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Fuzzy Linearity of the Seminormed Fuzzy Integrals of Interval-valued Functions

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요 약

일반적으로 Lebesgue 적분에서 성립하지만 퍼지적분에서 성립되지 않는 성질이 몇 가지 있다. 그 중 하나가 선형성이다. 본 논문에서는 선형성 표현식에서 덧셈을 supremum 으로 곱셈을 infimum으로 대신한 퍼지선형성의 정의를 소개하고 구간값을 갖는 함수의 준노름 퍼지적분이 퍼지가법성을 갖는 퍼지 측도와 연속인 준 노름이 saturated 조건을 만족할 때, [Max] 조건을 만족하는 가측함수에 대해 퍼지선형성이 성립함을 보였다.

Abstract

In general, the fuzzy integral lacks some important properties that Lebesgue integral possesses. One of them is linearity. In this paper, we introduce fuzzy linearity in which we use the supremum and the infimum instead of additon and scalar multiplication in the expression of linearity and show that the fuzzy linearity of the seminormed fuzzy integrals of interval-valued functions when the fuzzy measure g is fuzzy additive, the continuous t-seminorm is saturated and measurable functions satisfy the condition[**Max**].

Key words : Interval number, fuzzy linearity, condition[**Max**], saturated

1. Introduction

Since Sugeno[9] define a fuzzy measure as a monotone set function without additivity and a fuzzy integral with respect to fuzzy measure, and it was made much deeper by the authors[6,7,11], but their integrands are all functions (point-valued).

On the other hand, it is well known that set-valued functions have been used repeatedly in Economics[3]. Integral of set-valued functions had been studied by Aumann[1], Debreu[2], and others. But all the integrals are based on Lebesgue integrals. Zhang[12,13] has defined fuzzy integrals of set-valued functions by using fuzzy integrals. This fuzzy integral does not hold linearity as consequence of the non-additivity of the fuzzy measure,

Klement and Ralescu[4] showed that the fuzzy integral has some linearity properties only for small classes of fuzzy measures. In [5], we introduced fuzzy linearity in which we use the supremum and the infimum instead of addition and scalar multiplication in the expression of linearity. In this paper, we extend to the fuzzy integral of interval- valued function.

2. Preliminaries

We recall some notion which will be used in this paper and investigate elementary properties.

Let $P([0,1])$ be the power set of $[0,1]$, X be a nonempty set, $\mathcal{A}$ is a σ-algebra formed by the subsets of X, $(X, \mathcal{A})$ is a measurable space, and a set function $g: \mathcal{A} \to [0,1]$ is a fuzzy measure defined by Sugeno. From now on, we consider only the set

$$L^0(X) = \{ f: X \to [0,1] \mid f \text{ is measurable } \}$$

For $f \in L^0(X)$ the fuzzy integral of on A with respect to a fuzzy measure g defined by

$$\int_A h\ dg = \sup_{\alpha \in [0,1]} [\alpha \wedge g(A \cap H_\alpha)].$$

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When $A = X$, the fuzzy integral is denoted by $\int h\, dg$.

Definition 2.1. [7,8] *A t-seminorm* is a function $\top : [0, 1]\times[0, 1]$ which satisfies :

(1) $\top(x, 1) = \top(1, x) = x$ for each $x \in [0, 1]$,

(2) if $x_1 \le x_3,\ x_2 \le x_4$ for each $x_1, x_2, x_3, x_4 \in [0, 1]$,

then $\top(x_1, x_2) \le \top(x_3, x_4)$.

Example 2.2. The following functions are t-seminorm

(1) $\top(x, y) = x \wedge y$

(2) $\top(x, y) = xy$

(3) $\top(x, y) = 0 \vee (x + y - 1)$

Let $\top$ be a t-seminorm. For all $h \in L^0(X)$, the seminormed fuzzy integral of h over $A \in \mathcal{A}$ to fuzzy measure g is defined as

$$\int_A h \top g = \sup_{\alpha \in [0,1]} \top(\alpha,\ g(A \cap H_\alpha))$$

In what follows, $\int_X h \top g$ will be denote by $\int h \top g$ for short.

Clearly, the seminormed fuzzy integral is the fuzzy integral for the case $\top(x, y) = x \wedge y$.

Let

$$I([0,1]) = \{\bar{r} : \bar{r} = [r^-, r^+] \subset [0,1]\}$$

Then the elements in the set $I([0, 1])$ are called interval numbers.

Operations $* \in \{+, \cdot, \vee, \wedge\}$.

On the interval numbers set, we make following definitions :

$$\bar{r} * \bar{p} = [r^- * p^-,\ r^+ * p^+],$$

$$k \cdot \bar{r} = [k \cdot r^-,\ k \cdot r^+],$$

$$\bar{r} \le \bar{p} \quad \text{iff} \quad r^- \le p^-,\ r^+ \le p^+.$$

Let $\mathcal{B}$ is the σ-algebra of Borel subsets of [0, 1]. A set-valued function is mapping

$$F : X \to P([0, 1]) \backslash \{\varnothing\}.$$

F is said to be closed valued iff $F(x)$ is closed for every $x \in X$, and F is said to be measurable iff

$$F^W(B) = \{x \in X \mid F(x) \cap B \ne \varnothing\} \in \mathcal{A}$$

for every $B \in \mathcal{B}$

An interval-valued function is a set-valued function $\bar{f} : X \to I([0, 1])$. It is usually written as

$$\bar{f}(x) = [f^-(x), f^+(x)],$$

where

$$f^-(x) = \inf \bar{f}(x),\ f^+(x) = \sup \bar{f}(x).$$

The following lemma shows that the measurability of interval-valued functions is closed under any operations.

Lemma 2.3. [12] Let $\bar{f}(x) = [f^-(x), f^+(x)]$. Then $\bar{f}$ is measurable iff f^- and f^+ are measurable.

From the above Lemma 2.3, we can consider the set

$$\bar{L}^0(X) = \{\bar{f} : X \to I([0, 1]) \mid \bar{f} \text{ is measurable }\}$$

By D. Zhang and Z. Wang [12] define the fuzzy integral of set-valued functions as following :

Let $H : X \to P([0, 1]) \backslash \{\varnothing\}$ be a closed set-valued function. Then the set

$\left\{ \int_A f\, dg : f \text{ is a measurable selection of } H \right\}$ is called the fuzzy integral of H on A, where $A \in \mathcal{A}$. H is said to be integrable on A, if $\int_A H\, dg \ne \varnothing$.

It is clear that H is integrable on every $A \in$ A if H is a closed-valued measurable set-valued function. By this property, it is easy to see that a measurable interval-valued function $\bar{f}$ is integrable.

Proposition 2.4. [12] Let $\bar{f}$ be a measurable interval-valued function. Then

$$\int \bar{f}\, dg = \left[\int f^-\, dg,\ \int f^+\, dg \right].$$

The following theorem shows that extended results corresponding to Sugeno's fuzzy integral are obtained.

Theorem 2.5. [12] Fuzzy integrals of measurable interval-valued function have the following properties :

(1) $\bar{f}_1 \le \bar{f}_2$ implies $\int_A \bar{f}_1\, dg \le \int_A \bar{f}_2\, dg$.

(2) If $A, B \in \mathcal{A}$, then $A \subset B$ implies $\int_A \bar{f}\, dg \le \int_B \bar{f}\, dg$

(3) If $A \in \mathcal{A}, \bar{r} \in I([0,1])$, then $\int_A \bar{r}\, dg = \bar{r} \wedge g(A)$.

(4) If $\bar{r} \in I([0, 1])$, then

$$\int_A (\bar{f} + \bar{r})\, dg \le \int_A \bar{f}\, dg + \int_A \bar{r}\, dg.$$

3. Nonlinearity of the Fuzzy Integral of Interval-valued Function

In comparison with the classical measure, the fuzzy measure abandons the additivity, but reserves the monotonicity, the continuity (or partial continuity), and vanishing on the empty set. So the fuzzy integral with respect to fuzzy measure lack some important properties that Lebesgue integral possesses. For instance Lebesgue integral has linearity but the fuzzy integral does not. We can see this in the following example.

Example 3.1. Let $(X = I([0, 1]), \mathcal{A}, g)$ be the Lebesgue measure space.

(1) If we take $\bar{h}(x) = \left[\frac{1}{2}x,\ x\right]$ for any $x \in [0, 1]$,

and $a = \frac{1}{2}$, then we have

$$\begin{aligned}\int a\ \overline{h}\ dg &= \int \frac{1}{2}\cdot\left[\frac{1}{2}x, x\right]dg \\ &= \int\left[\frac{1}{4}x, \frac{1}{2}x\right]dg \\ &= \left[\frac{1}{5}, \frac{1}{3}\right]\end{aligned}$$

and

$$\begin{aligned}a\int \overline{h}(x)\ dg &= \frac{1}{2}\int\left[\frac{1}{2}x, x\right]dg \\ &= \frac{1}{2}\cdot\left[\frac{1}{3}, \frac{1}{2}\right] \\ &= \left[\frac{1}{6}, \frac{1}{4}\right]\end{aligned}.$$

Consequently, we have

$$\int \frac{1}{2}\ \overline{h}\ dg \neq \frac{1}{2}\int \overline{h}\ dg.$$

(2) If we take $\overline{h}_1(x) = \left[\frac{1}{4}x, \frac{1}{3}x\right]$

$\overline{h}_2(x) = \left[\frac{1}{4}x, \frac{1}{2}x\right]$, then we have

$$\begin{aligned}\int \overline{h}_1(x)\,dg &= \left[\int \frac{1}{4}x\,dg, \int \frac{1}{3}x\,dg\right], \\ &= \left[\frac{1}{5}, \frac{1}{4}\right],\end{aligned}$$

and

$$\begin{aligned}\int \overline{h}_2(x)\,dg &= \left[\int \frac{1}{4}x\,dg, \int \frac{1}{2}x\,dg\right] \\ &= \left[\frac{1}{5}, \frac{1}{3}\right],\end{aligned}$$

but

$$\begin{aligned}\int(\overline{h}_1(x)+\overline{h}_2(x))\ dg &= \left[\int \frac{1}{2}x\,dg, \int \frac{5}{6}x\,dg\right]. \\ &= \left[\frac{1}{3}, \frac{5}{11}\right]\end{aligned}$$

Hence

$$\int \overline{h}_1(x)\,dg + \int \overline{h}_2(x)\,dg \neq \int(\overline{h}_1(x) + \overline{h}_2(x))\,dg$$

(3) If we take $\overline{h} = [0, x]$ for all $x \in [0, 1]$ and $\overline{r} = \left[\frac{1}{5}, \frac{1}{2}\right]$, then we have

$$\begin{aligned}\int(\overline{r}\cdot\overline{f})\ dg &= \int\left(\left[\frac{1}{5}, \frac{1}{2}\right]\cdot[0, x]\right)dg \\ &= \int\left[0, \frac{x}{2}\right]dg \\ &= \left[\int 0\,dg, \int\frac{x}{2}\ dg\right] \\ &= \left[0, \frac{1}{3}\right]\end{aligned}$$

But

$$\begin{aligned}\overline{r}\cdot\int\overline{f}dg &= \left[\frac{1}{5}, \frac{1}{2}\right]\cdot\left[\int 0\,dg, \int x\,dg\right] \\ &= \left[\frac{1}{5}, \frac{1}{2}\right]\cdot\left[0, \frac{1}{2}\right] \\ &= \left[0, \frac{1}{4}\right].\end{aligned}$$

Therefore we get

$$\int(\overline{r}\cdot\overline{f})\ dg \neq \overline{r}\cdot\int\overline{f}dg$$

In [4], Klement and Ralescue showed that the fuzzy integral has some linearity properties only for small classes of fuzzy measures.

Proposition 3.2. [4] Let (X, A, g) be a fuzzy measure space. Then the following statements are equivalent :

(1) For any $h_1, h_2 \in L^0(X)$, $a, b \in R_+$:

$$a\cdot h_1 + b\cdot h_2 \in L^0(X) \Rightarrow \int_A(a\cdot h_1 + b\cdot h_2)\,dg = a\cdot\int_A h_1\,dg + b\cdot\int_A h_2\,dg$$

(2) g is a probability measure fulfilling $g(A) \in \{0, 1\}$ for all $A \in \mathcal{A}$.

We say that a function $F : \overline{L}^0(X) \to I([0, 1])$ is fuzzy linear of interval-valued function if

$$F[(\overline{r}\wedge\overline{h}_1) \vee (\overline{s}\wedge\overline{h}_2)] = [\overline{r}\wedge F(\overline{h}_1)] \vee [\overline{s}\wedge F(\overline{h}_2)]$$

where $\overline{r}, \overline{s} \in I([0, 1])$.

In [5], we showed the following result for the case real-valued function

Let (X, $\mathcal{A}$, g) be a fuzzy measure space. and let $h_1, h_2 \in L^0(X)$. Then the fuzzy linearity for the fuzzy integral

$$\int_A((a\wedge h_1)\vee(b\wedge h_2))\,dg = \left(a\wedge\int_A h_1\,dg\right)\vee\left(b\wedge\int_A h_2\,dg\right)$$

for any $A \in \mathcal{A}$ and every nonnegative constants a and b holds if g is a fuzzy additive measure.

We can extend the above property to the fuzzy integral of interval-valued functions under the same condition.

Theorem 3.3. Let (X, $\mathcal{A}$, g) be a fuzzy measure space. and let $\overline{h}_1, \overline{h}_2 \in \overline{L}^0(X)$. Then the fuzzy linearity for the fuzzy integral

$$\int_A((\overline{r}\wedge\overline{h}_1)\vee(\overline{s}\wedge\overline{h}_2))\,dg = \left(\overline{r}\wedge\int_A\overline{h}_1\,dg\right)\vee\left(\overline{s}\wedge\int_A\overline{h}_2\,dg\right)$$

for any $A \in \mathcal{A}$ and $\overline{r}, \overline{s} \in I([0, 1])$ holds if g is a fuzzy additive measure.

Proof. We amy assume that $A = X$ without loss of generality. Using the Proposition 2.4 and a fuzzy additivity measure of g, we have

$$\begin{aligned}&\int_A(\overline{f}\vee\overline{h})\,dg \\ &= \left[\int_A(f^-\vee h^-)\,dg, \int_A(f^+\vee h^+)\,dg\right]\end{aligned}$$

$$= \left[\int_A f^- dg \vee \int_A h^- dg \;,\; \int_A f^+ dg \vee \int_A h^+ dg \right]$$
$$= \left[\int_A f^- dg \;,\; \int_A f^+ dg \right] \vee \left[\int_A h^- dg \;,\; \int_A h^+ dg \right]$$
$$= \int_A \bar{f} dg \vee \int_A \bar{h}\, dg.$$

The remaining property $\int_A (\bar{a} \wedge \bar{f})\, dg = \bar{a} \wedge \int_A \bar{f} dg$ which does not need fuzzy additivity of g can be obtained in the same way as in [5].

4. Fuzzy Linearity of the Seminormed Fuzzy Integral of the Interval-valued Function.

In [5], to prove the equality

$$a \wedge \int_A h \top g = \int_A (a \wedge h) \top g$$

we introduced two concepts of saturated t-seminorm and the condition [Max].

There is an important subset of $I \times I$. The closure of the subset of $I \times I$ consisting of the point (x, y) with $\top(x, y) = x$ is called the maximal x-region of the t-seminorm $\top$. It is easy to see that every maximal x-region contains at least two line segments

$$\{(x, 1) : 0 \le x \le 1\} \text{ and } \{(0, y) : 0 \le y \le 1\}.$$

In Example 2.2, (1) has the maximal x-region the triangle $y \ge x$, and (2), (3) have the smallest possible maximal x-region

$$\{(x, 1) : 0 \le x \le 1\} \cup \{(0, y) : 0 \le y \le 1\}.$$

For a t-seminorm $\top$, define a map $\mathrm{t} : [0, 1] \to [0, 1]$ by

$$\mathrm{t}\,(x) = \inf\{y : \top(x, y) = x\}.$$

In general, since $0 \le \top(0, y) \le \top(0, 1) = 0$ for all $y \in [0, 1]$, $\mathrm{t}\,(0) = 0$; Since $\top(1, y) = y$ for all $y \in [0, 1]$, $\mathrm{t}\,(1) = 1$. For $\top(x, y) = x \wedge y$, $\mathrm{t}\,(x) = x$ for all $x \in [0, 1]$.

We say that a measurable function h satisfies the condition[Max] with respect to a t-seminorm $\top$ and a fuzzy measure g if the following condition holds:

$\mathrm{t}\,(\alpha_0) \le g(H_{\alpha_0} \cap A)$ whenever

$$\int_A h \top g = \alpha_0;$$

in other words,

$$\top(\alpha_0, \beta) = \alpha_0 \text{ for every } \beta > g(H_{\alpha_0} \cap A)$$

whenever $\int_A h \top g = \alpha_0$.

The concept of the condition[Max] is necessary for the fuzzy linearity. Now, we can consider the other definition.

A t-seminorm is said to be saturated if for all $x' \le x$, $x, x' \in [0, 1]$

$$\top(x', y) = x' \text{ whenever } \top(x, y) = x.$$

Many t-seminorm satisfy this condition. In fact, all the t-seminorms in Example 2.2 are saturated. The following example shows that a continuous t- seminorm $\top$ may not be saturated.

Example 4.1. Let A, B and C be the regions of the square $I \times I$ given as follows

(a) A is the triangle with vertices $(1, 0), (0, 1)$ and $(1, 1)$.

(b) B is the triangle with vertices $(1, 0)$, $(0, 1)$ and $\left(\frac{1}{4}, \frac{1}{4}\right)$.

(c) C is the tetrahedron with vertices $(1, 0)$, $(0, 0)$, $(0, 1)$, $\left(\frac{1}{4}, \frac{1}{4}\right)$.

For $(a, b) \in A$, $\top(a, b) = a \wedge b$. In B, the line segments joining the points $(a, 1-a)$ and $\left(\frac{2a+1}{4}, \frac{2a+1}{4}\right)$, for $0 \le a \le \frac{1}{2}$, with the $\top$-value a ; and extend it by symmetry. Finally, $\top$ is defined to be 0 in the region C. It is not hard to see that $\top$ is a continuous t-seminorm. But it is not saturated because $\top\left(\frac{1}{4}, \frac{1}{2}\right) \ne \frac{1}{4}$ even if $\top\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$ and $\frac{1}{4} < \frac{1}{2}$.

Theorem 4.2. Let $(X, \mathcal{A}, g)$ be a fuzzy measure space. Suppose the t-seminorm $\top$ is continuous and saturated. If $h \in L^0(X)$ satisfies the condition[Max], then

$$\int_A (a \wedge h) \top g = a \wedge \int_A h \top g$$

for every $a \in [0, 1]$ and $A \in \mathcal{A}$.

In [5], We shows that the fuzzy linearity for the seminormed fuzzy integral holds from Theorem 4.2 and fuzzy additivity of fuzzy measure g.

Now, we consider the extended concept of such linearity.

Theorem 4.3. Let $(X, \mathcal{A}, g)$ be a fuzzy measure space and let $\bar{h}_1, \bar{h}_2 \in \bar{L}^0(X)$, $A \in \mathcal{A}$, $\bar{r}, \bar{s} \in I([0, 1])$. Suppose the t-seminorm $\top$ is continuous and saturated. Then the fuzzy linearity for the seminormed fuzzy integral of the interval-valued function

$$\int_A ((\bar{r} \wedge \bar{h}_1) \vee (\bar{s} \wedge \bar{h}_2)) \top g = \left(\bar{r} \wedge \int_A \bar{h}_1 \top g\right) \vee \left(\bar{s} \wedge \int_A h_2 \top g\right)$$

holds if both $\bar{h}_1$ and $\bar{h}_2$ satisfy the condition[Max] and g is a fuzzy additive measure.

Proof. Without any loss of generality, we can assume that $A = X$. Since g is a fuzzy additive measure, it is clear that

$$\int(\overline{h}_1 \vee \overline{h}_2) \top g = \int \overline{h}_1 \top g \vee \int \overline{h}_2 \top g.$$

So, we only need to prove that

$$\int(\overline{r} \wedge \overline{h}) \top g = \overline{r} \wedge \int \overline{h} \top g.$$

Let $\overline{h}_1$ and $\overline{h}_2$ satisfy the condition[Max] and the t-seminorm $\top$ is continuous and saturated.

Using the Theorem 4.2, we have

$$\begin{aligned}
\int(\overline{r} \wedge \overline{h}) \top g &= \int([r^-, r^+] \wedge [h^-, h^+]) \top g \\
&= \int([r^- \wedge h^-, r^+ \wedge h^+]) \top g \\
&= \left[\int(r^- \wedge h^-) \top g, \int(r^+ \wedge h^+) \top g\right] \\
&= \left[r^- \wedge \int h^- \top g, r^+ \wedge \int h^+ \top g\right] \\
&= [r^-, r^+] \wedge \left[\int h^- \top g, \int h^+ \top g\right] \\
&= \overline{r} \wedge \int \overline{h} \top g
\end{aligned}$$

References

[1] J. Aumann, "Integrals of set-valued functions," J. Math. Anal. Appl. Vol. 12, pp.1-12. 1965

[2] G. Debreu, "Integration of correspondence," proc. 5th Berkeley Symp. On Math. Stat. and Prob., Vol.II pp.351-372, 1966

[3] E. Klein and A. Thompson, Theory of correspn-dence, Wiley-interscience, New York, 1984.

[4] E. P. Klement and D. Ralescu, " Nonlinearity of the Fuzzy Integral," Fuzzy Sets and Systems, Vol. 11, pp. 309-315, 1983.

[5] S. M. Im, M. H. Kim and S. S. Shin, "Fuzzy Linearity of the Seminormed Fuzzy Integral," Proc. IC-AI'02, Vol. II Las Vegas, pp.555--562. 2002

[6] D. Ralescu and G. Adams, "The Fuzzy Integral," J. Math. Anal. Appl., Vol. 75, pp. 562-570, 1980.

[7] F. Suarez Garcia and P. Gil. Alvarez, "Two Families of Fuzzy Integrals," Fuzzy Sets and Systems, Vol. 18, pp. 67-81, 1986.

[8] F. Suarez Garcia and P.Gil Alvarez, "Measures of Fuzziness of Fuzzy Events," Fuzzy Sets and System, Vol. 21, pp. 147-157, 1987.

[9] M. Sugeno, "Theory of Fuzzy Integrals and Its Applications," Ph. D Dissertation Thesis, Tokyo Institute of Technology, 1974.

[10] Z. Wang, G. J. Klir, "Fuzzy Measure Theory," Plenum Press, New York, 1992.

[11] L. Xuecheng, "Further Discussion on Convergence Theorems for Seminormed Fuzzy Integrals and Semiconormed Fuzzy Integrals," Fuzzy Sets and System, Vol. 55, pp. 219-226, 1993.

[12] D. Zhang and Z. Wang, "Fuzzy integrals of fuzzy-valued functions," Fuzzy Sets and Systems, Vol. 54, pp. 63-67, 1993.

[13] D. Zhang and Z. Wang, " On set-valuwd fuzzy integrals," Fuzzy Sets and Systems, Vol. 56, pp. 237-241, 1993.

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