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Properties of VSI CUSUM Chart for Monitoring Dispersion Matrix[1)]

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Abstract

Properties of the variable sampling interval(VSI) CUSUM chart for monitoring dispersion matrix of related quality characteristics are investigated. Performances of the proposed charts are evaluated for matched fixed sampling interval(FSI) and VSI charts in terms of average time to signal(ATS) and average number of samples to signal (ANSS). Average number of swiches(ANSW) of the proposed VSI Shewhart and CUSUM charts are also investigated.

Keywords : ANSS, ANSW, ATS, VSI control chart

1. Introduction

Statistical process control is an effective method to improve process quality and process productivity. One of the most common tools of statistical process control is the statistical control chart.

Control charts are usually used for monitoring quality variables from a process to detect any shifts in the production process and eliminate assignable causes in the parameters of the distribution of these quality variables. The ability of a control chart is determined by the length of the time required for the chart to signal when the process is out-of-control state, and the frequency of false alarm when the process is in-control state. When the detection of small or moderate shifts in the process parameters is important, CUSUM chart is a good alternatives

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to the Shewhart chart.

Recent years, applications of VSI procedure have become quite frequent. For the VSI chart, Reynolds(1989) showed that the use of two sampling intervals spaced as apart as possible is optimal. In this paper, we also consider VSI procedures with two sampling time intervals d_1 and d_2($d_1 < d_2$).

One disadvantage of VSI scheme is that frequent switching between different sampling intervals requires more cost and effort to administer the process than corresponding FSI scheme. Amin and Letsinger(1991) described general procedures for VSI scheme and examined switching behavior and runs rules for switching between different sampling intervals on univariate $\overline{X}$-chart.

In many situtations, there exist multiple quality variables to define the quality of output and the quality is often characterized by joint levels of quality variables rather than a single quality variable. And shifts in the components of dispersion matrix for the related quality variables are often important.

The multivariate approach to quality control was first introduced by Hotelling (1947) and became popular in recent years. Woodall and Ncube(1985) considered a single multivariate CUSUM procedure for monitoring the means of multivariate normal process. Croiser(1988) and Pignatiello and Runger(1990) considered new multivariate CUSUM control schemes that accumulate past sample information for each parameter and then form a univariate CUSUM statistic from the multivariate data for monitoring the mean vector of a multivariate normal process.

In this paper, we investigate the properties of multivariate VSI control charts for monitoring dispersion matrix Σ in terms of ATS and ANSS where the target process mean vector $\underline{\mu}$ remained known constant. And we also investigate the ANSW of the proposed chart.

2. Description of Some Control Procedures

Assume that the quality vector $\underline{X}' = (X_1, X_2, \cdots, X_p)$ are jointly distributed as p-variate normal distribution $N_p(\underline{\mu}, \Sigma)$. We take a sequence of independent random vectors $\underline{X}_1, \underline{X}_2, \underline{X}_3, \cdots,$ where $\underline{X}_i = (\underline{X}'_{i1}, \cdots, \underline{X}'_{in})'$ is a sample of observations at the sampling time i and $\underline{X}_{ij} = (X_{ij1}, \cdots, X_{ijp})'$. Let $\underline{\theta}_0 = (\underline{\mu}_0, \Sigma_0)$ be the known target process parameters for $\underline{\theta} = (\underline{\mu}, \Sigma)$ of p quality variables, where $\underline{\mu}$ is mean vector and Σ is dispersion matrix of $\underline{X}'$.

2.1 Evaluating Sample Statistic

The general multivariate statistical quality control chart can be considered as a repetitive tests of significance where each quality characteristic is defined by p

quality variables $X_1, X_2, \cdots, X_p$. Therefore, we can obtain a sample statistic for momitoring variance-covariance matrix Σ by using the likelihood ratio test(LRT) statistic for testing $H_0 : \Sigma = \Sigma_0$ vs $H_1 : \Sigma \neq \Sigma_0$ where target mean vector of the quality variables μ_0 is known. The regions above the upper control limit(UCL) corresponds to the LRT rejection region. For the ith sample, likelihood ratio λ_i can be expressed as

$$\lambda_i = n^{-\frac{np}{2}} \cdot |A_i \Sigma_0^{-1}|^{\frac{n}{2}} \cdot \exp\left[-\frac{1}{2} tr(\Sigma_0^{-1} A_i) + \frac{1}{2} np\right].$$

Let TV_i be $-2 \ln \lambda_i$. Then

$$TV_i = tr(A_i \Sigma_0^{-1}) - n \ln |A_i| + n \ln |\Sigma_0| + np \ln n - np, \tag{2.1}$$

and, we use the LRT statistic TV_i as the sample statistic for Σ. If the sample statistic TV_i plots above the UCL, the process is deemed out-of-control state and assignable causes are sought.

Since it is difficult to obtain the exact distribution of TV_i when the process is in-control or out-of-control states, the performances of the proposed charts based on the sample statistic TV_i are obtained by simulation.

2.2 ATS and ANSW of VSI Procedure

For VSI chart, the sampling times are random variables and the sampling interval depends on the past sample informations of $X_1, X_2, \cdots, X_i$. Reynolds(1989) investigated the theoretical aspects of a VSI one- and two-sided Shewhart charts.

To implement two sampling interval VSI control scheme, the in-control region is divided into 2 regions I_1, I_2 where I_i is the region in which the sampling interval d_i is used $(i=1,2)$. In this paper, we assume that the VSI chart is started at time 0 and the interval used before the first sample, is a fixed constant, say d_0. Then the ATS can be expressed as

$$ATS = d_0 + d_1 \psi_1 + d_2 \psi_2 \tag{2.2}$$

where ψ_i is the expected number of samples before the signal.

The VSI procedures have been shown to be more efficient when compared to the corresponding FSI procdures with respect to the ATS. But, frequent switching between the different sampling intervals d_1 and d_2 can be a complicating factor in the application of control charts with VSI procedures. Therefore, it is necessary to define the number of swiches(NSW) as the number of swiches made from the start of the process until the chart signals, and let the average number of swiches (ANSW) be the expected value of the NSW.

The ANSW can be obtained as follows

$$ANSW = ARL \cdot P(\text{swich}) \tag{2.3}$$

And, the probability of swich is given by

$$P(\text{swich}) = P(d_1) \cdot P(d_2 | d_1) + P(d_2) \cdot P(d_1 | d_2) \tag{2.4}$$

where $P(d_i)$ is the probability of using sampling interval d_i, and $P(d_i | d_j)$ is the conditional probability of using sampling interval d_i in the current sample given that the sampling interval d_j $(d_i \neq d_j)$ was used in the previous sample.

3. Shewhart Control Chart

The Shewhart chart has a good ability to detect large changes in monitored parameter quickly and is easy to implement the process. However, the Shewhart chart is slow to detect small or moderate changes of the parameters.

The control limits for the FSI Shewhart chart based on the LRT statistic TV_i would be set by using percentage point of TV_i, and the chart signals whenever

$$TV_i \geq h_{TV(S)}. \tag{3.1}$$

And for VSI Shewhart chart based on TV_i, suppose that the sampling interval ;

$$\begin{aligned} d_1 &\text{ is used when } TV_i \in (g_{TV(S)},\, h_{TV(S)}], \\ d_2 &\text{ is used when } TV_i \in (0,\, g_{TV(S)}], \end{aligned}$$

where $g_{TV(S)} \leq h_{TV(S)}$ and $d_1 < d_2$.

Since it is difficult to obtain the exact distribution of LRT statistic TV_i, the design parameters $g_{TV(S)}$ and $h_{TV(S)}$ can be obtained to satisfy a desired ATS and ANSS by simulation.

4. CUSUM Control Chart

CUSUM chart is maintained by taking samples and plotting a cumulative sum of differences between sample statistic and the target value in time order on the chart. The CUSUM chart is efficient when the detection of small shifts in a production process is important.

For FSI CUSUM chart based on TV_i can also be constructed as

$$Y_{TV,i} = \max\{Y_{TV,i-1}, 0\} + (TV_i - k_{TV}), \tag{4.1}$$

where $Y_{TV,i} = \omega_{TV} \cdot I_{(\omega_{TV} \ge 0)}$ and the reference value $k_{TV} \ge 0$. This multivariate CUSUM chart signals whenever $Y_{TV,i} \ge h_{TV(C)}$.

Since it is difficult to obtain the exact distribution of the chart statistic in (4.1), the percentage point and properties of this chart can be evaluated by simulation under the process parameters of the process are on-target or changed.

And for VSI CUSUM chart based on TV_i, suppose that the sampling interval ;

$$\begin{aligned} d_1 &\text{ is used when } Y_{TV,i} \in (g_{TV(C)}, h_{TV(C)}], \\ d_2 &\text{ is used when } Y_{TV,i} \in (-k_{TV}, g_{TV(C)}], \end{aligned}$$

where $g_{TV(C)} \le h_{TV(C)}$ and $d_1 < d_2$.

The design parameters $g_{TV(C)}$ and $h_{TV(C)}$ can be obtained to satisfy a desired ATS and ANSS by simulation.

5. Concluding Remarks

In order to evaluate the performances and compare the properties of the proposed charts, the charts should have the same ANSS and ATS when the process is in-control and some kinds of standards for comparison are necessary.

For simplicity in our numerical computation, we assume that target mean vector $\underline{\mu}_0 = \underline{0}'$, all diagonal elements of Σ_0 are 1 and off-diagonal elements of Σ_0 are 0.3. The numerical results were obtained when the ANSS and ATS of the in-control state was approximately equal to 370.4, $d_0 = 1$ and the sample size for each variable was five for $p = 3$ and 4.

Since the performance of the charts depends on the components of Σ, it is not possible to investigate all of the different ways in which Σ could change. Hence, we consider the following typical types of shifts for comparison in the process parameters.

(1) V_i : σ_{10} of Σ_0 is increased to $[1+(4i-3)/10]$.
(2) C_i : ρ_{120} and ρ_{210} of Σ_0 are changed to $[0.3+(2i-1)/10]$
(3) (V_i, C_i) for $i=1,2,3$.
(4) S_i : Σ_0 is changed to $c_i\Sigma_0$ where $c_i=[1+(3i-2)/10]^2$.

[Table 1] Properties of the Shewhart chart for Σ

types of shift	$p=3$			$p=4$		
	ANSS	ATS	ANSS	ANSS	ATS	ANSW
in-control	370.4	370.4	184.39	370.4	370.4	184.13
V_1	340.6	335.2	169.40	357.3	353.4	177.59
V_2	53.9	39.9	23.78	118.7	94.0	55.21
V_3	8.8	4.9	2.39	19.3	11.1	6.51
C_1	354.4	350.1	176.26	359.7	356.6	178.70
C_2	244.7	213.2	118.84	291.6	266.1	143.51
C_3	101.3	55.2	36.70	166.4	113.1	71.66
(V_1, C_1)	328.2	320.4	163.09	349.6	343.6	173.69
(V_2, C_2)	46.3	32.3	19.56	100.7	75.7	45.54
(V_3, C_3)	6.9	3.2	1.26	14.2	6.7	3.70
S_1	299.6	286.4	148.50	326.6	313.6	161.94
S_2	27.2	15.7	9.60	47.9	27.1	17.36
S_3	4.3	1.9	0.47	5.7	2.2	0.62
S_4	1.8	1.1	0.03	2.0	1.2	0.03

[Table 2] Properties of the CUSUM chart for Σ ($p=3$)

types of shift	$k_{TV}=9.0$			$k_{TV}=9.5$			$k_{TV}=10.0$		
	ANSS	ATS	ANSS	ANSS	ATS	ANSW	ANSS	ATS	ANSW
in-control	370.3	370.4	71.98	370.3	370.4	94.69	370.3	370.4	114.99
V_1	302.2	289.5	57.87	311.9	301.0	78.81	318.0	308.3	97.76
V_2	24.5	12.3	2.92	25.2	12.9	3.91	26.4	14.2	5.17
V_3	7.2	3.5	0.73	6.6	3.1	0.70	6.3	3.0	0.72
C_1	321.6	311.7	61.89	329.1	320.7	83.45	334.2	326.9	103.06
C_2	103.5	72.7	16.55	119.9	89.1	26.33	136.4	105.5	37.70
C_3	18.5	7.6	1.66	19.1	7.4	2.05	20.9	8.1	2.78
(V_1, C_1)	274.5	257.6	52.16	288.6	273.7	72.39	296.8	283.6	90.88
(V_2, C_2)	20.3	9.7	2.27	20.4	9.8	2.89	21.3	10.6	3.77
(V_3, C_3)	5.7	2.6	0.47	5.2	2.3	0.4	4.9	2.2	0.41
S_1	213.8	189.0	39.57	229.0	206.9	56.31	242.0	221.7	72.8
S_2	12.3	5.5	1.22	11.9	5.1	1.34	11.9	5.1	1.55
S_3	4.2	1.9	0.28	3.8	1.8	0.23	3.5	1.6	0.20
S_4	2.4	1.3	0.07	2.1	1.2	0.04	2.0	1.2	0.03

[Table 3] Properties of the CUSUM chart for Σ ($p=4$)

	$k_{TV}=16.0$			$k_{TV}=16.5$			$k_{TV}=17.0$		
types of shift	ANSS	ATS	ANSS	ANSS	ATS	ANSW	ANSS	ATS	ANSW
in-control	370.4	370.4	38.81	370.4	370.4	54.67	370.4	370.4	70.33
V_1	316.3	303.7	32.75	322.8	311.9	47.12	325.8	316.7	61.36
V_2	43.5	24.5	3.35	41.8	22.8	4.29	42.7	23.5	5.53
V_3	13.8	7.2	1.19	12.0	5.7	1.12	11.0	5.0	1.12
C_1	331.8	321.3	34.41	334.7	326.3	48.95	339.9	332.9	64.11
C_2	130.8	95.8	11.74	140.9	107.3	18.15	153.6	121.2	25.94
C_3	33.8	17.4	2.29	32.1	15.2	2.72	32.8	15.0	3.36
(V_1, C_1)	290.7	272.9	29.86	298.0	282.8	43.19	305.9	292.8	57.30
(V_2, C_2)	35.7	19.6	2.71	33.6	17.5	3.29	33.9	17.4	4.09
(V_3, C_3)	10.7	5.4	0.96	9.2	4.2	0.82	8.4	3.6	0.75
S_1	215.1	185.3	21.34	226.6	199.8	31.82	237.2	213.3	43.15
S_2	17.9	9.0	1.41	15.7	7.1	1.36	14.7	6.3	1.39
S_3	6.3	3.0	0.58	5.3	2.4	0.40	4.8	2.0	0.31
S_4	3.5	1.7	0.21	2.9	1.4	0.11	2.6	1.3	0.07

After the design parameters h and g of the proposed Shewhart and CUSUM charts have been determined, the ANSS, ATS and ANSW values of the proposed types of shifts were obtained by simulation with 10,000 runs.

The properties and comparison of the proposed procedures are given in [Table 1] through [Table 3]. From the numerical results, we found the following properties. VSI schemes are more efficient than FSI schemes in terms of ATS. The results in [Table 2] and [Table 3] show that large reference value k is efficient for large shifts from the target value and smaller reference value k is efficient for small shifts of the parameters in terms of ANSS and ATS.

We also found that the CUSUM procedures exhibit far fewer switches when compared to the Shewhart procedure. Also, it was established that smaller values of the parameters k for CUSUM procedures will reduce the NSW between two sampling intervals, respectively.

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