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A Simulation Study on The Discounted Cost Distribution under Age Replacement Policy

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Abstract. During the last three decades, a few attentions have been paid for investigating the cost distribution for the optimal maintenance problems. In this article, we derive the moment of the discounted cost distribution over an infinite time horizon for the basic age replacement problem. With first two moments of the discounted cost distribution, we approximate the underlying distribution function by three theoretical distributions. Through a Monte Carlo simulation, we conclude that the log-normal distribution is the best fitted one to approximate the discounted cost distribution.

Keywords: preventive maintenance, age replacement, discounted cost distribution, simulation, parametric approximation, χ^2 goodness-of-fit test

1. INTRODUCTION

It is of great importance to determine the preventive maintenance schedule effectively for an unreliable system. For instance, consider a typical production machine for which the replacement of a significant part has to be made. If the failure of the production machine occurs during the operation phase, the larger corrective maintenance cost will be required to stop the production lot. Conversely, if the preventive maintenance is frequently executed, the cumulative preventive maintenance cost will increase. The most practical and important replacement problem is *the age replacement problem*, and it has been studied extensively in the context of reliability theory (Barlow *et al*. 1960, Barlow *et al*. 1965, Cleroux *et al*. 1974, Fox 1966, Osaki *et al*. 1975 and Schaeffer 1971). In fact, if one can observe the age of the production machine directly, it is well known that the age replacement is the best policy (Bergman 1980), *i.e.* if the failure does not occur up to a pre-specified age, replace the system unit by a new spare one at that time, otherwise, replace it at the failure time.

In general, the age replacement problem is treated as a simple algebraic one to determine the optimal age which minimizes the expected cost per unit time in the steady-state or the expected total discounted cost over an infinite time horizon. However, the earlier works on the age replacement problems have never paid their attentions

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to examine the probability distribution properties for the cost with the optimal age replacement schedule. The main reason is that the optimal age replacement policy which minimizes the expected cost necessarily maximizes the cost distribution in the sense of ordinary stochastic ordering. However, once the optimal age replacement schedule is determined, of the most practical interest is the statistical properties of the cost distribution. In other words, if the risk averter prefers an extremely small variance of the maintenance cost as well as the minimum expected cost, the age replacement schedule based on only the expected cost is not always suitable. Kawai *et al*. (1987) derived the variance of the discounted cost distribution and considered the mean-variance age replacement problem.

The present article focuses on the age replacement problem under the total discounted cost criterion, and evaluates its probability distribution function approximately. First, we obtain the first three moments of the discounted cost distribution analytically. The method used here is quite different from that based on the renewal-type equation in Kawai *et al*. (1987). Second, we perform the Monte Carlo simulation and calculate the discounted cost distribution over an infinite time horizon numerically. Also, we approximate the discounted cost distribution by three parametric (theoretical) probability distributions; the normal distribution, the log-normal distribution and the Weibull distribution. Then, the moment matching is used to determine parameters involved in the probability distributions. Finally, the approximated cost distributions are compared with the simulation results. The goodness-of-fit test is carried out to assess the parametric approximation methods. The attempt in this article gives a new insight to evaluate the resulting cost distribution with the optimal maintenance schedule and, at the same time, provides an economical perspective to the spare part management practice.

2. THE AGE REPLACEMENT MODEL

Let us consider a one-unit system operated in continuous time. Let X be the continuous lifetime with probability distribution function $F(x)$, density function $f(x)$, failure rate $h(x)=f(x)/\overline{F}(x)$ and finite mean $1/\lambda$ (>0), where, in general, $\overline{\psi}(\cdot)=1-\psi(\cdot)$. In the typical age replacement model, if the unit does not fail until a pre-specified age t_0 (≥ 0), then it is replaced by a new one preventively, otherwise, it is replaced at the failure time. Denote the corrective and preventive replacement costs by c_1 and c_2, respectively, where, without any loss of generality, $c_1 > c_2 > 0$. Of the most practical interest is the derivation of the optimal age replacement time t_0^* which minimizes any relevant cost criterion.

Following Fox (1966), we consider the case where the cost is discounted by the discount rate $r\,(>0)$. Define the total discounted cost during the time interval [0, t] with the scheduled age t_0 as $N(t \mid t_0)$. Then, the mean value $M(t \mid t_0) = \mathrm{E}[N(t \mid t_0)]$ for arbitrary time $t \geq 0$ is as follows. (i) When $t_0 > t$, $M(t \mid t_0) = c_1\exp(-rt)$. (ii) When $t_0 = t$, $M(t \mid t_0) = c_2\exp(-rt_0)$. (iii) when $t_0 < t$,

$$M(t\mid t_0)=\int_0^{t_0}e^{-rx}\big[c_1+M(t-x\mid t_0)\big]dF(x) + e^{-rt_0}\big[c_2+M(t-t_0\mid t_0)\big]\overline{F}(t_0). \qquad (1)$$

Then the problem is to derive the optimal age t_0^* which minimizes the expected total discounted cost over an infinite time horizon $C(t_0)=\lim_{t\to\infty}M(t\mid t_0)$, *i.e.* $\min_{0\leq t_0\leq\infty}C(t_0)$, where

$$C(t_0)=\frac{c_1\int_0^{t_0}e^{-rt}dF(t)+c_2e^{-rt_0}\overline{F}(t_0)}{r\int_0^{t_0}e^{-rt}\overline{F}(t)dt}. \qquad (2)$$

Define the numerator of the derivative of $C(t_0)$ with respect to t_0, divided by $\overline{F}(t_0)\exp(-rt_0)$ as $q(t_0)$,

$$q(t_0)=h(t_0)\int_0^{t_0}e^{-rt}\overline{F}(t)dt - \int_0^{t_0}e^{-rt}dF(t)-c_2/(c_1-c_2). \qquad (3)$$

Then, we have the optimal age replacement time t_0^* which minimizes the expected total discounted cost over an infinite time horizon $C(t_0)$.

Proposition 2.1. (1) Suppose that the lifetime distribution F(x) is strictly IFR (Increasing Failure Rate). If $h(\infty)>K(r)$, then there exists a unique optimal age replacement time t_0^* ($0<t_0^*<\infty$) which satisfies $q(t_0^*)=0$, where $K(r)=\{c_1F^*(r)+c_2\overline{F}^*(r)\}/\{(c_1-c_2)\overline{F}^*(r)/r\}$ and $F^*(r)=\int_0^\infty\exp(-rt)dF(t)$. The corresponding minimum expected cost is $C(t_0^*)=\{(c_1-c_2)h(t_0^*)\}/r-c_2$. (2) If $h(\infty)\leq K(r)$ under the strictly IFR assumption or if F(t) is DFR (decreasing failure rate), then $t_0^*\to\infty$ and $C(\infty)=c_1F^*(r)/\overline{F}^*(r)$.

For the proof, see Fox (1966) and Osaki and Nakagawa (1975).

Corollary 2.2. *Define the discounted cost distribution in the steady-state,* $Q(y\mid t_0)=\Pr\{\lim_{t\to\infty}N(t\mid t_0)\leq y\}$. *Then the optimal age replacement time* t_0^* *in Proposition 2.1 maximizes* $Q(y\mid t_0)$, *i.e.* $Q(y\mid t_0^*)\geq Q(y\mid t_0)$ *iff* $C(t_0^*)\leq C(t_0)$ *for arbitrary* t_0.

The proof is omitted for brevity. From Corollary 2.2, it is found that the optimal age replacement time which maximizes the discounted cost distribution is reduced to the optimal policy based on the expected cost with discounting. In the following section, we further characterize the discounted cost distribution $Q(y\mid t_0^*)$.

3. PROBABILITY DISTRIBUTION OF THE DISCOUNTED COST

From an intuitive insight, it can be seen that

$$Q(y\,|\,t_0) = 1 - \int_0^{t_0} \overline{Q}(e^{-rx}(y-c_1)\,|\,t_0)dF(x) - \overline{Q}(e^{-rt_0}(y-c_2)\,|\,t_0)\overline{F}(t_0). \quad (4)$$

However, it is very difficult to solve the above integral equation with the boundary conditions $Q(0\,|\,t_0) = 0$ and $Q(\infty\,|\,t_0) = 1$. Therefore, we attempt to derive the n-th moment of the discounted cost distribution $Q(y\,|\,t_0^*)$ in this section.

Letting $A_i(t_0)$ denote the time interval (r.v.) from one replacement to the next preventive or corrective replacement whichever occurs first during i-th (i = 1, 2,…) cycle, the total spent time up to the end of i-th cycle is $S_i(t_0) = \sum_{j=1}^{i} A_j(t_0)$. Similarly, the incurred cost during i-th cycle (r.v.) is $B_i(t_0) = c_1 I_{\{A_i(t_0)\le t_0\}} + c_2 I_{\{A_i(t_0)>t_0\}}$, where, $I_{\{A\}}$ is the indicator function for the event A. Then, the total discounted cost over an infinite time horizon (r.v.) can be represented as

$$\xi(t_0) = \lim_{t\to\infty} N(t\,|\,t_0) = \sum_{i=1}^{\infty} e^{-rS_i(t_0)} B_i(t_0) = e^{-rA_1(t_0)}\{B_1(t_0) + \sum_{i=2}^{\infty} B_i(t_0) e^{-r\{S_i(t_0)-A_1(t_0)\}}\}. \quad (5)$$

Since both $A_i(t_0)$ and $B_i(t_0)$ (i = 1, 2,…) are i.i.d. random variables, we drop the subscript i in the following part. Hence, it is seen that $\xi(t_0) = \exp(-rA(t_0))\{B(t_0) + \xi(t_0)\}$ in the sense of probability distribution.

Proposition 3.1. *Let* $\overline{m}_n(t_0)$ *be the n-th* (n = 1, 2,…) *central moment for the random variable* $\xi(t_0)$, *i.e.* $\overline{m}_n(t_0) = \mathrm{E}\left[(\xi(t_0) - \overline{m}_1(t_0))^n\right]$. *Then, we have*

$$\overline{m}_n(t_0) = \sum_{j=0}^{n} \binom{n}{j} \overline{m}(t_0) \times \mathrm{E}\left[e^{-jrA(t_0)}\left\{C(t_0)e^{-rA(t_0)} + B(t_0)e^{-rA(t_0)} - C(t_0)\right\}^{n-j}\right]. \quad (6)$$

Proof. From the definition, we have

$$\begin{aligned}\overline{m}_n(t_0) &= \mathrm{E}\left[\{\xi(t_0) - C(t_0)\}^n\right] \\ &= \mathrm{E}\left[\{e^{-rA(t_0)}(B(t_0) + \xi(t_0)) - C(t_0)\}^n\right] \\ &= \mathrm{E}\left[\{e^{-rA(t_0)}(\xi(t_0) - C(t_0)) + C(t_0)e^{-rA(t_0)} + B(t_0)e^{-rA(t_0)} - C(t_0)\}^n\right] \\ &= \sum_{j=0}^{n} \binom{n}{j} \overline{m}(t_0) \mathrm{E}\left[e^{-jrA(t_0)}\left\{C(t_0)e^{-rA(t_0)} + B(t_0)e^{-rA(t_0)} - C(t_0)\right\}^{n-j}\right]. \quad (7)\end{aligned}$$

This completes the proof.

Corollary 3.2. *The variance* $\overline{m}_2(t_0)$ *and the third moment around the mean* $\overline{m}_3(t_0)$ *for* $Q(y\,|\,t_0)$ *are, respectively, given by*

$$\overline{m_2}(t_0) = \frac{\{c_1 + C(t_0)\}^2 \int_0^{t_0} e^{-2rt} dF(t)}{2r\int_0^{t_0} e^{-2rt}\overline{F}(t)dt} + \frac{\{c_2 + C(t_0)\}^2 e^{-2rt_0}\overline{F}(t_0) - C(t_0)^2}{2r\int_0^{t_0} e^{-2rt}\overline{F}(t)dt} \quad (8)$$

and

$$\begin{aligned}\overline{m_3}(t_0) &= \frac{\left[\left[\{c_1 + C(t_0)\}^3 + 3\overline{m_2}(t_0)\{c_1 + C(t_0)\}\right]\int_0^{t_0} e^{-3rt}dF(t)\right.}{3r\int_0^{t_0} e^{-3rt}\overline{F}(t)dt} \\ &+ \frac{\left[\{c_1 + C(t_0)\}^3 + 3\overline{m_2}(t_0)\{c_1 + C(t_0)\}\right]e^{-3rt_0}\overline{F}(t_0)}{3r\int_0^{t_0} e^{-3rt}\overline{F}(t)dt} \\ &- \frac{\left.C(t_0)^3 - C(t_0)\overline{m_2}(t_0)\right]}{3r\int_0^{t_0} e^{-3rt}\overline{F}(t)dt}. \quad (9)\end{aligned}$$

The second moment $\overline{m}_2(t_0)$ in Corollary 3.2 was derived first by Kawai *et al*. (1987), and is called *the stationary variance* for the discounted cost. However, notice that their approach is based on the direct approach to solve an integral equation and is quite different from ours. On the other hand, our result is due to a general result on random Dirichlet series by Stadje (1997). In fact, Stadje (1997) proved the asymptotic normality of the central moment as $r \to 0$ for more general cases.

4. APPROXIMATING THE DISCOUNTED COST DISTRIBUTION

4.1 Simulation

In order to characterize the probability distribution properties for the discounted cost over an infinite time horizon, we carry out the Monte Carlo simulation. Suppose that the lifetime obeys the following Weibull distribution;

$$F(t) = 1 - \exp\{-(\frac{t}{\theta})^{\beta}\}, \quad (10)$$

where θ(> 0) and β (> 0) are the scale and shape parameters, respectively. We summarize the simulation procedure in the following:

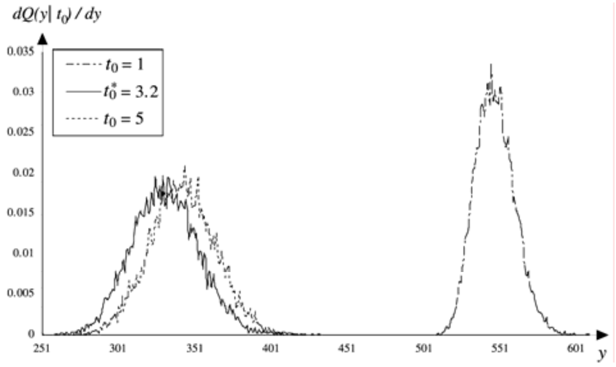

Figure 1. Discounted cost distribution with different age (c_1 = 10, c_2 = 5, θ= 3.2, β= 2.2, r = 0.01).

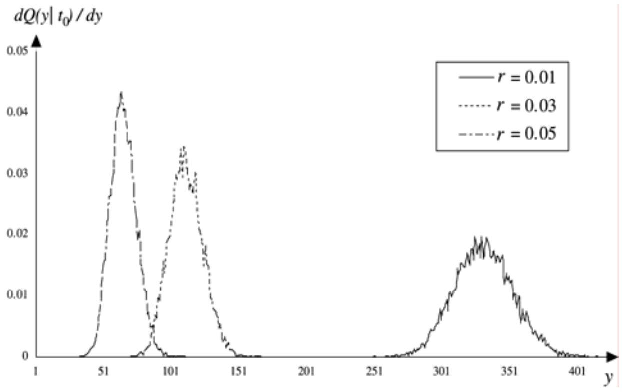

Figure 2. Discounted cost distribution with different discount rate (c_1 = 10, c_2 = 5, θ= 3.2, β= 2.2, $t_0 = t_0^*$).

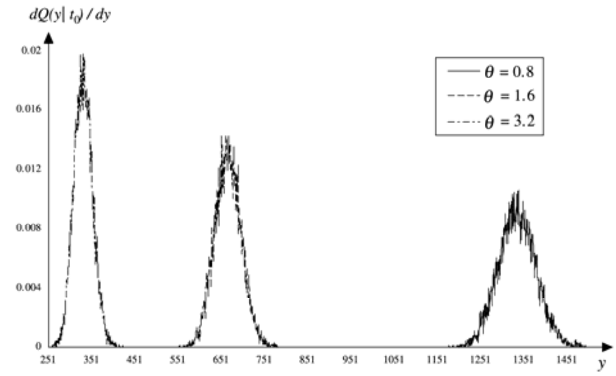

Figure 3. Discounted cost distribution with different scale parameter of the lifetime (c_1 = 10, c_2 = 5, β= 2.2, r = 0.01, $t_0 = t_0^*$).

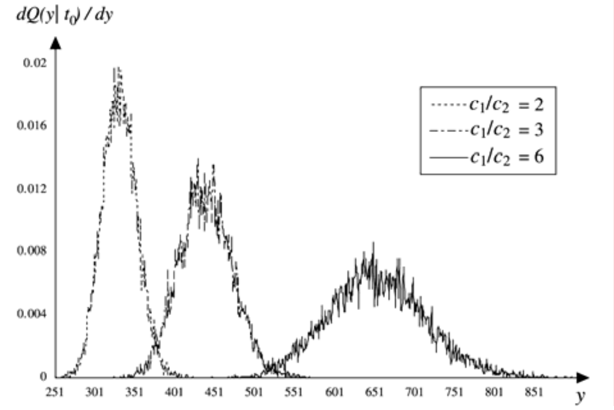

Figure 4. Discounted cost distribution with different discount ratio c_1 / c_2 (θ = 3.2, β = 2.2, r = 0.01, $t_0 = t_0^*$).

Simulation Procedure:

(Step 1) Generate the pusedo random number following the probability distribution function $F(t)$ by the usual mixed congruential method.
(Step 2) If the realization of the lifetime is less than the fixed age t_0, we regard it as the corrective failure time, and record the discounted corrective maintenance cost, otherwise, record the discounted preventive maintenance cost at the age t_0.
(Step 3) After similar procedure is repeated 10,000 times, we sum all discounted costs occurred at each cycle, and obtain a realization of the discounted cost over an infinite time horizon.
(Step 4) From 10,000 execution results by the simulation, *i.e.*, from 10,000 realizations of the discounted cost, we approximate the cost distribution for the fixed age.

Figure 1 illustrates the discounted cost distribution with different age, where c_1 = 10, c_2 = 5, θ= 3.2, β= 2.2 and r = 0.01. Roughly speaking, it is observed that the resulting discounted cost distribution is symmetric. As the fixed age is far from the optimal age t_0^* = 3.2, the mode and the mean for the resulting distribution function are shifted to the right, and the tail of the distribution is shorter. This implies that the total discounted cost distribution tends to become smaller as the age replacement time is far from the optimal age t_0^*. In Fig. 2, we examine the discounted cost distribution with different discount rate, where c_1 = 10, c_2 = 5, q = 3.2, β= 2.2 and $t_0 = t_0^*$. From this result, if the discount rate r becomes larger, then the mean and the mode are shifted to the left, but the tail of the distribution becomes shorter. This point is quite different from Figure 1.

Next, we investigate the dependence of the lifetime distribution to the discounted cost distribution function. Figure 3 shows the discounted cost distribution with different scale parameter of the lifetime, where c_1 = 10, c_2 = 5, β= 2.2, r = 0.01 and $t_0 = t_0^*$. As the scale parameter θ for the lifetime distribution increases monotonically , *i.e.* as the corresponding failure rate is small, both the mode and mean of the discounted cost distribution are shifted to the left and its tail becomes shorter. Also, Fig. 4 depicts the discounted cost distribution with different cost ratio c_1/c_2, where θ= 3.2, β = 2.2, r = 0.01 and $t_0 = t_0^*$. When the cost ratio c_1/c_2 increases, the distribution shows the reverse tendency to Fig. 3.

In this way, the sensitivity analyses for the other model parameters tell us some useful probability distribution properties. However, the Monte Carlo simulation is often troublesome to perform, especially for a sequential situation in which the lifetime distribution has to be estimated from the failure data. In addition, if one wish to know the tail probability of the discounted cost, the simulation approach may be unreliable to evaluate it. In other words, the practical interest is to approximate the discounted cost distribution by any theoretical probability distribution function, and is to realize an economically stable manufacturing system.

4 . 2 Goodness-of-Fit Test by Some Theoretical Distributions

Here, we approximate the discounted cost distribution by three theoretical distribution functions with first two moments around the mean, and compare them with the Monte Carlo simulation results. Suppose that the discounted cost distribution with Weibull lifetime can be approximated by one of three theoretical probability distributions;

(i) Normal distribution:

$$Q(y\,|\,t_0) \sim N(C(t_0), \overline{m}_2(t_0)), \tag{11}$$

(ii) Log-normal distribution:

$$Q(y\,|\,t_0) \sim LN(\mu(t_0), \sigma^2(t_0)), \tag{12}$$

where $\mu(t_0)$ and $\sigma(t_0)$ are the unique solutions for the following simultaneous equations;

$$C(t_0) = \exp\left(\mu(t_0) + \frac{\sigma^2(t_0)}{2}\right), \tag{13}$$

$$\overline{m}_2(t_0) = \{\exp(2\mu(t_0) + \sigma^2(t_0))\} \times \{\exp(\sigma^2(t_0)) - 1\}. \tag{14}$$

(iii) Weibull distribution:

$$Q(y\,|\,t_0) \sim W(\alpha(t_0), \gamma(t_0)), \tag{15}$$

where $\alpha(t_0)$ and $\gamma(t_0)$ are the unique solutions for the simultaneous equations;

$$C(t_0) = \gamma(t_0)\Gamma\left(1 + \frac{1}{\alpha(t_0)}\right), \tag{16}$$

$$V(t_0) = \gamma^2(t_0)\left[\Gamma\left(1 + \frac{2}{\alpha(t_0)}\right) - \left\{\Gamma\left(1 + \frac{1}{\alpha(t_0)}\right)\right\}^2\right] \tag{17}$$

and $\Gamma(\cdot)$ is the standard gamma function.

In Fig. 5, we compare the simulation results with three theoretical distributions. It is found from this result that both the normal and the log-normal distributions are better fitted than the Weibull distribution.

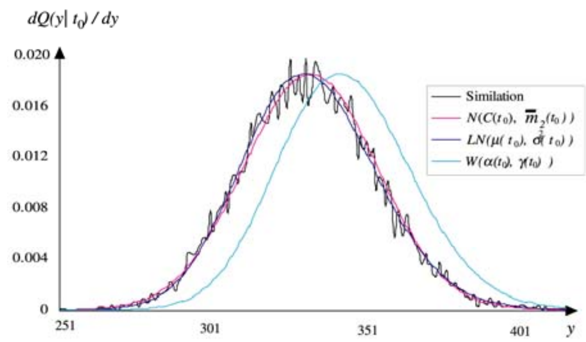

Figure 5. Comparison of parametric approximations (c_1 = 10, c_2 = 5, θ = 3.2, β = 2.2, r = 0.01, t_0 = 3.2).

We make 25 data sets with different combinations of model parameters c_1, c_2, θ, β, r and t_0^*, where each data set involves 10,000 simulation runs. For each data set, we execute the χ^2 goodness-of-fit tests with two kinds of significant level (5% and 0.5%) between the simulation result and three theoretical distributions. Table 1 presents the number of data sets accepted at each significant level. From this result, we conclude that the log-normal distribution is the best fitted one to approximate the discounted cost distribution.

Table 1. χ^2 goodness-of-fit tests.

significant level	N	LN	W
more than 5%	6	14	0
more than 0.5%	12	19	0
less than 0.5%	13	6	25

5. CONCLUSION

In this article, we have derived the moment around the origin of the discounted cost distribution over an infinite time horizon for the basic age replacement problem. With first two moments of the discounted cost distribution, we have approximated the underlying probability distribution function by some theoretical distributions. Through a Monte Carlo simulation, we have concluded that the log-normal distribution is the best fitted one to approximate the discounted cost distribution. This result will be useful to investigate the statistical properties of the maintenance cost and to design the economic maintenance plan for real manufacturing systems.

Recently, Chen and Jin (2003) consider a cost-variability-sensitive age replacement policy based on the variamce of discounted cost, although their result involves serious mistakes. In the forthcoming paper, we will correct the Chen and Jin's result based on some analytical results derived in this paper.

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