A Procedural Theory of Concepts and the Problem of Synthetic a priori

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Abstract: The Kantian idea that some judgments are synthetic even in the area of a priori judgments cannot be accepted in its original version, but a modification of the notions 'analytic' and 'synthetic' discovers a rational core of that idea. The new definition of 'analytic' concerns concepts and makes it possible to distinguish between analytic concepts, which are effective ways of computing recursive functions, and synthetic concepts, which either define non-recursive functions, or define recursive functions in an ineffective way. To justify this claim we have to construe concepts as abstract procedures not reducible to set-theoretical entities.
Key words: Concept, a priori, analytic, synthetic, construction, transparent intensional logic, intuitionism.

Introduction

Whereas the common conviction of contemporary logicians has it that the extension of the concept a priori is the same as the extension of the concept analytic (Martin-Löf in his [1992] being an exception) Kant believed that some a priori judgments are synthetic. One of his famous formulations that should define analyticity vs. syntheticity is:

Entweder das Prädikat B gehört zum Subjekt A als etwas, was in diesem Begriffe A (versteckterweise) enthalten ist; oder B liegt ganz ausser dem Begriff A, ob es zwar mit demselben in Verknüpfung steht. Im ersten Fall nenne ich das Urteil analytisch, im andern synthetisch.

(Critique of Pure Reason, A: 6-7: "Either the predicate B belongs to the subject A as something that is (covertly) contained in this concept A; or B lies entirely outside the concept A, though to be sure it stands in connection with it. In the first case, I call the judgement analytic, in the second synthetic.")

On the other hand, he demonstrated his idea by arguing that the sentence

$$7 + 5 = 12$$

being, of course, *a priori*, is at the same time synthetic according to his definition.

We want to show that

- Kant's definition is untenable, as well as his argument;
- there is a rational core in his attempt to distinguish analytic from synthetic even in the area of *a priori* concepts;
- then an essential modification of the definition of *analytic* and *synthetic* is necessary;
- such a modification should concern *concepts* in general rather than judgments;
- without a *procedural* theory of concepts we can hardly imagine what to do;
- concepts as abstract procedures can be confronted with the notion of *effective procedure*(or: *algorithm*);
- analytic concepts *a priori* are effective procedures that compute some (general/partial) recursive function;
- the other concepts, which either define a recursive function in a non-effective way, or define a non-recursive function, can be said to be *synthetic*;
- there are more synthetic concepts a priori than analytic concepts *a priori*.
- the analytic vs. synthetic distinction is (in the area of mathematical concepts) deeply connected with the actual vs. potential infinity distinction.

1. Untenability of Kant's definition and example

In [Couturat 1908] it has been shown that Kant's example, i.e.,

$$7 + 5 = 12,$$

cannot demonstrate syntheticity *a priori* according to Kant's definition. This is because the definition itself is untenable due to Kant's reducing the form of a sentence to subject-predicate form. What could be a predicate and a subject in the above example?

There are two options here:

a) The predicate is the expression '= 12':

b) The predicate is '='.

Ad a): Then the subject expression is '7 + 5'. The respective concept leads to the number 12 (as Couturat shows in details), and no other possibility can be discussed. Thus we can say that the (concept given by the) predicate is contained in the (concept given by the) subject.

Ad b): In this case the 'subject concept' is a pair of concepts ⟨concept of 7 + 5, concept of 12⟩. Since both these concepts lead to the same number and the identity relation contains just those pairs of numbers where the first number is the same as the second number we can say that the 'predicate', i.e., the concept of identity, extensionally contains the 'subject pair', so it is intensionally contained in it.

To sum up, Kant's attempt to define analyticity as well as his attempt to prove that some *a priori* sentences are synthetic had to break down because they were based on the simplifying subject-predicate analysis.

2. A rational core of Kant's idea

Informally, we can all the same state that all mathematical concepts do not 'behave' in the same way. Some of them are sufficient to identify the object whose concepts they are, some other need some 'help' (given by other concepts) so that their way to identify the object is in this sense incomplete. To make this suspicion exact we need some results of the contemporary logic. The intuitionistic approach to this problem can be found in [Martin-Löf 1992]. Here we will try to preserve the classical, realistic viewpoint.

As we have seen, the synthetic character of such simple arithmetical sentences as Kant adduces can be hardly proved. Consider, however, some non-trivial mathematical problems like Goldbach's conjecture or Fermat's Last Hypothesis, the former not yet solved, the latter only recently. Is the distinction between such claims and the elementary arithmetic claims essential? If so, it surely cannot be connected with the question of *a priori*: all mathematical claims are *a priori* true or *a priori* false. Perhaps a modification of Kant's definition of analytic vs. synthetic-a modification that would lack any reference to the subject-predicate dogma-could explain the distinction and make our intuition exact.

3. Concepts

Such a modification can be realized as soon as we decide to generalize the problem: to consider *concepts* rather than judgements as for analyticity.

What is logically interesting on expressions is primarily their meaning. We accept the Fregean (originally vague) idea that between an expression and its denotation there is sense, which leads us to the denotation ("die Art des Gegebenseins"). Further, instead of 'sense' we use the customary term *meaning* and-with Church in his [1956] we say that

the meaning of an expression E is a concept of the denotation of E.

Saying that 7 + 5 = 12 we say that the concept expressed by this statements is a concept of the truth-value **T**, similarly '7 + 5' expresses a concept of the number 12 and '12' expresses another concept of the same number. Our modification of the notion of analyticity will be connected with this kind of generalization. Our question is then:

Are some a priori (in particular, mathematical) concepts synthetic?

What do we mean by 'concept'?

The depsychologisation of the category CONCEPT (starting with Bolzano in [1837] and Frege in the 90s) can be realized in two fundamentally distinct ways (see [Materna 1998], [Materna 2000]). One of them consists in some such precisification which results in defining concept as a set-theoretical entity. (See [Frege 1891], [Frege 1892] as a classical example.) This way is connected with essential problems. One of them is that the distinction between an universal and a way of its grasping vanishes: a concept of the set of primes would be, for example, identical with this very set, which contradicts the intuitive idea that one and the same object can be identified by many distinct concepts. (For this important idea see [Bolzano 1837].) A general objection to the set-theoretical explication of *concept* can be formulated by a quotation from [Zalta 1988, 183]:

> *[A]lthough sets may be useful for describing certain structural relationships, they are not the kind of thing that would help us to understand the nature of presentation. There is nothing about a set in virtue of which it may be said to present something to us.*

Bolzano in [1837] and Bealer in [1982] suggest (not very explicitly) another way. Concepts should be *structured*. Considering concepts as being identical with (possible) meanings we can see some link to Cresswell's idea of *structured* meaning (see, e.g.,[Cresswell 1985]). A systematic realization of this idea (based on transparent intensional logic) can be found in [Materna 1998], [Duzi 2003].

The resulting theory could be characterized as *a procedural theory of concepts* (so 'PTC'). Before we say more in terms of transparent intensional logic (TIL), let us informally suggest the core of PTC.

Concepts are conceived of as abstract procedures that (in the better case) result in identifying an object. So they are structured in the 'algorithmic' sense rather than in the mereological sense (as consisting of parts). As abstract procedures they are not spatially and temporally localizable, as abstract *procedures* ('instructions') they are not reducible to set-theoretical objects. They consist of 'intellectual steps', 'instructions'(both these expressions are metaphors, of course), which are in *principle* derivable from the structure of the given expression. This idea can be made precise when the respective definitions are formulated within the system of TIL.

3a. Constructions

Exact definitions of constructions can be found in several books and articles: here we refer to [Tichý 1988] and [Materna 1998]. The most relevant points will be now informally explained.

The TIL language is fully transparent: all the semantically salient features are inherently present. In particular, we need to know the type of the denoted entity and the way of its identifying, i.e. the construction of the denoted entity. Moreover, we need not only to use constructions, but also to mention them within the theory. In such a case the denoted entity is a (less-structured) construction, and it is identified by a more structured construction of a higher order.

Thus the ontology of TIL is organised into a two-dimensional infinite hierarchy, which at the lowest level has to stem from a certain base. TIL is an open system and the choice of the base is arbitrary. Here we will need only the following atomic (basic) types: o (the set {T,F} of truth-values), τ (the set of real numbers), and/or ν (the set of natural numbers). The remaining atomic types, ι (the set of individuals) and ω (the set of possible worlds) are needed when empirical objects are constructed (propositions, properties, relations-in-intension, magnitudes, etc.: in general, intensions in the sense of Possible-World Semantics), which is not the case in the area of mathematical objects.

The "horizontal level" of a type is increased by the rule of forming partial functions: if α, β_1, $\cdot$, β_n are types then the collection of partial functions $\beta_1 \times \cdot \times \beta_n \rightarrow \alpha$ is a type, denoted $(\alpha\ \beta_1 \cdot \beta_n)$.

The "vertical line" increases the order of a construction: Entities of the 1[st] **order** type are not structured from an algorithmic point of view, they are not procedures-constructions, nor do they involve constructions in their domains. Collection of constructions identifying 1^{st} order entities forms a type $*_1$ (of ***order 2***). Constructions identifying entities of order 1 and 2 form a type $*_2$ (of ***order 3***). And so on · $*_3$, $*_4$, ·

The technical inspiration goes from typed λ -calculus. To be a little bit more formal, let us characterize four most important kinds of construction.

- For every type there are countably infinitely many *variables* at our disposal, where a variable is a kind of construction which constructs objects dependently on valuation; they *v*-construct objects, where *v* is a parameter of valuation. The usual letters *x, y, z,* x_1, x_2,... are names of variables.1)
- Where X is any object (including constructions), ^0X is a construction called *trivialisation*. It constructs X without any change. (The importance of trivialisation can be appreciated in particular in the ramified hierarchy.)
- *Composition*, [X $X_1...X_m$], consists in applying the function (type scheme (α $\beta_1...\beta_m$)) *v*-constructed by X to arguments (of types β_1, · ,β_m, respectively) *v-constructed* by $X_1,...,X_m$. It can be *v*-improper, i.e., *v*-construct nothing (the constructed function may be undefined on the given argument).
- *Closure*, [λ $x_1...x_m$ X] *v*-constructs a function in the well-known way (see λ -calculus).

(Notation. An object O is of a type α : O/α .

A construction X identifies an entity of type α : X → α .)

Classes of objects of type α belong to the type (o α) (as characteristic functions), relations of objects of types $\beta_1,...,\beta_m$ belong to the type (o β_1.. β_m), the basic arithmetical operations (on real numbers) belong to the type (τ τ τ) etc. Also logical objects like truth functions, quantifiers etc. are a

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1) Constructions are extra-linguistic objects, so variables have to be defined as above. No construction contains a1 linguistic expression!

kind of function (types, $(o\ o)$, $(o\ o\ o)$, $(o\ (o\ \tau))$ etc.).[2)] Higher-order entities are *hyperintensional* involving constructions: $*_n$, $(o\ *_n)$, ·

All the objects of the types of order 1 are set-theoretical objects. The idea of constructions has been motivated by the important fact that there is a fundamental difference between an object and the way it is given (constructed). One and the same object can be 'given' in (theoretically infinitely) many ways, and ignoring this fact leads very often to misunderstandings (to a 'doubletalk'). (An innocent example: How many logically necessary propositions are there? Just one: it can be given by infinitely many constructions, encoded by sentences of a language.) Thus constructions are no more set-theoretical objects. They are structured, and their components are particular instructions: the construction as a whole determines particular roles of its components.

Constructions themselves do not contain any letter or bracket, so the construction

$$[\lambda\, x\, [{}^0\!>\, x\, {}^0 0]]$$

constructs the (characteristic function of the) set of positive numbers, it is an instruction whose particular components and their 'roles' are unambiguously given by the definitions of closure and composition. This abstract instruction cannot contain letters, brackets, 'λ'.

Constructions define *hyperintensionality* (see [Cresswell 1975]). Concepts (potential meanings) are hyperintensional. Now we can define concepts.

3b. Concepts as closed constructions

Since constructions have been introduced as explications of abstract procedures it is obvious that our explication of concepts is done in terms of constructions. Concepts should identify objects, constructions (v-)construct

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2) Some resemblance to Montague is obvious. The distinctions are relevant but there is no space here for explaining why TIL rather than Montague has been chosen here.

objects. Some concepts are empty, some constructions are (v-)improper. Constructions, as representatives of hyperintensionality, are (potential) meanings[3], concepts should be identified with (potential) meanings (Church). So it seems that concepts are at least a kind of construction. The only specific feature of concepts is that they should identify objects without any dependence on a parameter, i.e., in our case, on the valuation (v-constructing). To be a concept a construction should contain no occurrence of a variable that would not be bound (either *via* λ, or via trivialisation, see [Materna 1998]). Thus concepts (of order n) are closed constructions (of order n).

(A complication can be stated here: constructions

$$\lambda x\ [{}^{0}{>}\ x\ {}^{0}0]$$

and

$$\lambda y\ [{}^{0}{>}\ y\ {}^{0}0]$$

are distinct but they represent one and the same concept. Similarly we would like to say the same in the case of constructions

$${}^{0}{>}$$

and

$$\lambda xy\ [{}^{0}{>}\ xy].$$

This problem has been solved in one way in [Materna 1998] and in another, better way in [Horák 2001], see also [Duzi 2003]. The result is that our identification of concepts with closed constructions can be maintained.)

An important **definition**:

Let X be an object that is not a construction. Then ^{0}X is a *simple concept*.

Notice that a simple concept identifies the respective object immediately, without using any other concept. It is simple also because it consists of just one instruction.

3) Sense (meaning) as an algorithm: see [Moschovakis 1990].

4. Finitism

Let us return to the most popular case of analytic vs. synthetic statements (*a priori*). Any such statement (we confine ourselves to mathematical statements) expresses a concept of a truthvalue. We could perhaps propose the following explication:

Explication I. (First attempt)

A concept C of a truth-value is analytic if it identifies the truth-value in finitely many steps using just those concepts that are its components (i.e., subconstructions of C).

The obvious generalization:

Explication I .

An a priori concept C is ***analytic*** if it identifies the object in finitely many steps using just those concepts that are its components. Otherwise C is ***synthetic***.

To see that this explication is not suitable let us consider simple concepts (see 3.b.). The immediate identification means simply that the object has been identified, and that finitely many steps (viz. one) have been applied. *Thus every simple concept would be analytic.*

This result seems to contradict our basic intuition. For take, e.g., the simple concept ^{0}Prime(-numbers). Only one instruction has been applied and the set of primes has been identified. Is it thinkable? Well, it is, namely if he who possesses this concept is God himself. Nobody other can grasp *actual infinity*.

To handle the problems of this kind we have to make more precise the notion of *step*. Every construction consists of only a finite number of steps, *instructions*. Yet every instruction contained in a construction is connected with some (number of particular abstract actions, which we will call) *executing steps*. Returning to our elementary example, the construction λx $[{}^0\!> x\ {}^0 0]$ $(x \rightarrow \tau)$ consists of three instructions:

- *Identify the relation $>$,*
- *identify the number 0,*
- *for any real number k apply $>$ to the pair $\langle k, 0 \rangle$.*

Here for example the third instruction is a rather suspect: though it does not appeal to the actual but only potential infinity, it commands to grasp any *real number*! Doesn't it involve an infinite number of 'executing steps'? As soon as we replace the type $\boldsymbol{\tau}$ (of real numbers) by type $\boldsymbol{\nu}$ (the set of natural numbers), so that the variable x ranges over $\boldsymbol{\nu}$, the third instruction is no doubt easily executable.

Knowing the concept, we understand these instructions. Does it mean that these *instructions are always executable*? The realist (Platonic) answer would be YES: if it is not executable by a human being with limited cognitive capacities, then it suffices that it can be executed by a hypothetical being whose intellectual capacities may exceed our limited ones. However, an intuitionist finds the knowledge of instructions that are always executable (possibly by a hypothetical being) to be a rather obscure notion. The difference between (Platonic) *realism and intuitionism* consists in the following claim: where the intuitionist says that no object is present, the realist says that the respective concept is synthetic[4]. (Cf. ^{0}Prime as the instruction "grasp the (actually infinite) set of primes"). It follows from this that, in order to objectively distinguish between realist / intuitionistic view, we should avoid an explicit appeal to our intellectual capabilities. In other words, we are going to define "feasibly executable concepts" without any reference to psychological content of any-being's capabilities. We also rule out some candidates for the "feasibility criterion" as formulated by the "strict finitism"[5], which cover two (conflicting) theses: a) that there is an upper bound B, independently of time, to the size of mathematical inscriptions such that if some $n > 2^B$ then the numeral n cannot be

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4) True, some intuitionists say that the object is not well-defined, which is, after all, not as distinct from the realist view.

5) See [Dubucs, Marion 2003]

presented, i.e., it is not feasible, and b) the thesis that identifies feasible computability with polynomial-time computability. In our approach to the problem *any bounds should remain hidden* and we want to reflect limitations to human cognitive capabilities in a structural manner - in terms of a structured concept / construction.

A new explication could be:

Explication II. (Second attempt)

An a priori concept C is ***analytic*** *if it identifies the object in an effective way using just those concepts that are its components. Otherwise C is* ***synthetic.***

This explication needs, however, some comment: we have to define in a more precise way the notion of an *effective identifying construction.* Intuitively, any instruction involving actual infinity is not effective. Thus the concept ^{0}Prime seems to be synthetic: the respective executing step is only one: grasp the infinite set as an actual infinity. Since we know, however, that the set of primes is recursively enumerable, there must be an analytic concept equivalent to the concept ^{0}Prime. If we have other simple concepts at our disposal, for instance, the concept of Card(inality of a set of natural numbers / ν $(o\ \nu)$) and the concept of Div(isibility-relation between natural numbers / $o\ \nu\ \nu$), we can replace the synthetic concept ^{0}Prime by an analytic complex concept **numbers that have exactly two factors:**

$$\lambda x\ [^0{=}\ ^0\mathrm{Card}\ \lambda y\ [^0\mathrm{Div}\ x\ y]\ ^02]$$

This construction identifies the set of primes in an effective way, because it actually identifies only a *potential infinity.* The procedure consists of the following steps:

- Take any natural number (x)
- Compute the (finite) set F of factors of x (λ *y...*)
- Compute the number N of elements of F (^{0}*Card* λ *y...*)
- If N = 2 then return True, otherwise False

Note that our identifying sets with their characteristic functions is important; it makes it possible to specify infinite sets in a *recursive way*. The procedure does not supply the actual infinity; it only 'decides', answers Yes or No to the question "Is x a prime"? for *any* x. True, to obtain the *whole set*, we would have to repeat the above steps infinitely many times (λ x), which is not as bad, because each time the steps are perfectly executable.

5. Trivialisation and "grasping".

We have to admit that the way the trivialisation has been defined is not definite enough in the present context. Imagine, for example, that the trivialisation of identity in the concept of primes (see above) were interpreted as follows:

grasp the relation =

(and apply it to the pair ⟨N, 2⟩, where N is the result, for the given x, of ^{0}Card λ y [^{0}Div x y]).

It could be interpreted in this way: trivialisation should "construct = without any change" according to our definition. Yet the relation = is an infinite relation, i.e., its characteristic function (over the type ν) is infinite in the sense that it is defined on infinitely many ordered pairs of (natural) numbers. "Grasping" this function means "grasping" actual infinity, which is incompatible with finitism; besides, before to apply this function to some pair of numbers this whole infinite function would have to be constructed, which is certainly not an effective way of constructing.

The same considerations hold for any trivialisation of infinite objects. The procedures conceived of as constructions should, however, bear finite character whenever it is possible. Consider, e.g., such an infinite function as the *successor* in Peano's arithmetic. Suc / (ν ν) is a function that associates *each* natural number with its immediate successor. Thus the construction

$$[^0\text{Suc}\ ^0 0]$$

constructs the number 1 (in other words, it is a concept of the number 1).

The trivialisation ^{0}Suc does not mean that the whole function is constructed with its $\aleph_0$ arguments, and the above construction of the number 1 cannot be interpreted in such a way, i.e., as if first this whole function Suc were to be constructed and *then* (!) applied to 0. The trivialisation ^{0}Suc must be conceived of as the recursive procedure, for example, a respective Turing machine. This is well compatible with the definition: the recursive procedure really constructs Suc without any change. The only distinction from the "$\aleph_0$"-variant consists in taking into account only *potential infinity.*

Interpreting trivialisation in this way, i.e., as avoiding actual infinity wherever it is possible, we can state that the following explication should be the adequate one:

Explication III:

The ***analytic concepts*** *are just those constructions which construct the respective objects in a finitary way: neither actual infinity nor calling other concepts is present.*

(Compare with a formulation in [Fletcher 1998], p.52:

> The very simplest type of construction allows just a single atom (call it '0') and a single combination rule (given a construction x we may construct S(x)) *with no associated conditions:* ... (Emphasis ours - D.+M.))

Remembering Church's Thesis we can state that our identification of concepts with constructions (i.e., a 'procedural theory of concepts') makes it possible to identify analytic concepts with *constructions that finitely construct recursive functions.* This formulation admits that a concept can construct a recursive function in a non-finitary way[6]. This will be shown in the next section.

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6) An important remark: Functions are construed here as being mappings. So recursive functions are also mappings, but such that there is an effective way of computing these mappings. This distinction is sometimes captured by terms 'extensional function' vs. 'intensional function'.

6. Analyticity and conceptual systems

Consider a relation λxy R with R effectively computable and the following function:

$$\varepsilon y\, R(x, y) = \begin{cases} \text{the least number y such that } R(x, y) \text{ if } \exists y\, R(x, y) \\ 0 \text{ otherwise} \end{cases}$$

This definition (i.e., the underlying concept of it) as such does not determine an effective procedure (see [Kleene 1952], p.317). In some cases, however, R is such a relation that the respective effective procedure exists, but this procedure is not attainable by the above definition: there is another definition the underlying concept of which is an effective procedure. (Such a case is given by the equation

$$ax + by = c$$

for a, b, c positive integers: the respective effective procedure is the concept underlying the well-known Euclid s algorithm.)

So we have two concepts C and C' , where C is equivalent to C' but is-unlike C-an effective procedure.

Another such case are the concepts of the number π . Let α be the type of circles. The type of C(ircumference of a circle), as well as of D(iameter of a circle) is then (τ α). π is an abbreviation the definiens of which can be simply written as (ι is this time the "descriptive operator", x ranges over τ , y ranges over α):

$$\iota\, x\ \forall y\ x = C(y) \ /\ D(y).$$

In the respective concept C and D can occur as ^{0}C, ^{0}D, which-together with the quantifier-makes it impossible to effectively calculate the number π . The same number, as an irrational number, can be construed as a function F, type (ν ν), that associates the k^{th} natural number with the k^{th} number in the infinite expansion of π . This function is recursive in that there is an effective procedure that computes for any natural number the value of F in finitely many steps. Now the above definition cannot determine this procedure if ^{0}C, ^{0}D are simple concepts. But a more realistic idea consists in the assumption that the concepts of C, D are not simple, that C, D are defined in some system that is based on other simple concepts so that the *terms diameter, circumference* are only abbreviations the definiens of which are such that the effective procedure computing F could be determined by the above definition. Actually, this assumption can be refuted. It suffices to look at the quantifier over circles. On the other hand, the effective concept is given by the formula for calculating the sum of an infinite series. The respective concept handles potential infinity only. Again, we have got some 'ineffective' concepts and a genuine 'algorithmic' concept.

An interesting question arises in connection with the procedural theory of concepts. Let us accept the last explication of analytic concepts (*a priori*). Then *synthetic concepts a priori* are of two kinds: Either they define a non-recursive function or they define a recursive function in a non-effective way (using actual infinity). Assuming-as we do-that concepts are objective and are discovered rather than created we can formulate an interesting consequence:

There are more synthetic than analytic concepts a priori.

The proof is simple: There are uncountably many functions while only countably many (partial) recursive functions. -

The second kind of synthetic concepts *a priori*, viz. the concepts that define recursive functions in an ineffective way, is interesting and inspires us to the following question:

(Q) *Under which conditions can an 'effective' counterpart of an 'ineffective' concept be discovered?*

This question looks like a methodological one, and its answering can be of some interest for methodology but primarily it belongs to the class of problems connected with the procedural theory of concepts, so to the class of logical questions. We will need the notion of *conceptual system* (see [Materna 1998]).

Definition

Let $C_1,...,C_n$ be simple concepts (see the end of section 3) and **V** the set of all variables. A *conceptual system* (over $C_1,...,C_n$) is a tuple with { $C_1,...,C_n$, **V** } as the first member and the inductive definition of constructions (confined to handling $C_1,...,C_n$) as the second member. The tuple makes it possible to create all and only such concepts whose simple subconcepts are members of { $C_1,...,C_n$ }. Where $C_{n+1},...$ are such 'derived' concepts we will briefly speak about the set

$$\{ C_1,...,C_n \} \cup \{C_{n+1}, ... \}$$

as about a conceptual system. The members of the set { $C_1,...,C_n$ } are *primitive concepts* of the given conceptual system, the members of the second set can be called *derived members*. -

It is obvious that the set of primitive concepts unambiguously determines the set of derived concepts.

To illustrate the connection between our question (Q) and the above definition let us briefly comment the case of Fermat's Last Theorem (FLT). FLT can be written in a usual way as the statement (a, b, c, n positive integers):

$$\forall abcn\ (\ n > 2 \supset \neg\ (\ a^n + b^n = c^n))$$

The history of the proof of FLT is fascinating particularly because of the fact that between Fermat's formulation and the final proof about three hundred years went by. This fact can be seen as a good motivation of our question (Q); we can also ask whether this fact is a contingent historical fact, i.e., whether it could have been possible to discover the proof earlier, in particular at the time of the first formulation of FLT (as Fermat himself suggested). This last question is a Yes-No question, and the answer will be deducible from the answer to the question (Q), which is a Wh-question.

Now the frame of the proof of FLT is extremely simple. The scheme thereof is:

$\neg$FLT implies a sentence **A**

A sentence **B** implies $\neg$**A**

B holds

$\therefore$FLT

The problem is that the first and third premises require acquaintance with some concepts that can be derived in such a conceptual system that nobody before the second half of the 20th century had at his disposal. For take, e.g., the simple concepts that are subconcepts of the concept that underlies the first formulation of FLT:

$^{0}\forall$, $^{0}*$(multiplication), $^{0}+$, $^{0}=$, ^{0}exp (exp xy = x^{y});

Clearly, neither these concepts nor their decompositions nor their combinations suffice to derive those concepts that were necessary for the final proof (elliptic curves, modularity, stability, etc.). In the eternal Platonic realm (excuse me, please) all potential conceptual systems 'exist' but in our vale of tears time is important, so we have to state that an analytic concept underlying FLT could not have been discovered before other conceptual systems than those ones that were known before the 20th century became a part of mathematics.

(In a sense, of course, the fact of this 'late' discovery *is* a contingent fact. For imagine an ingenious mathematician who was able to define all those concepts that were necessary for proving FLT. Would the existence of such a mathematician be logically impossible? Surely not: we could at most claim that it was historically impossible, assuming that historical modalities were well defined. Thus there are possible worlds where a mathematician has proved FLT, say, in the 18th century. This is, however, not an important problem from our viewpoint.)

Fermat's formulation represents a *non-effective*, i.e., a *synthetic concept a* *priori*, due to the universal quantifier, whereas the contemporary *proof of FLT* represents its *analytic* counterpart. Here I would like to quote again Fletcher [1998], p.53:

[t]he distinction between finitism (based on mere 'combinatorial' considerations) and intuitionism (which includes non-combinatorial notions such as proof...

(See the following ***Remark.***)

If proof is admitted as representing a concept (no great problem for TIL) then the concept underlying the proof of FLT is an analytic counterpart of the synthetic concept that underlies Fermat's first formulation thereof.

Remark: Ad *analytic counterparts of synthetic concepts*: Let us compare this conception with what Fletcher (in [1998, 105]) characterizes as the "usual account of the distinction" between *finitist* and *intuitionist* reasoning. The latter admits - unlike the former -

> properties, such as being a proof of a formula and being a meaningful rule, that are decidable (in the sense that we can decide them using intelligent judgement) but are not necessarily recursive.

To the suspicion that this conception is incompatible with Church's Thesis Fletcher answers that one can still believe that

> for every 'abstract' function on numbers there is a recursive function extensionally equivalent to it;

where, of course, the term function is used in the intuitionist sense, i.e., not as a mapping but as a procedure.

From this viewpoint a following question can be raised:

Compare the concept C1 that is expressed by the standard formulation

$$\lambda abcn\ (n > 2 \supset \neg (a^n + b^n = c^n))$$

and the concept C2 underlying the recent proof of FLT. Finally, consider the concept C3 that underlies the original formulation of FLT, viz.

$$\forall abcn\ (n > 2 \supset \neg (a^n + b^n = c^n)).$$

Once more, C3 is in our terminology - synthetic: the concept ${}^0\forall$ at least presupposes a finished actually infinite set of quadruples. C2 constructs the solution T without presupposing actual infinity. As for C1, it is certainly analytic (it effectively calculates the value of the respective recursive function for any quadruple of natural numbers). C1 does not solve FLT, of course. Our question concerns C2 and C3 as confronted with the above quotation from [Fletcher 1998]: Can the quoted view be interpreted as follows?

C3 is the 'abstract function on numbers' whereas C2 is the recursive function extensionally equivalent to it.

The positive answer depends on whether the proof of FLT is or is not intuitionistically sound. In connection therewith another interpretation is possible: C2 could correspond to a *decidable but not recursive* property: the proof is then admissible for an intuitionist but not for a finitist. Then the above hypothesis makes us expect that there should be some C4, viz. a genuinely recursive function extensionally equivalent to C2.

(Fletcher's objectives are, among other things, oriented to making the characterizations of finitism and intuitionism more precise and "to diminish the apparent distance between intuitionism and finitism" (*ibidem*).)

7. Conclusion

seems that at least some mathematical discoveries consist in replacing synthetic concepts (a priori) with analytic concepts extensionally equivalent to them. Better to say: development of solving mathematical problems can be described in terms of conceptual systems as follows: A mathematical problem is formulated, i.e., a concept representing this problem is expressed; if the concept is analytic the problem is solved; otherwise, the problem is equivalently reformulated in terms of some other concepts belonging to the given conceptual system; if the resulting concept is analytic, the problem is solved; otherwise, some other conceptual system is discovered and used, till the resulting concept is analytic. Indeed, if the problem consists in calculating a function that is not recursive the problem cannot be solved, and any concept that should identify this function is synthetic.

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