+1} \cdots$ where $i_{k+1} \ne j_{k+1}$ for some $k = 0, 1, 2 \cdots$. Since $x, y \in E_k$, we see $|x-y| \le (\frac{1}{3})^k$ and $\rho(f(x), f(y)) = (\frac{1}{2})^k$. Therefore we have $|x-y|^{\frac{\log 2}{\log 3}} \le [(\frac{1}{3})^k]^{\frac{\log 2}{\log 3}} \le \rho(f(x), f(y)) = (\frac{1}{2})^k \le 2[(\frac{1}{3})^{k+1}]^{\frac{\log 2}{\log 3}} \le 2|x-y|^{\frac{\log 2}{\log 3}}$ for each $x, y \in F$. □

COROLLARY 3. *If $G \subset \{1,2\}^{\mathbb{N}}$, then $\dim_H(G) = s/\frac{\log 2}{\log 3}$, where $s = \dim_H(f^{-1}(G))$.*

Proof. We note that f is a bijection. It follows from Propositions 1 and 2. □

COROLLARY 4. *If $G \subset \{1,2\}^{\mathbb{N}}$, then $\dim_p(G) = s/\frac{\log 2}{\log 3}$, where $s = \dim_p(f^{-1}(G))$.*

Proof. It follows from Propositions 1 and 2 and f is a bijection. . □

Proposition 5. *([1, 3, 4]) For a distribution set $F(r)$ where $r \in [0,1]$,*

$$\dim_H(F(r)) = \dim_p(F(r)) = \frac{r\log r + (1-r)\log(1-r)}{-\log 3}.$$

Proof. From [1, 3, 4], we note that $\dim_H(F(r)) = \dim_p(F(r)) = \frac{r\log r+(1-r)\log(1-r)}{r\log a+(1-r)\log b}$ for a self-similar Cantor set with contraction ratios a, b. It is obtained for $a = b = \frac{1}{3}$. □

Corollary 6. *For each $r \in [0,1]$,*

$$\dim_H(f(F(r))) = \dim_p(f(F(r))) = \frac{r\log r + (1-r)\log(1-r)}{-\log 2}.$$

Proof. It follows from Proposition 5 and Corollaries 3 and 4. □

Remark 7. Since $P(G) = p(f^{-1}(G))$([5]), if $P(G) > 0$ where $G \subset \{1,2\}^{\mathbb{N}}$ then $\dim_H(G) = 1$ from Corollary 3. Also we see that if $p(E) > 0$ where $E \subset F$ then $\dim_H(E) = \frac{\log 2}{\log 3}$.

Remark 8. Since $P(G) = p(f^{-1}(G))$, if $P(G) > 0$ where $G \subset \{1,2\}^{\mathbb{N}}$ then $\dim_p(G) = 1$ from Corollary 4 and the fact that if $p(E) > 0$ where $E \subset F$ then $\dim_p(E) = \frac{\log 2}{\log 3}$.

Remark 9. In the above Corollary, we see that $\dim_H(f(F(\frac{1}{2}))) = \dim_p(f(F(\frac{1}{2}))) = 1$. But we note that $p(F(\frac{1}{2})) = 1 > 0$ by the strong law of large numbers, which gives also the information that $\dim_H(f(F(\frac{1}{2}))) = \dim_p(f(F(\frac{1}{2}))) = 1$ from above Remarks. Combining the above facts and the fact that in Corollaries 3 and 4 $f^{-1}(G) \subset F$ and $\dim_H(f^{-1}(G)) \leq \frac{\log 2}{\log 3}$ and $\dim_p(f^{-1}(G)) \leq \frac{\log 2}{\log 3}$, we easily see that $\dim_H(\{1,2\}^{\mathbb{N}}) = \dim_p(\{1,2\}^{\mathbb{N}}) = 1$(cf. [5]).

Remark 10. We clearly see that $P(f(F(r))) = 0$ for all $r(\neq \frac{1}{2}) \in [0,1]$ from Remarks 7 and 8 and Corollary 6. We note that $\{f(F(r)) : r \in [0,1]\}$ forms a multifractal spectral class of a coding space $\{1,2\}^{\mathbb{N}}$ with a non-Euclidean metric giving $\dim_H(f(F(r))) = \dim_p(f(F(r))) = \frac{r\log r+(1-r)\log(1-r)}{-\log 2}$ for its members.

EXAMPLE 11. Let $E = \cup_{r(\neq\frac{1}{2})\in[0,1]} f(F(r))$. We see that $P(E) = 0$ since $P(f(F(\frac{1}{2}))) = 1$ and $P(\{1,2\}^{\mathbb{N}}) = 1$. However we see that $\dim_H(E) = \dim_p(E) = 1$ without the condition that $P(E) > 0$. It follows from that $\dim_H(E) \geq \sup_{r(\neq\frac{1}{2})\in[0,1]} \dim_H(f(F(r)))$ by monotonicity and

$$\sup_{r(\neq\frac{1}{2})\in[0,1]} \dim_H(f(F(r))) = \sup_{r(\neq\frac{1}{2})\in[0,1]} \frac{r\log r + (1-r)\log(1-r)}{-\log 2} = 1$$

by Proposition 5. Similarly it holds for packing case.

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