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On The Product of Laplace and Bessel Random Variables

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Abstract

The distribution of the product $|XY|$ is derived when X and Y are Laplace and Bessel random variables distributed independently of each other.

1. INTRODUCTION

For given random variables X and Y, the distribution of the product $|XY|$ is of interest in problems in biological and physical sciences, econometrics, and classification. As an example in Physics, Sornette (1998) mentions:

> " … To mimic system size limitation, Takayasu, Sato, and Takayasu introduced a threshold x_c… and found a stretched exponential truncating the power-law pdf beyond x_c Frisch and Sornette recently developed a theory of extreme deviations generalizing the central limit theorem which, when applied to multiplication of random variables, predicts the generic presence of stretched exponential pdfs. The problem thus boils down to determining the tail of the pdf for a product of random variables … "

The distribution of $|XY|$ has been studied by several authors especially when X and Y are independent random variables and come from the same family. For instance, see Sakamoto (1943) for uniform family, Harter (1951) and Wallgren (1980) for Student's t family, Springer and Thompson (1970) for normal family, Stuart (1962) and Podolski (1972) for gamma family, Steece (1976), Bhargava and Khatri (1981) and Tang and Gupta (1984) for beta family, Abu-Salih (1983) for

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power function family, and Malik and Trudel (1986) for exponential family (see also Rathie and Rohrer (1987) for a comprehensive review of known results). However, there is relatively little work of this kind when X and Y belong to different families. In the applications mentioned above, it is quite possible that X and Y could arise from different but similar distributions.

In this note, we study the distribution of $|XY|$ when X and Y are independent random variables having the Laplace and Bessel function distributions with pdfs

$$f(x) = \frac{\lambda}{2} \exp(-\lambda |x|) \tag{1}$$

and

$$f(y) = \frac{|y|^m}{\sqrt{\pi} 2^m b^{m+1} \Gamma(m+1/2)} K_m\left(\left| \frac{y}{b} \right| \right) \tag{2}$$

respectively, for $\infty < x < \infty$, $\infty < y < \infty$, $\lambda > 0$, $b > 0$ and $m > 1$, where

$$K_m(x) = \frac{\sqrt{\pi} x^m}{2^m \Gamma(m+1/2)} \int_1^{\infty} (t^2 - 1)^{m-1/2} \exp(-xt) dt$$

is the modified Bessel function of the third kind. The calculations involve the Bessel function of the first kind defined by

$$J_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{1}{(\nu+1)_k k!} \left(-\frac{x^2}{4} \right)^k,$$

the modified Bessel function of the first kind defined by

$$I_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{1}{(\nu+1)_k k!} \left(\frac{x^2}{4} \right)^k,$$

and the hypergeometric function defined by

$$G(a, b, c; x) = \sum_{k=0}^{\infty} \frac{1}{(a)_k (b)_k (c)_k} \frac{x^k}{k!}$$

where $(e)_k = e(e+1) \cdots (e+k-1)$ denotes the ascending factorial. We also need the following important lemma.

LEMMA 1 (*Equation (2.16.8.9), Prudnikov et al., 1986, volume 2*) For $c > 0$ and $p > 0$,

$$
\begin{aligned}
&\int_0^{\infty} x^{\alpha-1} \exp(-p/x) K_{\nu}(cx)\, dx \\
&= \frac{2^{\alpha-2}}{c^{\alpha}} \Gamma\left(\frac{\alpha+\nu}{2}\right) \Gamma\left(\frac{\alpha-\nu}{2}\right) G\left(\frac{1}{2}, 1-\frac{\alpha+\nu}{2}, 1-\frac{\alpha-\nu}{2}; \frac{c^2 p^2}{16}\right) \\
&\quad - \frac{2^{\alpha-3} p}{c^{\alpha-1}} \Gamma\left(\frac{\alpha+\nu-1}{2}\right) \Gamma\left(\frac{\alpha-\nu-1}{2}\right) G\left(\frac{3}{2}, \frac{3-\alpha-\nu}{2}, \frac{3+\nu-\alpha}{2}; \frac{c^2 p^2}{16}\right) \\
&\quad + \frac{p^{\alpha+\nu} c^{\nu}}{2^{\nu+1}} \Gamma(-\nu) \Gamma(-\nu-\alpha) G\left(1+\nu, 1+\frac{\alpha+\nu}{2}, \frac{1+\nu+\alpha}{2}; \frac{c^2 p^2}{16}\right) \\
&\quad + \frac{p^{\alpha-\nu}}{2^{1-\nu} c^{\nu}} \Gamma(\nu) \Gamma(\nu-\alpha) G\left(1-\nu, 1+\frac{\alpha-\nu}{2}, \frac{1+\alpha-\nu}{2}; \frac{c^2 p^2}{16}\right).
\end{aligned}
$$

Further properties of the above special functions can be found in Prudnikov *et al.* (1986) and Gradshteyn and Ryzhik (2000).

2. CDF

Theorem 1 derives an explicit expression for the cdf of $|XY|$ in terms of the hypergeometric function.

THEOREM 1 *Suppose X and Y are distributed according to (1) and (2), respectively. The cdf of $Z = |XY|$ can be expressed as*

$$
\begin{aligned}
F(z) &= 1 - \left\{ \sqrt{\pi} 2^m b^{m+1} \Gamma\left(m+\frac{1}{2}\right) G\left(\frac{1}{2}, \frac{1}{2}-m, \frac{1}{2}; \frac{\lambda^2 z^2}{16 b^2}\right) \right. \\
&\quad + (2b)^{-m} (\lambda z)^{2m+1} \Gamma(-m) \Gamma(-2m-1)\, G\left(1+m, \frac{3}{2}+m, 1+m; \frac{\lambda^2 z^2}{16 b^2}\right) \\
&\quad \left. + \left(\frac{3C}{2}-1\right) (2b)^m \lambda z \Gamma(m)\, G\left(1-m, \frac{3}{2}, 1; \frac{\lambda^2 z^2}{16 b^2}\right) \right\} \\
&\quad \Big/ \left\{ \sqrt{\pi} 2^m b^{m+1} \Gamma\left(m+\frac{1}{2}\right) \right\}
\end{aligned}
\tag{3}
$$

where C denotes the Euler's constant.

PROOF: The cdf $F(z)=\Pr(|XY|\leq z)$ can be expressed as

$$
\begin{aligned}
F(z) &= \Pr(|X|\leq z/|Y|)\\
&= \frac{1}{\sqrt{\pi}2^{m}b^{m+1}\Gamma(m+1/2)}\int_{-\infty}^{\infty}\left\{1-\exp\left(-\frac{\lambda z}{|y|}\right)\right\}|y|^{m}K_{m}\left(\left|\frac{y}{b}\right|\right)dy\\
&= 1-\frac{1}{\sqrt{\pi}2^{m}b^{m+1}\Gamma(m+1/2)}\int_{-\infty}^{\infty}\exp\left(-\frac{\lambda z}{|y|}\right)|y|^{m}K_{m}\left(\left|\frac{y}{b}\right|\right)dy\\
&= 1-\frac{1}{\sqrt{\pi}2^{m-1}b^{m+1}\Gamma(m+1/2)}\int_{0}^{\infty}\exp\left(-\frac{\lambda z}{y}\right)y^{m}K_{m}\left(\frac{y}{b}\right)dy. \qquad (4)
\end{aligned}
$$

Application of Lemma 1 shows that the integral in (4) can be expressed as

$$
\begin{aligned}
&\int_{0}^{\infty}\exp\left(-\frac{\lambda z}{y}\right)y^{m}K_{m}\left(\frac{y}{b}\right)dy\\
&= \sqrt{\pi}2^{m-1}b^{m+1}\Gamma\left(m+\frac{1}{2}\right)G\left(\frac{1}{2},\frac{1}{2}-m,\frac{1}{2};\frac{\lambda^{2}z^{2}}{16b^{2}}\right)\\
&\quad -2^{m-2}b^{m}\lambda z\Gamma(m)\lim_{w\to 0}\Gamma(w)G\left(\frac{3}{2},1-m,1;\frac{\lambda^{2}z^{2}}{16b^{2}}\right)\\
&\quad +2^{-(m+1)}b^{-m}(\lambda z)^{2m+1}\Gamma(-m)\Gamma(-2m-1)G\left(1+m,\frac{3}{2}+m,1+m;\frac{\lambda^{2}z^{2}}{16b^{2}}\right)\\
&\quad +2^{m-1}b^{m}\lambda z\Gamma(m)\lim_{w\to -1}\Gamma(w)G\left(1-m,\frac{3}{2},1;\frac{\lambda^{2}z^{2}}{16b^{2}}\right)\\
&= \sqrt{\pi}2^{m-1}b^{m+1}\Gamma\left(m+\frac{1}{2}\right)G\left(\frac{1}{2},\frac{1}{2}-m,\frac{1}{2};\frac{\lambda^{2}z^{2}}{16b^{2}}\right)\\
&\quad +2^{m-2}b^{m}\lambda z\Gamma(m)\lim_{w\to 0}\{-\Gamma(w)+2\Gamma(w-1)\}G\left(\frac{3}{2},1-m,1;\frac{\lambda^{2}z^{2}}{16b^{2}}\right)\\
&\quad +2^{-(m+1)}b^{-m}(\lambda z)^{2m+1}\Gamma(-m)\Gamma(-2m-1)G\left(1+m,\frac{3}{2}+m,1+m;\frac{\lambda^{2}z^{2}}{16b^{2}}\right)
\end{aligned}
$$

where $\Gamma(0)$ and $\Gamma(-1)$ are interpreted as limits and the gamma function for negative numbers is defined through the relation $\Gamma(1-w)\Gamma(w)=\pi/(\sin(\pi w))$. The result of the theorem follows by noting that the limit of $-\Gamma(w)+2\Gamma(w-1)$ is $3C-2$, where C denotes the Euler's constant. ■

Using special properties of the hypergeometric function, one can derive simpler forms for the distribution of $|XY|$ when m takes half integer values.
This is illustrated in the corollary below.

<u>COROLLARY 1</u> *If* m = 3/2, 5/2, 7/2, 9/2, 11/2 *then (3) reduces to*

$$F(z)= -1/(8y)\{-8y-4I_0(2y)y-3I_0(2y)y^3C+2I_0(2y)y^3-4J_0(2y)y-3J_0(y)y^3C +2J_0(2y)y^3+8J_1(2y)+6I_1(2y)y^2C-4I_1(2y)y^2+8J_1(2y)+6J_1(2y)y^2C -4J_1(2y)y^2\},$$

$$F(z)= -1/(96y)\{-96y-80I_0(2y)y-45I_0(2y)y^3C+30I_0(2y)y^3-80J_0(2y)y -45J_0(2y)y^3C+30J_0(2y)y^3+128I_1(2y)+72I_1(2y)y^2C-32I_1(2y)y^2 +9I_1(2y)y^4C-6I_1(2y)y^4+128J_1(2y)+72J_1(2y)y^2C-64J_1(2y)y^2 -9J_1(2y)y^4C+6J_1(2y)y^4\},$$

$$F(z)= -1/(960y)\{-960y-1056I_0(2y)y-495I_0(2y)y^3C+298I_0(2y)y^3-15I_0(2y)y^5C +10I_0(2y)y^5-1056J_0(2y)y-495J_0(2y)y^3C+362J_0(2y)y^3+15J_0(2y)y^5C -10J_0(2y)y^5+1536I_1(2y)+720I_1(2y)y^2C-160I_1(2y)y^2+150I_1(2y)y^4C -100I_1(2y)y^4+1536J_1(2y)+720J_1(2y)y^2C-800J_1(2y)y^2-150J_1(2y)y^4C +100J_1(2y)y^4\},$$

$$F(z)= -1/(53760y)\{-1120J_0(2y)y^5+23626J_0(2y)y^3+15434I_0(2y)y^3-512I_1(2y)y^2 -6954I_1(2y)y^2-53248J_1(2y)y^2+7466J_1(2y)y^4-70I_1(2y)y^6-70J_1(2y)y^6 -71424I_0(2y)y-71424J_0(2y)y-29295I_0(2y)y^3-1680I_0(2y)y^5C -29295J_0(2y)y^3C+1680J_0(2y)y^5C+105I_1(2y)y^6C-10815J_1(2y)y^4C +105J_1(2y)y^6C+40320I_1(2y)y^2C+40320J_1(2y)y^2C+10815I_1(2y)y^4C +1120I_0(2y)y^5+98304I_1(2y)-53760y+98304J_1(2y)\},$$

$$F(z)= -1/(967680y)\{1966080I_1(2y)+1966080J_1(2y)-967680y+483042J_0(2y)y^3 -29618J_0(2y)y^5+126J_0(2y)y^7+227934I_0(2y)y^4C+4536I_1(2y)y^6C -227934J_1(2y)y^4C+4536J_1(2y)y^6C+725760I_1(2y)y^2C+725760J_1(2y)y^2C -547155I_0(2y)y^3C-43659I_0(2y)y^5C-189I_0(2y)y^7C-547155J_0(2y)y^3C +43659J_0(2y)y^5C-189J_0(2y)y^7C+28594I_0(2y)y^5-1101312J_1(2y)y^2 +164244J_1(2y)y^4+246498I_0(2y)y^3+126I_0(2y)y^7-3024I_1(2y)y^6 -1482240I_0(2y)y-3024J_1(2y)y^6-1482240J_0(2y)y+133632I_1(2y)y^2 -139668I(2y)y^4\},$$

where $y=\sqrt{\lambda z/b}$ *and* C *denotes the Euler's constant.*

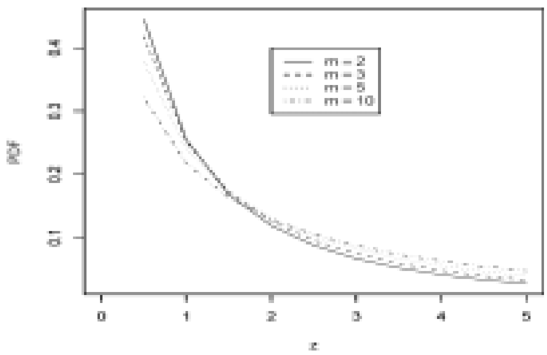

Figure 1. Plots of the pdf of (3) for $\lambda=1$, $b=1$ and $m=2,3,5,10$.

Figure 1 illustrates possible shapes of the pdf of (3) for $\lambda=1$, $b=1$, and a range of values of m. Note that the shapes are unimodal and that the value of m largely dictates the behavior of the pdf near $z=0$.

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