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An Analytical Calculation of the Transport of the Solute Dumped in a Homogeneous Open Sea with Mean and Oscillatory Flows

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An analytical model for predicting the convection-diffusion of solute dumped in a homogeneous open sea of constant water depth has been developed in a time-integral form. The model incorporates spatially uniform, uni-directional, mean and oscillatory currents for horizontal convection, the settling velocity for the vertical convection, and the anisotropic turbulent diffusion. Two transformations were introduced to reduce the convection-diffusion equation to the Fickian type diffusion equation, and then the Galerkin method was then applied via the expansion of eigenfunctions over the water column derived from the Sturm-Liouville problem. A series of calculations has been performed to demonstrate the applicability of the model.

Key words: Analytical model, Solute transport, Convection-diffusion, Galerkin method

Introduction

The problem of predicting the transport of solutes such as dissolved toxic matters, radionuclides and suspended sediments on the basis of a convection-diffusion equation appear in various disciplines such as hydrology, environmental and chemical engineering and oceanography. Since the basic equation is nonlinear, numerical methods are usually adopted as a predictive tool. A very limited number of analytical solutions are available in idealized conditions; the depth is spatially constant, the convection velocity and eddy diffusivity are constant or take simple functional forms (Prakash, 1977; Smith, 1982; Wilson and Okubo, 1978; Yasuda, 1988; Zoppou and Knight, 1997). For some collection of solutions, see Noye (1987). There is obviously a continuing need to develop analytical solutions of the convection-diffusion equation because of its fundamental and practical importance; analytical solutions are valuable not only for the better understanding on the transport processes but the verification of the numerical schemes.

In this study an analytical solution of time-integral form has been derived to predict the transport of solutes released at the sea surface in a homogeneous open sea, keeping in mind the application to dumping of dredged sediments, industrial wastes and feed stuffs, vice versa. In detail, we have combined the horizontally two-dimensional solution, derived by Jung et al. (2003a) for the build-up of the heat field due to a point source in coastal regions with an oscillatory cross-flow, with the vertically one-dimensional solution, derived by Jung et al. (2003b), for the determination of local distribution of suspended sediment. The model incorporates spatially uniform, uni-directional horizontal convection, anisotropic horizontal diffusion, the vertical convection due to settling velocity, and the vertical diffusion. The presence of a point source at the sea surface is assumed as the dumping rate of solute. No net flux condition is applied at the sea surface (except for the source point), while the downward net flux is considered through the introduction of a depositional velocity. In the model by Yasuda (1988) a finite water depth was assumed but the horizontal diffusion was neglected and no net flux condition was applied at the sea surface and sea bottom boundaries. The solution is sought as a salient feature of this study by applying the Galerkin method in time domain via the eigenfunction expansion over the water column. In applications the direct time-integration of the solution is made rather than the calculation of Arie's moments (1956) to investigate the solution behaviors. A series of calculations are carried out

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to examine the contribution of settling velocity, mean and oscillatory currents along with the role of longitudinal turbulent diffusion to the determination of the solute distribution.

Basic equation and Solution

Basic form of convection-diffusion equation

We consider a horizontally infinite ocean of constant water depth with a point source at the sea surface. The ambient flow is assumed to compose of spatially-invariant mean and oscillatory currents in the x-direction. A non-conservative form of the convection-diffusion (transport) equation for the solute transport may be written as:

$$\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x} - w_s\frac{\partial T}{\partial z} + k_x\frac{\partial^2 T}{\partial x^2} + \frac{k_x}{\beta^2}\frac{\partial^2 T}{\partial y^2} + k_z\frac{\partial^2 T}{\partial z^2} \quad (1)$$

where,

$$u = u_m + u_{max}\sin\omega t \quad (2)$$

In the foregoing equations t is time, x, y and (zero at the sea surface and $-h$ at the sea bottom) are the Cartesian coordinates, T is the concentration of solute, u is the -directed ambient flow velocity, w_s is the settling velocity of solute (assumed to be negative), k_x, k_x/β^2 are the horizontal diffusion coefficients, β is the coefficient for non-isotropic diffusion, k_z is the vertical diffusion coefficient, u_m denotes mean current, u_{max} is the amplitude of the oscillatory current, ω $(=2\pi/T_P)$ is the angular frequency of the oscillatory current (T_P is the period). The ambient flow varies as a function of time only and the diffusion coefficients are all assumed to be time and space-invariant.

Assuming that the source is located at the origin $((x,y,z=(0,0,0))$, boundary conditions at the water surface ($z=0$) and sea bottom ($z=-h$) are given by

$$-w_s T(0) = k_z\left(\frac{\partial T}{\partial z}\right)_0 = F_s\delta(x,y,z),$$
$$-w_s T(-h) = k_z\left(\frac{\partial T}{\partial z}\right)_h = v_d T(-h) \quad (3)$$

where F_s is the solute influx at the sea surface, v_d is the depositional velocity and $\delta(x,y,z)$ is the delta function. Subscripts, $-h$ and 0, denote the levels derivatives are defined.

Transformation to a simple diffusion equation

As a first step of the solution procedure, two transformations are introduced as follows:

$$T(x,y,z,t) = C(x,y,z,t)\cdot\exp\left[\frac{w_s}{2k_z}z - \frac{w_s^2}{4k_z}t\right] \quad (4)$$

and

$$\chi = x - \int_\tau^t u(t)\,dt \quad (5)$$

where $C(x,y,z,t)$ is a new concentration variable. With the two transformations, Equation (1) reduces to a simple diffusion equation.

$$\frac{\partial C}{\partial t} = k_x\frac{\partial^2 C}{\partial \chi^2} + \frac{k_x}{\beta^2}\frac{\partial^2 C}{\partial y^2} + k_z\frac{\partial^2 C}{\partial z^2} \quad (6)$$

The boundary conditions given in Equation (3) reduce to

$$-\frac{w_s}{2}C(0) + k_z\left(\frac{\partial C}{\partial z}\right)_0 = F_{sT}$$
$$-\frac{w_s}{2}C(-h) + k_z\left(\frac{\partial C}{\partial z}\right)_h = v_d C(-h) \quad (7)$$

where,

$$F_{sT} = F_s\delta(x,y)\exp\left(\frac{w_s^2}{4k_z}t\right) \quad (8)$$

Galerkin expansion over the vertical space domain

We seek a solution for the diffusion equation (6) in the form,

$$C(\chi,y,z,t) = \sum_{r=1}^{m}\hat{C}_r(\chi,y,t)f_r(z) \quad (9)$$

where $\hat{C}_r(\chi,y,t)$, $r=1,2,\cdots,m$, are the unknown coefficients, $f_r(z)$, $r=1,2,\cdots,m$, are depth-varying basis functions, and m is the number of basis functions used. A Galerkin-eigenfunction technique used by Heaps (1972), Davies (1980) and Jung (1989) for the vertical variation of horizontal currents is applied to derive the series solution. Detailed procedure is described below.

Taking first an inner product with f_k in Equation (6) and applying integration by part twice to the vertical diffusion term (2nd term on the right hand side) gives,

$$\int_{-h}^{0}\frac{\partial C}{\partial t}f_k dz = \int_{-h}^{0}\left\{k_x\frac{\partial^2 C}{\partial \chi^2} + \frac{k_x}{\beta^2}\frac{\partial^2 C}{\partial y^2}\right\}f_k\,dz$$
$$+\left[k_z\frac{\partial C}{\partial z}f_k\right]_{-h}^{0} - \left[k_z C\frac{df_k}{dz}f_k\right]_{-h}^{0} + k_z C\frac{d^2 f_k}{dz^2}dz \quad (10)$$

Incorporating the boundary conditions given as (7) and (8) and substituting the expansion given in (9) lead to,

$$\sum_{r=1}^{m}\frac{\partial \hat{C}_r}{\partial t}\int_{-h}^{0} f_r f_k\,dz =$$
$$\sum_{r=1}^{m}\left\{k_x\frac{\partial \hat{C}_r}{\partial \chi^2}+\frac{k_x}{\beta^2}\frac{\partial^2 \hat{C}_r}{\partial y^2}\right\}\int_{-h}^{0} f_r f_k\,dz + F_{sT} f_k(0)$$
$$-\sum_{r=1}^{m}\hat{C}_r f_r(0)\left\{-\frac{w_s}{2}f_k(0)+k_z\left(\frac{df_k}{dz}\right)_0\right\}$$
$$+\sum_{r=1}^{m}\hat{C}_r f_r(-h)\left\{\left(-\frac{w_s}{2}-v_d\right)f_k(-h)+k_z\left(\frac{df_k}{dz}\right)_{-h}\right\}$$
$$+\sum_{r=1}^{m}\hat{C}_r k_z\int_{-h}^{0} f_r\frac{d^2 f_k}{dz^2}\,dz \qquad (11)$$

Choosing f_r as a set of solutions (eigenfunctions) deduced from the well-known Sturm-Liouville system,

$$k_z\frac{d^2 f_r}{dz^2}+\lambda f_r=0,\ (r=1,2,\cdots m) \qquad (12)$$

subject to,

$$-\frac{w_s}{2}f_r(0)+k_z\left(\frac{df_r}{dz}\right)_0=0,$$
$$\left(-\frac{w_s}{2}-v_d\right)f_r(-h)+k_z\left(\frac{df_r}{dz}\right)_{-h}=0,$$
$$f_r(0)=1 \qquad (13)$$

then the 3rd and 4th terms of the right-hand side of Equation (11) are eliminated and we get

$$\frac{\partial \hat{C}_r}{\partial t}=\left\{k_x\frac{\partial^2 \hat{C}_r}{\partial \chi^2}+\frac{k_x}{\beta^2}\frac{\partial^2 \hat{C}_r}{\partial y^2}\right\}+\Phi_r\cdot F_{sT} f_r(0)-\lambda_r \hat{C}_r \qquad (14)$$

where λ_r is the real-valued r-th eigenvalue and,

$$\Phi_r=1/\int_{-h}^{0} f_r^2\,dz \qquad (15)$$

In deriving Equation (14) the well-known orthogonal condition of the eigenfunctions is used. That is,

$$\int_{-h}^{0} f_r f_k\,dz=0 \quad \text{if} \quad r\neq k \qquad (16)$$

Eigenfunctions and eigenvalues determined from equations (12) and (13) are:

$$f_r(z)=\cos\alpha_r z+\left(\frac{w_s}{2k_z\alpha_r}\right)\sin\alpha_r z,\ \lambda_r=k_z\alpha_r^2$$
$$(r=1,2,\cdots,m) \qquad (17)$$

where α_r satisfies the following transcendental equation.

$$v_d\cos\alpha_r h=\left\{\left(\frac{w_s}{2}+v_d\right)\frac{w_s}{2k_z\alpha_r}+k_z\alpha_r\right\}\sin\alpha_r h \qquad (18)$$

In case $w_s=v_d=0$, r-th eigenfunction and eigenvalue are given by,

$$f_r=\cos(\sqrt{\lambda_r/k_z}z),\ \lambda_r=k_z(r-1)^2\pi^2/h^2 \qquad (19)$$

Solutions in terms of original variables

Assuming that the solute field is initially at zero throughout the water column, an appropriate solution of Equation (14) to an instantaneous release of solute dumped at the origin (0,0) at time $t=\tau$ might be

$$\hat{C}_r(\chi,y,t)=\frac{\beta F_{sT}(\tau)\Phi_r}{4\pi k_x(t-\tau)}\cdot\exp[-\lambda_r(t-\tau)]\cdot I_{es} \qquad (20)$$

where,

$$I_{es}=\exp\left[-\frac{\chi^2+(\beta y)^2}{4k_x(t-\tau)}\right],\ F_{sT}(\tau)=F_s\exp\left[\frac{w_s^2}{4k_z}\tau\right] \qquad (21)$$

The convolution integral can be applied for the continuous release of solute started from $t=0$. That is,

$$\hat{C}_r(\chi,y,t)=\Phi_r\int_0^t\frac{\beta F_s}{4\pi k_x(t-\tau)}\exp\left[\frac{w_s^2}{4k_z}\tau\right]$$
$$\cdot\exp[-\lambda_r(t-\tau)]\cdot I_{es}\,d\tau \qquad (22)$$

Summing up the contribution of all eigenfunctions using Equation (9) then gives

$$\hat{C}_r(\chi,y,z,t)=\sum_{r=1}^{m}\Phi_r f_r(z)\int_0^t\frac{\beta F_s}{4\pi k_x(t-\tau)}\exp\left[\frac{w_s^2}{4k_z}\tau\right]$$
$$\cdot\exp[-\lambda_r(t-\tau)]\cdot I_{es}\,d\tau \qquad (23)$$

Substituting the relation between the original and transformed coordinates, that is,

$$\chi=x-u_m+u_{\max}/\omega\cdot(\cos\omega t-\cos\omega\tau) \qquad (24)$$

and substituting equation (23) into equation (4) finally give,

$$T(x,y,z,t)=\exp\left[\frac{w_s}{2k_z}z\right]\cdot\sum_{r=1}^{m}\Phi_r f_r(z)\int_0^t\frac{\beta F_s}{4\pi k_x(t-\tau)}$$
$$\cdot\exp\left[-\left(\frac{w_s^2}{4k_z}+\lambda_r\right)(t-\tau)\right]\cdot I_{eu}\,d\tau \qquad (25)$$

where

$$I_{eu}=\exp\left[-\frac{((x-u_m+u_{\max}/\omega\cdot(\cos\omega t-\cos\omega\tau))^2+(\beta y)^2}{4k_x(t-\tau)}\right] \qquad (26)$$

We note that in case of $w_s=v_d=0$, $\lambda_1=0$, and $f_1=1$. Correspondingly, $\Phi_1=1/h$ and $\Phi_r=2/h$ for $r\geq 2$. Therefore,

$$T(x,y,z,t)=\sum_{r=2}^{m}\left(1+2\cos\frac{(r-1)\pi z}{h}\right)\int_0^t\frac{\beta F_s}{4\pi h k_x(t-\tau)}$$
$$\cdot\exp[-\lambda_r(t-\tau)]\cdot I_{eu}\,d\tau \qquad (27)$$

which is identical to the form shown in Carslaw and Jaeger (1959) in case the turbulent diffusion is assumed to be isotropic ($\beta=1$) and the oscillatory current is absent.

Results

A total of three sets of calculations have been performed; the first set is the calculations in stagnant water, the second set is the calculations with mean flow condi:ion and thee final set includes the calculations with mean and oscillatory flows. Throughout the calculations the water depth is assumed to be 70 m and the value of m is taken as 50. $k_y(=k_x/\beta^2)$ is set to be ten times smaller than β^2 by choosing

A simple mid-ordinate method has been used to evaluate the integral (25). Calculations have been continued over a month with source strength F_s=100 kg/m^2/s. The time increment is approximated to be a finite value, $\Delta\tau$, chosen as T_P/1800 seconds. Results are the solute concentrations on the x-z cross-section on the side of positive x axis passing through the source point.

Calculations in stagnant water

To get an idea on the role of the settling velocity and the vertical eddy diffusion, calculations denoted by Cal-S1 to Cal-S4 have been performed in the absence of the horizontal convection. In these calculations the horizontal diffusion is set to k_x=0.2 m^2/s.

Fig. 1 shows the cross-sectional distribution of solute computed with k_z=0.05 m^2/s (Cal-S1) and k_z=0.0005 m^2/s (Cal-S2). The settling velocity w_s= -4×10^{-5} m/s and the depositional velocity v_d=2×10^{-5} m/s (that is, half of the settling velocity as in Prandle, 1997) has been used.

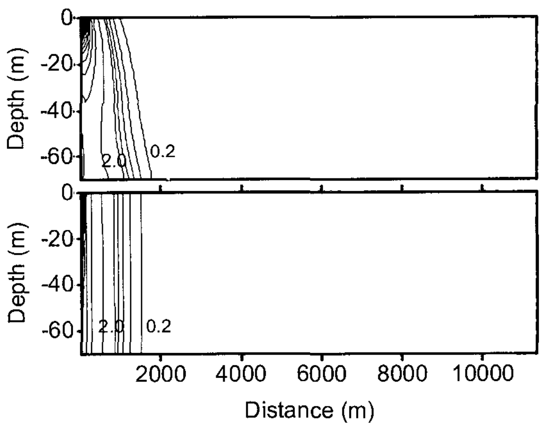

Fig. 1. Concentration of solute computed in the absence of horizontal convection. Upper panel: k_z=0.05 m^2/s, Lower panel: k_z=0.0005 m^2/s.

It is seen that a core of high concentration is formed near the source point when the vertical mixing is significantly suppressed, while a vertically well-mixed feature is formed when the vertical mixing is intense. Negative values of concentration, although it is not clearly shown in the figures, appear below the source point, which is known to be Gibbs phenomenon.

Fig. 2 shows the cross-sectional distribution of solute computed with the w_s=-4×10^{-6} m/s (Cal-S3) and -4×10^{-4} m/s (Cal-S4). The depositional velocities have been again set to be half of each value of the settling velocity.

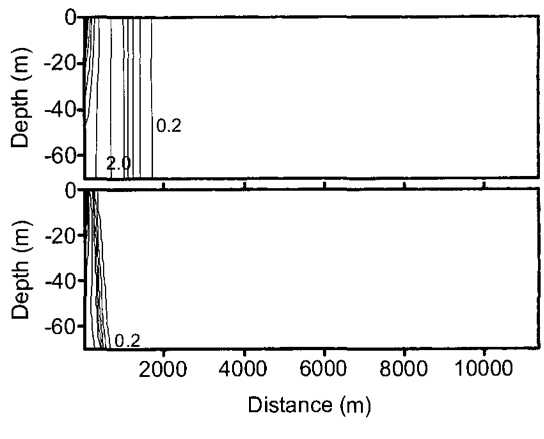

Fig. 2 Concentrations of solute computed in the absence of horizontal convection. Upper panel: w_s=-4×10^{-6} m/s, Lower panel: w_s=-4×10^{-4} m/s.

It is noted that there is remarkable change in the distribution patterns. A horizontal diffusive process becomes pronounced when a small value of settling velocity is used, revealing a vertically well-mixed structure except for the core region. When the settling velocity is high, solute rapidly falls down to the bottom, forming a concentration field with a narrow band. It is expected that the concentration near the bottom tends to increases in the absence of the depositional velocity, that is, in case of no flux condition used by previous workers (for example, Yasuda, 1988).

Calculations in the presence of mean current

Three calculations denoted by Cal-M1 to Cal-M3 have been carried out in the presence of the horizontal convection by the mean current. In Cal-M1 u_m=0.02 m/s is used with k_x=0.2 m^2/s, in Cal-M2 the mean current is same as Cal-M1 but the horizontal diffusion is reduced by one order (u_m=0.02 m/s with k_x=0.02 m^2/s). And in Cal-M3 u_m=0.2 m/s is used with k_x=0.2 m^2/s. In these calculations the settling and depositional velocities used are same as in Cal-S1 and Cal-S2

(w_s=-4×10^{-5} m/s and v_d=2×10^{-5} m/s).

We can clearly see in Fig. 3 that the solute is transported in the mean flow direction. It seems that the concentration inversely decreases as the distance from the source point increases. From the comparison of Cal-M1 and Cal-M2, it is seen that the overall patterns are more or less similar, but the concentration including the core region is increased as the horizontal diffusion is reduced. It is interesting to note that the main axis of the transport is formed with an angle to the horizontal level. This attributes to the presence of convective velocity. Consequently, contours reveal the pattern reducing near the sea surface. In the presence of large mean horizontal convective velocity (Cal-M3), the solute evidently remains near the sea surface. No wiggles have been noted in the solution except for the appearance of negative oscillations below the core region.

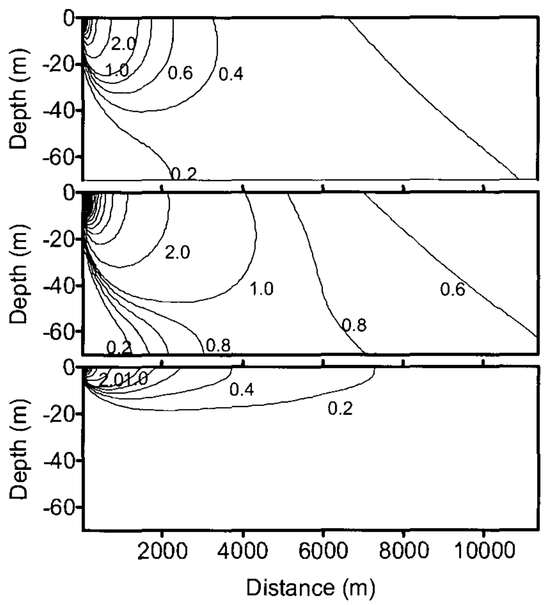

Fig. 3 Concentrations of solute computed in the presence of mean currents. Upper panel: u_m=0.02 m/s and k_x=0.2 m^2/s, Middle panel: u_m=0.02 m/s and k_x=0.02 m^2/s, Lower panel: u_m=0.2 m/s and k_x=0.2 m^2/s.

Calculations in the presence of mean and oscillatory flows

Three calculations denoted by Cal-T1 to Cal-T3 have been carried out in the presence of the horizontal convection via the mean and oscillatory currents. In these calculations u_m is set to 0.02 m/s, and the settling and the depositional velocity are set to w_s=-4×10^{-5} m/s and v_d=2×10^{-5} m/s, respectively. In Cal-T1 $u_{\max}$= 0.2 m/s is used with k_x=0.2 m^2/s, in Cal-T2 $u_{\max}$=0.2 m/s with k_x=2 m^2/s, and in Cal-T3 $u_{\max}$=0.4 m/s with k_x=2 m^2/s. Results shown in Fig. 4 are cross-sectional concentrations when the excursion in the x-direction reaches its maximum (that is, just at the time when the oscillatory current changes signs to the negative x-direction).

It is noted that Cal-T1 gives a very complicated pattern; the wiggles are pronounced and an unrealistic rise of concentration appears in an isolated form near the tidal excursion distance. Although results are not shown here, the wiggles in the result of Cal-T1 is still alive even with m=100. It has been found that use of a relatively high value of k_x almost eliminates the pattern (Cal-T2). However, with the use of increased $u_{\max}$ the concentration rise in an isolated form reappears (Cal-T3). In fact, this pattern has been previously noted in the depth-averaged two-dimensional solution derived by Jung et al (2003a) when the tidal convection dominates over the dispersion process. The longitudinal diffusion should be sufficiently large to get physically realistic features.

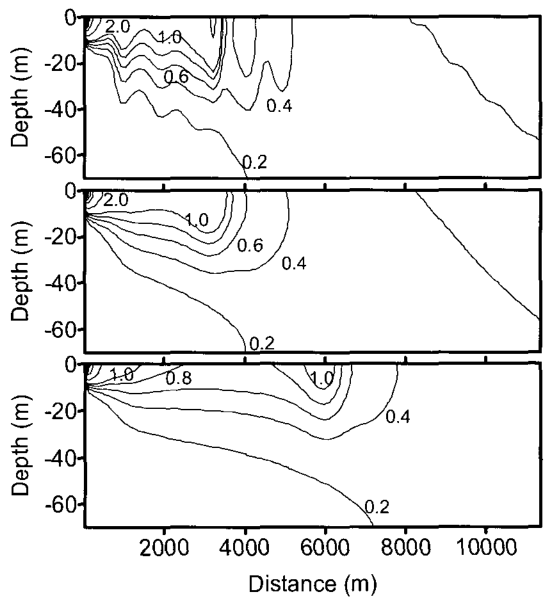

Fig. 4 Concentration computed in the presence of mean and oscillatory flows with u_m=0.02 m/s. Upper panel: $u_{\max}$=0.2 m/s and k_x=0.2 m^2/s, Middle panel: $u_{\max}$=0.2 m/s and k_x=2.0 m^2/s, Lower panel: $u_{\max}$=0.4 m/s and k_x=2.0 m^2/s.

Discussion

By adopting the approaches developed by Jung et al. (2003a,b), an analytical model for predicting the convection-diffusion of solute dumped in a homogeneous open sea of constant water depth has been developed in a time-integral form. The model incorporates spatially uniform, uni-directional, mean and oscillatory currents for horizontal convection, the settling velocity for the vertical convection and the anisotropic horizontal turbulent diffusion. The convection-diffusion equation has been reduced to Fickian type diffusion equation and then the Galerkin method is then applied via the expansion of eigenfunctions over the water column derived from the well-known Sturm-Liouville system.

A series of calculations has been performed to demonstrate the applicability of the model, including the calculations in stagnant water, the calculations with mean water flow, and the calculations with mean and oscillatory flows.

In the course of sensitivity calculations it has been found that the trickiest problem is to define the horizontal and the vertical eddy diffusivity coefficients, particularly in the presence of the oscillatory flow. The wiggles are pronounced in the concentration field and an unrealistic rise of concentration appears in an isolated form near the tidal excursion distance when the tidal convection dominates over the dispersion process. No spurious wiggles appear in the presence of mean current. More thorough comparison with field and laboratory experiments are required in the future but it might be needed to develop a variable horizontal diffusion model in which the coefficient increases downstream direction $k_x=\varepsilon ux$ (u is the velocity and x is the distance) as proposed by Hunt (1999).

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