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Optimal M-level Constant Stress Design with K-stress Variables for Weibull Distribution

Gyoung-Ae Moon[1)]

Abstract

Most of the accelerated life tests deal with tests that use only one accelerating variable and no other explanatory variables. Frequently, however, there is a test to use more than one accelerating or other experimental variables, such as, for examples, a test of capacitors at higher than usual conditions of temperature and voltage, a test of circuit boards at higher than usual conditions of temperature, humidity and voltage. A accelerated life test is extended to M-level stress accelerated life test with k-stress variables. The optimal design for Weibull distribution is studied with k-stress variables.

Keywords : Accelerated life test, Constant stress, Weibull distribution.

1. Introduction

In many reliability studies, it may require a long testing times because the lifetimes of test units under the usual conditions tend to be long for extremely reliable units. As a common approach to shorten the lifetimes of test units, the accelerated life testings are widely used. Accelerated life testing quickly yields information on test unit. Testing units are subjected to conditions of greater stress and fail sooner than the usual conditions.

Using data from accelerated conditions, a model is fitted and then extrapolated to make inferences on the lifetimes, the reliability, failure rates, etc. under the usual conditions. Widely used methods of applying stress to test units are the constant stress test and the step stress test.

In constant stress testing, a test unit is subjected to a fixed stress and observed until it fails or is removed(censored). Meeker(1984) and Meeker and

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1) Assistant Professor, Department of Computer Engineering, Donghae University, Donghae, Gangwondo, 240-713, Korea
E-Mail : diana62@donghae.ac.kr

Nelson(1975) considered the design for Type I censored constant stress accelerated life tests(ALTs) and gave the design the optimal test conditions and sample allocation. Nelson(1980, 1983, 1990) presented the cumulative exposure model analyzing the data from the step stress ALTs and studied the design to determine the optimal stress change time. Bai, Kim and Lee(1989) and Miller and Nelson(1983) obtained the stress change time which minimizes the asymptotic variance of maximum likelihood estimate of the log scale parameter at the design condition. Bai and Chung(1992) studied two optimal designs and compared the performances of two-step stress and constant stress partially ALTs under the tampered random variable model proposed DeGroot and Goel(1979). Khamis and Higgins(1996) derived optimum three-step stress test and evaluated several compromise plans when the lifetime of test unit for any stress is exponential. Khamis(1997a) studied the optimum designs for two-step stress and constant stress ALTs, and compared two tests under Weibull models. Khamis(1997b) also studied the M-step stress test with K-stress variables for exponential distribution.

In this paper, we consider the optimal designs for M-level constant stress ALTs with k-stress variables under Weibull distribution. In constant stress tests, one might expect to have censored data especially at the lower levels, while there would be little or no censoring at higher levels. We derive the optimal sample allocations when censoring occurs at the lower stress and the shape parameter is known. An appropriate model for log mean failure time is fit to data and the estimate of log mean failure time at usual condition is obtained. Maximum likelihood estimators(MLEs) of the parameters are also obtained, and the Fisher information matrix is derived.

2. Optimal M-level constant stress design with k-stress variables

Testing is done with M-level constant stress vectors $(x_{11}, x_{21}, \cdots, x_{k1})$, $(x_{12}, x_{22}, \cdots, x_{k2})$, $\cdots$, $(x_{1m}, x_{2m}, \cdots, x_{km})$, where $x_{li} \leq x_{lj}$, $i \leq j$ and $k+1 \leq m$, and the life distribution of the test unit for any stress is Weibull with known shape parameter.

For the M-level constant stress ALTs, n_i, $i = 1, 2 \cdots, m$ units randomly chosen from n test units are put on each stress, and they are run until either failure occurs or censoring occurs at preassigned censoring time W. n_{ui} is the number of test units failed at lower stress and n_{ci} is the number of test units that are censored at a fixed censoring time W.

The scale parameter θ_i at stress $(x_{1i}, x_{2i}, \cdots, x_{ki})$ is given by

$$\log\theta_i = \gamma_0 + \sum_{l=1}^{k} \gamma_i x_{li}, \quad i = 1, 2, \cdots, m. \tag{1}$$

In the presentation of our results, and without loss of generality, we use the

$$y_{li} = \frac{x_{li} - x_{l0}}{x_{lm} - x_{l0}}, \ l = 1, 2, \cdots, k, \ , i = 1, 2, \cdots, m.$$

The model is

$$\log\theta_i = \beta_0 + \sum_{l=1}^{k} \beta_i y_{li}, \quad i = 1, 2, \cdots, m. \tag{2}$$

The probability density function(p.d.f) for Weibull distribution under the constant stress ALTs with Type-I censoring at lower stress level can be written as

$$f_i(w) = \left[\frac{\delta_i}{\theta_i} w_i^{\delta_i - 1} exp\left(-\frac{w_i^{\delta_i}}{\theta_i}\right)\right]^{c_i} \left[\exp\left(-\frac{w_i^{\delta_i}}{\theta_i}\right)\right]^{1 - c_i}, \quad 0 \leqq w_i \leqq W, \tag{3}$$

where $c_i = \begin{cases} 1, & w_i \leqq W \\ 0, & w_i > W \end{cases}, \ i = 1, 2, \cdots, m.$

The lifetimes of test units are independent and identically distributed. The Weibull distribution with known shape parameter δ_i, $i = 1, 2, \cdots, m$ is transformed to the exponential distribution using the transformation $T_{ij} = W_{ij}^{\delta_i}$. That is

$$f_i(t) = \left[\frac{1}{\theta_i} \exp\left(-\frac{t_i}{\theta_i}\right)\right]^{c_i} \left[\exp\left(-\frac{t_i}{\theta_i}\right)\right]^{1 - c_i}, \quad 0 \leqq t_i \leqq T, \tag{4}$$

where $c_i = \begin{cases} 1, & t_i \leqq T \\ 0, & t_i > T \end{cases}, \ i = 1, 2, \cdots, m.$

The likelihood function from observation $T_{ij} = t_{ij}$, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n_i$ is

$$L(\theta_1, \theta_2, \cdots, \theta_m) = \Pi_{i=1}^{m} \Pi_{j=1}^{n_{ui}} \left(\frac{1}{\theta_i} \exp\left(-\frac{t_{ij}}{\theta_i}\right)\right) \Pi_{j=1}^{n_{ci}} \left(\exp\left(-\frac{T}{\theta_i}\right)\right), \tag{5}$$

where n_{ui} = the number of units failed at i-th stress vector,

n_{ci} = the number of censored at i-th stress vector.

Substituting (2) for θ_i, $i=1,2,\cdots,m$ in (5), the log-likelihood function is given with unknown parameters $\beta_0,\beta_1,\cdots,\beta_k$ as follows;

$$\begin{aligned}\log L(\beta_0,\beta_1,\cdots,\beta_k) &= -\sum_{i=1}^{m} n_{ui}(\beta_0+\sum_{l=1}^{k}\beta_l y_{li}) - \sum_{i=1}^{m}\sum_{j=1}^{n_{ui}} t_{ij}\exp(-\beta_0-\sum_{l=1}^{k}\beta_l y_{li}) \\ &\quad -\sum_{i=1}^{m} n_{ci}T\exp(-\beta_0-\sum_{l=1}^{k}\beta_l y_{li}) \qquad (6) \\ &= -\beta_0\sum_{i=1}^{m} n_{ui} - \sum_{l=1}^{k}\beta_l\sum_{i=1}^{m} n_{ui}y_{li} - \sum_{i=1}^{m} U_i\exp(-\beta_0-\sum_{l=1}^{k}\beta_l y_{li}),\end{aligned}$$

where $U_i=\sum_{j=1}^{n_{ui}} t_{ij}+n_{ci}T$ and it is the total test time at i-th stress vector $(x_{1i},x_{2i},\cdots,x_{ki})$, $i=1,2,\cdots,m$.

Maximum likelihood estimators(MLEs) for the model parameters $\beta_0,\beta_1,\cdots,\beta_k$ can be obtained by solving the following equation using the Newton Raphson method.

$$\frac{\partial\log L(\beta_0,\beta_1,\cdots,\beta_k)}{\partial\beta_0} = -\sum_{i=1}^{m} n_{ui} + \sum_{i=1}^{m} U_i\exp(-\beta_0-\sum_{l=1}^{k}\beta_l y_{li}),$$
$$\frac{\partial\log L(\beta_0,\beta_1,\cdots,\beta_k)}{\partial\beta_s} = -\sum_{i=1}^{m} n_{ui}y_{si} + \sum_{i=1}^{m} U_i y_{si}\exp(-\beta_0-\sum_{l=1}^{k}\beta_l y_{li}),$$

for $s=1,2,\cdots,k$.

The Fisher information matrix is obtained by taking the expected value of the second partial and mixed partial derivatives of $\log L(\beta_0,\beta_1,\cdots,\beta_k)$ with respect to $\beta_0,\beta_1,\cdots,\beta_k$.

$$\frac{\partial\log L(\beta_0,\beta_1,\cdots,\beta_k)}{\partial\beta_0^2} = -\sum_{i=1}^{m} U_i\exp(-\beta_0-\sum_{l=1}^{k}\beta_l y_{li}),$$
$$\frac{\partial\log L(\beta_0,\beta_1,\cdots,\beta_k)}{\partial\beta_0\partial\beta_s} = -\sum_{i=1}^{m} U_i y_{si}\exp(-\beta_0-\sum_{l=1}^{k}\beta_l y_{li}),$$
$$\frac{\partial\log L(\beta_0,\beta_1,\cdots,\beta_k)}{\partial\beta_s^2} = -\sum_{i=1}^{m} U_i y_{si}^2\exp(-\beta_0-\sum_{l=1}^{k}\beta_l y_{li}),$$
$$\frac{\partial\log L(\beta_0,\beta_1,\cdots,\beta_k)}{\partial\beta_s\partial\beta_t} = -\sum_{i=1}^{m} U_i y_{si}y_{ti}\exp(-\beta_0-\sum_{l=1}^{k}\beta_l y_{li}),$$

where $s\neq t=1,2,\cdots,k$. It can be seen that

$$F = n\begin{pmatrix} \sum_{i=1}^{m} A_i, & \sum_{i=1}^{m} A_i y_{1i}, & \sum_{i=1}^{m} A_i y_{2i}, & \cdots, & \sum_{i=1}^{m} A_i y_{ki} \\ \sum_{i=1}^{m} A_i y_{1i}, & \sum_{i=1}^{m} A_i y_{1i}^2, & \sum_{i=1}^{m} A_i y_{1i} y_{2i}, & \cdots, & \sum_{i=1}^{m} A_i y_{1i} y_{ki} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{i=1}^{m} A_i y_{ki}, & \sum_{i=1}^{m} A_i y_{1i} y_{ki}, & \sum_{i=1}^{m} A_i y_{2i} y_{ki}, & \cdots, & \sum_{i=1}^{m} A_i y_{ki}^2 \end{pmatrix}, \tag{7}$$

where $A_i = E(\frac{n_{ui}}{n}) = \frac{n_i}{n}\left(1-\exp\left(-\frac{T}{\theta_i}\right)\right) = \phi_i p_i,\ i=1,2,\cdots,m.$

We consider the optimal design with $k+1$ stress vectors, which is the minimum number of stress vectors needed to fit the model in (2). The asymptotic variance multiplied by sample size, $nAVC$, of the MLEs of the log scale parameter at the usual stress $(y_{10}, y_{20}, \cdots, y_{k0})$ is then given by

$$nAVC = n(1, y_{10}, y_{20}, \cdots, y_{k0})F^{-1}(1, y_{10}, y_{20}, \cdots, y_{k0})^t. \tag{8}$$

By differentiating (8) with respect to $\phi_i,\ i=1,2,\cdots,m$ and equating to zero, the optimal sample proportions $\phi_i^*,\ i=1,2,\cdots,m$ to be allocated at stress vector $(y_{1i}, y_{2i}, \cdots, y_{ki}),\ i=1,2,\cdots,m$ can be found, which minimize the asymptotic variance.

In practice, to find the optimal design, we must approximate the parameters by experience, similar data or preliminary test.

Example 1 : we consider $k=2$ as a special case. Suppose an optimal 3-level stress design with two stress variables is to be planned when the model is

$$\log\theta_i = \beta_0 + \beta_1 y_{1i} + \beta_2 y_{2i}.$$

Then the asymptotic variance multiplied by the sample size at the usual stress is given by

$$nAVC = \frac{d_1^2}{A_1} + \frac{d_2^2}{A_2} + \frac{d_3^2}{A_3},$$

where $A_i = \phi_i p_i,\ i=1,2,3$ and

$$d_1^2 = \frac{[(y_{12}y_{23} - y_{13}y_{22}) - y_{10}(y_{23} - y_{22}) - y_{20}(y_{12} - y_{13})]^2}{[y_{11}(y_{22} - y_{23}) + y_{12}(y_{23} - y_{21}) + y_{13}(y_{21} - y_{22})]^2},$$
$$d_2^2 = \frac{[(y_{11}y_{23} - y_{13}y_{21}) - y_{10}(y_{23} - y_{21}) - y_{20}(y_{11} - y_{13})]^2}{[y_{11}(y_{22} - y_{23}) + y_{12}(y_{23} - y_{21}) + y_{13}(y_{21} - y_{22})]^2},$$
$$d_3^2 = \frac{[(y_{11}y_{22} - y_{12}y_{21}) - y_{10}(y_{22} - y_{21}) - y_{20}(y_{11} - y_{12})]^2}{[y_{11}(y_{22} - y_{23}) + y_{12}(y_{23} - y_{21}) + y_{13}(y_{21} - y_{22})]^2}.$$

Suppose that $\beta_0 = 0$, $\beta_1 = -1.0$ and $\beta_2 = -5.0$, and the stress vectors are $(0.2, 0.3)$, $(0.2, 0.6)$ and $(1.0, 1.0)$. If $p_1 = 0.6$ is assumed, then $p_2 = 0.98$ and $p_3 = 1.0$. The optimal sample proportions are obtained by $\phi_1^* \approx 0.8$, $\phi_2^* \approx 0.1$ and $\phi_3^* \approx 0.1$, respectively when $nAVC = 8.053$.

Example 2 : The 40 simulated sample is given in Table 1 when β_0, β_1 and β_2 have the same values as in Example 1 and the number of data on 3-level stress are $n_1 = 32$, $n_2 = 4$ and $n_3 = 4$. If $p_1 = 0.6$ is assumed, then the censoring time is given by $T = 0.16739$. Then we fit the following model

$$\log \theta_i = \beta_0 + \beta_1 y_{1i} + \beta_2 y_{2i}.$$

The MLEs of β_0, β_1 and β_2 by Newton-Raphson methods are obtained as

$$\hat{\beta}_0 = 0.0162, \quad \hat{\beta}_1 = -1.0482, \quad \hat{\beta}_2 = -4.8626.$$

And the observed information matrix, $\hat{F}$, and covariance matrix, $\hat{F}^{-1}$, are given as follows:

$$\hat{F} = \begin{pmatrix} 26.00 & 8.40 & 11.80 \\ 8.40 & 4.88 & 5.56 \\ 11.80 & 5.56 & 7.06 \end{pmatrix},$$

and

$$\hat{F}^{-1} = \begin{pmatrix} 0.2133 & 0.3800 & -0.6559 \\ 0.3800 & 2.6717 & -2.7392 \\ -0.6559 & -2.7392 & 3.3951 \end{pmatrix},$$

which was determined by substituting the estimated values $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\beta}_2$ in the asymptotic covariance matrix.

To find the 95% confidence intervals for parameters, β_0, β_1 and β_2, we can get the standard errors of $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\beta}_2$ by taking the square root of the diagonal elements of $\hat{F}^{-1}$, and the 95% confidence intervals, $\hat{\beta}_i \pm z_{0.025} SE(\hat{\beta}_i)$ for β_0, β_1 and β_2 are given by

$$-0.889 \leqq \beta_0 \leqq 0.921,$$
$$-4.252 \leqq \beta_1 \leqq 2.155,$$
$$-8.474 \leqq \beta_2 \leqq -1.251.$$

Table 1. Simulated data with 2-stress variables based on $\beta_0 = 0$, $\beta_1 = -1.0$, $\beta_2 = -5.0$ and $T = 0.1674$

level	stress	Failure times				
1	$(y_{11}, y_{21}) = (0.2, 0.3)$	0.0042	0.0154	0.0165	0.0196	0.0236
		0.0283	0.0378	0.0451	0.0504	0.0553
		0.0575	0.0701	0.0793	0.0854	0.1022
		0.1181	0.1315	0.1654	0.1674	0.1674
		0.1674	0.1674	0.1674	0.1674	0.1674
		0.1674	0.1674	0.1674	0.1674	0.1674
		0.1674	0.1674			
2	$(y_{12}, y_{22}) = (0.2, 0.6)$	0.0026	0.0305	0.0543	0.0908	
3	$(y_{13}, y_{23}) = (1.0, 1.0)$	0.0005	0.0016	0.0037	0.0052	

In order to use this optimal design, unknown parameters β_0, β_1 and β_2 must be approximated by the past data set or preliminary test. However, the wrong pre-estimated values of parameters may not lead to optimal sample proportions and result in the poor estimators of parameters at the usual condition. Thus, the effects of the pre-estimated values of parameters are investigated. The wrong values of β_0, β_1 and β_2 lead to wrong values of θ_i and then p_i, $i = 1, 2, 3$. So, the true values of p_1 and p_2 are assumed to be 0.6 and 0.8, respectively.

The behaviors of $nAVC$ relative to optimal $nAVC$ due to wrong pre-estimated values of p_1 and p_2 are shown in Figure 1. If the pre-estimated value of p_1 is not too far from true value, the $nAVC$ is likely to be stable even though the pre-estimated values of p_2 is far from true value whereas the $nAVC$ seems to be very sensitive to wrong values of p_1. This means that data gathered from the lower stresses plays more important role in estimating parameters at the usual conditions.

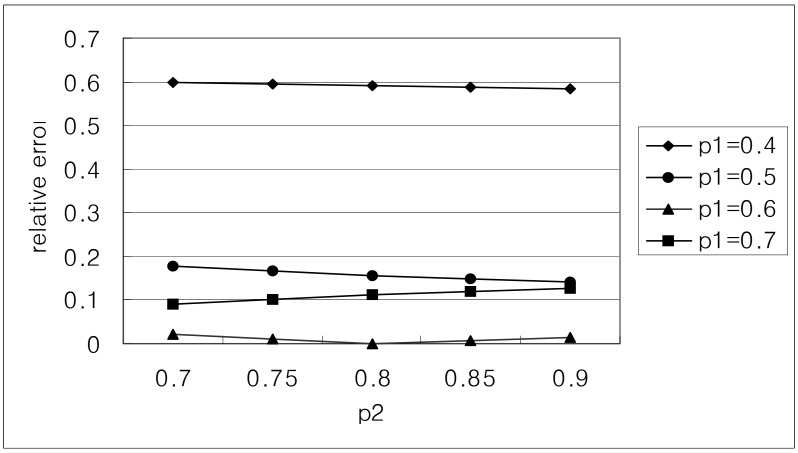

Figure 1. The relative errors of $nAVC$

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