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The First Passage Time of Stock Price under Stochastic Volatility

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Abstract

This paper gives an approximation to the distribution function of the .rst passage time of stock price when volatility of stock price is modeled by a function of Ornstein-Uhlenbeck process. It also shows how to obtain the error of the approximation.

Keywords : First Passage Time, Stochastic Volatility

1. Introduction

Traditionally, we assume that a stock price process X_t satisfies

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t.$$

For example, Black and Scholes assumed that $\mu(t,x) = \mu x$ and $\sigma(t,x) = \sigma x$ with σ, the volatility, is a constant. However, empirical data show that the constant volatility assumption is generally not true. Therefore, as suggested by Fouque-Papanicolaou-Siscar [4], we now try to refine Black and Scholes model by modeling the volatility σ as a stochastic process driven by some other process. In other words, we assume $\sigma = \sigma_t = f(Y_t)$ where Y_t is some stochastic process that drives σ and f is some positive continuous function. From modeling point of view and from tractability, Ornstein-Uhlenbeck is one of the good candidates for Y. So let Y be a mean-reverting Ornstein-Uhlenbeck process satisfying

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$$dY_t = \alpha(m - Y_t)dt + \beta\sqrt{2\alpha}dB_t, \quad Y_0 = y,$$

where m is the long run mean level of Y and α is the rate of mean reverting. There is a minor problem with this choice for Y. The model implies that the stock price X and the volatility are perfectly correlated, and this is not true in general.

Instead, volatility is modeled to have an independent random component of its own. Thus we will assume Y is an Ornstein-Uhlenbeck process satisfying

$$dY_t = \alpha(m - Y_t)dt + \beta\sqrt{2\alpha}d\hat{Z}_t, \quad Y_0 = y,$$

where $\hat{Z}$ is another Brownian motion typically correlated with the Brownian motion B_t driving the stock price. The instantaneous correlation coefficient defined by

$$d\langle B, \hat{Z}\rangle t = \rho dt,$$

for some $\rho \in [-1, 1]$ It is also convenient to write

$$\hat{Z}_t = \rho B_t + \sqrt{1-\rho^2}Z_t,$$

where Z is a standard Brownian motion independent of B. Z can be thought of the source of additional randomness in the volatility fluctuations [5]. As a result, we now look at a stock price process X_t satisfying

$$dX_t = \mu X_t dt + f(Y_t)X_t dB_t \quad \text{with}$$
$$dY_t = \alpha(m - Y_t)dt + \beta\sqrt{2\alpha}(\rho dB_t + \sqrt{1-\rho^2}dZ_t)$$

Before we go on, let us state a few well-known facts about the process Y.

Proposition 1.1 : Suppose that Y satisfies

$$dY_t = \alpha(m - Y_t)dt + \beta\sqrt{2\alpha}d\tilde{B}_t, \quad Y_0 = y$$

Then :

a. $Y_t = m + e^{-\alpha t}(y - m) + \beta\sqrt{2\alpha}\int_0^t e^{-\alpha(t-s)}d\tilde{B}_s.$

b. So that $Y_t \sim N(m + e^{-\alpha t}(y-m),\ \beta^2(1-e^{-2\alpha t}))$.

c. The process Y is stationary, ergodic. In fact, with the appropriate sampling distribution for Y_0, $Cov(Y_t, Y_s) = \rho(t,s) = \beta^2 e^{-\alpha|t-s|}$.

d. The invariant distribution of Y_t (obtained by letting $t \to \infty$) is $N(m, \beta^2)$.

e. $Y_t \sim \widetilde{Y}_{\alpha t}$ where $\widetilde{Y}$ satisfies $d\widetilde{Y}_t = (m - \widetilde{Y}_t)dt + \beta\sqrt{2}d\widetilde{B}_t$, $\widetilde{Y}_0 = y$

Proof :

The proof of this proposition can be found in many standard text books. (See [1], [2],...). In particular, part (e) is a straight forward application of the time change theorem.

2. Partial Solution to The Problem

For any stochastic process X, let T_b^X be the first passage time of X to a level b, i.e., $T_b^X = \inf\{t > 0 : X_t \geq b\}$.

We are going to approximate the distribution of T_b^X for the above problem. We first need a result that is similar to the egordic theorem.

Lemma 2.1 : Suppose $\widetilde{Y}$ is an Ornstein-Uhlenbeck process satisfying

$$d\widetilde{Y}_t = (m - \widetilde{Y}_t)dt + \beta\sqrt{2}d\widetilde{B}_t, \quad \widetilde{Y}_0 = y$$

Then, for any bounded function g

$$\lim_{\alpha\to\infty}\int_0^t g(\widetilde{Y}_{\alpha s})ds = tE_\infty[g(\widetilde{Y})] = tE_\infty[g(Y)]$$

where $E_\infty[\cdot]$ denotes the expectation with respect to the invariant distribution.

Proof :

From the ergodic property of $\widetilde{Y}$; we have

$$\lim_{\alpha\to\infty}\int_0^t g(\widetilde{Y}_{\alpha s})ds = \lim_{\alpha\to\infty} t\frac{1}{\alpha t}\int_0^{\alpha t} g(\widetilde{Y}_\tau)d\tau = tE_\infty[g(\widetilde{Y})].$$

Also, since the invariant distribution of Y is the same as that of $\widetilde{Y}$; the second equality in the lemma holds.

From now on, we will denote $E_{\infty}[\cdot]$ as $<\ \cdot\ >$. The spirit of this lemma is that: Since the distribution of $\widetilde{Y}_{\alpha t}$ (or Y_t) depends only on the product αt, allowing t to become large is the same in distribution as allowing the rate of mean reversion α to become large. [2]

It is not too hard to show that $X_t = X_0 \exp\{-\frac{1}{2}\int_0^t f^2(Y_s)ds + \int_0^t f(Y_s)dB_s\}$g solves the above stochastic differential equation $dX_t = \mu X_t dt + f(Y_t)X_t dB_t$. As a result, we want to approximate $P[T_b^X \le a]$ where

$$\begin{aligned} T_b^X &= \inf\{t:\ X_t \ge\ b\} \\ &= \inf\{t:\ X_0 \exp\{\mu t - \frac{1}{2}\int_0^t f^2(Y_s)ds + \int_0^t f(Y_s)dB_s\} \ge b\}\ . \end{aligned}$$

For simplicity, we first look at the case where $X_t = \mu t - \frac{1}{2}\int_0^t f^2(Y_s)ds + \int_0^t f(Y_s)dB_s$.

Theorem 2.1 : Suppose $X_t = \mu t - \frac{1}{2}\int_0^t f^2(Y_s)ds + \int_0^t f(Y_s)dB_s$ where Y is a stationary, ergodic process satisfying $dY_t = \alpha(m - Y_t)dt + \beta\sqrt{2\alpha}d\widetilde{B}_t$, $Y_0 = y$. Then, for large α,

$$\begin{aligned} P[T_b^X \le a] &\approx P[\sup_{0 \le t \le a}(\mu t - \frac{1}{2}\langle f^2 \rangle t + W_{\langle f^2 \rangle t}) \ge b] \\ &= \int_0^a \frac{b'}{\sqrt{2\pi x^3}} e^{-\frac{1}{2x}(b' - \mu' x)^2} dx \end{aligned}$$

where $b' = \frac{b}{\sqrt{\langle f^2 \rangle}}$ and $\mu' = \frac{\mu - \frac{1}{2}\langle f^2 \rangle}{\sqrt{\langle f^2 \rangle}}$.

Proof :

Since $X_t = \mu t - \frac{1}{2}\int_0^t f^2(Y_s)ds + \int_0^t f(Y_s)dB_s$,

$$\begin{aligned} P[T_b^X \le a] &= P[\sup_{0\le t\le a} X_t \ge b] \\ &= P[\sup_{0\le t\le a} \mu t - \frac{1}{2}\int_0^t f^2(Y_s)ds + \int_0^t f(Y_s)dB_s) \ge b] \end{aligned}$$

From part (e) of proposition 1.1,

$$= P[\sup_{0\le t\le a} (\mu t - \frac{1}{2}\int_0^t f^2(\tilde{Y}_{as})ds + \int_0^t f(\tilde{Y}_{as})dB_s) \ge b]$$

Applying the time change theorem,

$$= P[\sup_{0\le t\le a} (\mu t - \frac{1}{2}\int_0^t f^2(\tilde{Y}_{as})ds + W_{\int_0^t f(\tilde{Y}_{as})ds}) \ge b]$$

Motivated by lemma 2.1, for large a

$$\begin{aligned} &\approx P[\sup_{0\le t\le a} (\mu t - \frac{1}{2} t\langle f^2\rangle + W_{t\langle f^2\rangle}) \ge b] \\ &= P[\sup_{0\le t\le a} \{(\frac{\mu - \frac{1}{2}\langle f^2\rangle}{\sqrt{\langle f^2\rangle}})t + W_t\} \ge \frac{b}{\sqrt{\langle f^2\rangle}}] \\ &= P[\sup_{0\le t\le a} (\mu' t + W_t) \ge b'] \end{aligned}$$

where $\mu' = \frac{\mu - \frac{1}{2}\langle f^2\rangle}{\sqrt{\langle f^2\rangle}}$ and $b' = \frac{b}{\sqrt{\langle f^2\rangle}}$. Applying theorem A.1 in the appendix, we obtain

$$\begin{aligned} P[T_b^X \le a] &\approx P[T_L^W \le a] \\ &= \int_0^a \frac{|b'|}{\sqrt{2\pi y^3}} e^{-\frac{1}{2y}(b' - \mu' y)^2} dy \end{aligned}$$

where $T_L^W = \inf\{s: W_s \le L\}$ with L is some line of the form $\mu t + b$. The next theorem shows how we can control the error in the above approximation.

Theorem 2.2 : Let E denotes the error in the above approximation, i.e., $E = |I(t) - J(t)|$, where

$$I(t) \ = \ P[\sup_{0\le t\le a}(\mu t-\frac{1}{2}\int_0^t f^2(\widetilde{Y}_{as})ds+W_{\int_0^t f(\widetilde{Y}_{as})ds})\ge b]$$

and

$$J(t) \ = \ P[\sup_{0\le t\le a}(\mu t-\frac{1}{2}\langle f^2\rangle t+W_{\langle f^2\rangle t})\ge b].$$

Then for any given ε, $\delta > 0$, there exists a_0, large enough, such that

$$\begin{aligned} E \ &\le P[\sup_{0\le t\le a}(-\frac{1}{2}(\langle f^2\rangle t+\varepsilon t)+W_{\langle f^2\rangle t+\varepsilon t}\ge b-\mu a] \\ &- P[\sup_{0\le t\le a}(-\frac{1}{2}(\langle f^2\rangle t-\varepsilon t)+W_{\langle f^2\rangle t-\varepsilon t}\ge b]+\delta \end{aligned}$$

whenever $a\ge a_0$, and $\mu\ge 0$.

We shall prove this theorem through a couple of propositions.

Proposition 2.1 : Assume $\mu\ge 0$. Then for any $\varepsilon\ge 0$,

$$\begin{aligned} &P[\sup_{0\le t\le a}(-\frac{1}{2}(\langle f^2\rangle t-\varepsilon t)+W_{\langle f^2\rangle t-\varepsilon t}\ge b] \\ &\le P[\sup_{0\le t\le a}(\mu t-\frac{1}{2}\langle f^2\rangle t+W_{\langle f^2\rangle t})\ge b] \\ &\le P[\sup_{0\le t\le a}(-\frac{1}{2}(\langle f^2\rangle t+\varepsilon t)+W_{\langle f^2\rangle t+\varepsilon t}\ge b-\mu a] \end{aligned}$$

Proof:

First of all, since $\mu\ge 0$,

$$\begin{aligned} &P[\sup_{0\le t\le a}(\mu t-\frac{1}{2}\langle f^2\rangle t+W_{\langle f^2\rangle t}\ge b] \\ &\quad\ge \ P[\sup_{0\le t\le a}(-\frac{1}{2}\langle f^2\rangle t+W_{\langle f^2\rangle t})\ge b] \\ &\quad= P[\sup_{0\le t\le a\langle f^2\rangle}(-\frac{1}{2}t+W_t)\ge b] \\ &\quad\ge \ P[\sup_{0\le t\le(\langle f^2\rangle-\varepsilon)a}(\frac{1}{2}t+W_t\ge b] \\ &\quad= P[\sup_{0\le t\le a}(-\frac{1}{2}(\langle f^2\rangle t-\varepsilon t)+W_{\langle f^2\rangle t-\varepsilon t}\ge b] \end{aligned}$$

This is the first inequality. In addition,

$$
\begin{aligned}
&P[\sup_{0\le t\le a}(\mu t-\frac{1}{2}\langle f^2\rangle t+W_{\langle f^2\rangle t})\ge b]\\
&\le P[\sup_{0\le t\le a}(-\frac{1}{2}\langle f^2\rangle t+W_{\langle f^2\rangle t})\ge b-\mu a]\\
&=P[\sup_{0\le t\le a\langle f^2\rangle}(-\frac{1}{2}t+W_t)\ge b-\mu a]\\
&\le P[\sup_{0\le t\le (\langle f^2\rangle+\varepsilon)a}(-\frac{1}{2}t+W_t)\ge b-\mu a]\\
&=P[\sup_{0\le t\le a}(-\frac{1}{2}(\langle f^2\rangle t+\varepsilon t)+W_{\langle f^2\rangle t+\varepsilon t}\ge b-\mu a]
\end{aligned}
$$

as required.

Proposition 2.2 : For any given $\varepsilon>0$, $\delta>0$ there exists an α_0 such that

$$
\begin{aligned}
&P[\sup_{0\le t\le a}(-\frac{1}{2}(\langle f^2\rangle t-\varepsilon t)+W_{\langle f^2\rangle t-\varepsilon t}\ge b]-\delta\\
&\le P[\sup_{0\le t\le a}(\mu t-\frac{1}{2}\int_0^t f^2(\widetilde{Y}_{\alpha s})ds+W_{\int_0^t f^2(\widetilde{Y}_{\alpha s})ds})\ge b]\\
&\le P[\sup_{0\le t\le a}(-\frac{1}{2}(\langle f^2\rangle t+\varepsilon t)+W_{\langle f^2\rangle t+\varepsilon t}\ge b-\mu a]+\delta
\end{aligned}
$$

provided that $\alpha\ge\alpha_0$, and $\mu\ge 0$.

Proof:

Since $\lim_{\alpha\to\infty}\int_0^t f^2(\widetilde{Y}_{\alpha s})ds=t\langle f^2\rangle$ a.s, $\lim_{\alpha\to\infty}\int_0^t f^2(\widetilde{Y}_{\alpha s})ds=t\langle f^2\rangle$ in probability. Therefore, given $\varepsilon>0$ and $\delta>0$, there exists α_0 such that whenever $\alpha\ge\alpha_0$,

$$P[\left|\int_0^t f^2(\widetilde{Y}_{\alpha s})ds-\langle f^2\rangle t\right|\ge\varepsilon t]\le\delta .$$

Let $A=[\left|\int_0^t f^2(\widetilde{Y}_{\alpha s})ds-\langle f^2\rangle t\right|\ge\varepsilon t]$. Then partitioning by A, we have

$$
\begin{aligned}
&P[\sup_{0\le t\le a}(\mu t-\frac{1}{2}\int_0^t f^2(\widetilde{Y}_{\alpha s})ds+W_{\int_0^t f^2(\widetilde{Y}_{\alpha s})ds})\ge b]\\
&=P([\sup_{0\le t\le a}(\mu t-\frac{1}{2}\int_0^t f^2(\widetilde{Y}_{\alpha s})ds+W_{\int_0^t f^2(\widetilde{Y}_{\alpha s})ds})\ge b]\cap A)\\
&\quad+P([\sup_{0\le t\le a}(\mu t-\frac{1}{2}\int_0^t f^2(\widetilde{Y}_{\alpha s})ds+W_{\int_0^t f^2(\widetilde{Y}_{\alpha s})ds})\ge b]\cap\overline{A}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&P[\sup_{0\le t\le a}(\mu t-\frac{1}{2}\int_0^t f^2(\tilde{Y}_{as})ds+W_{\int_0^t f^2(\tilde{Y}_{as})ds})\ge b] \\
&\le \delta+P([\sup_{0\le t\le a}(\mu t-\frac{1}{2}\int_0^t f^2(\tilde{Y}_{as})ds+W_{\int_0^t f^2(\tilde{Y}_{as})ds})\ge b]\cap\overline{A}) \\
&\le \delta+P([\sup_{0\le t\le a}(-\frac{1}{2}\int_0^t f^2(\tilde{Y}_{as})ds+W_{\int_0^t f^2(\tilde{Y}_{as})ds})\ge b-\mu a]\cap\overline{A}) \\
&= \delta+P([\sup_{0\le t\le \int_0^a f^2(\tilde{Y}_{as})ds}(-\frac{1}{2}t+W_t)\ge b-\mu a]\cap\overline{A}) \\
&\le \delta+P([\sup_{0\le t\le (\langle f^2\rangle+\varepsilon)a}(-\frac{1}{2}t+W_t)\ge b-\mu a]\cap\overline{A}) \\
&\le P[\sup_{0\le t\le a}(-\frac{1}{2}(\langle f^2\rangle t+\varepsilon t)+W_{\langle f^2\rangle t+\varepsilon t}\ge b-\mu a]+\delta.
\end{aligned}
$$

This is the second inequality. For the first inequality,

$$
\begin{aligned}
&P[\sup_{0\le t\le a}(\mu t-\frac{1}{2}\int_0^t f^2(\tilde{Y}_{as})ds+W_{\int_0^t f^2(\tilde{Y}_{as})ds})\ge b] \\
&\ge P[\sup_{0\le t\le a}(-\frac{1}{2}\int_0^t f^2(\tilde{Y}_{as})ds+W_{\int_0^t f^2(\tilde{Y}_{as})ds})\ge b] \\
&\ge P([\sup_{0\le t\le a}(-\frac{1}{2}\int_0^t f^2(\tilde{Y}_{as})ds+W_{\int_0^t f^2(\tilde{Y}_{as})ds})\ge b]\cap\overline{A}) \\
&= P([\sup_{0\le t\le \int_0^a f^2(\tilde{Y}_{as})ds}(-\frac{1}{2}t+W_t\ge b]\cap\overline{A}) \\
&\ge P([\sup_{0\le t\le (\langle f^2\rangle-\varepsilon)t}(-\frac{1}{2}t+W_t)\ge b]\cap\overline{A}) \\
&= P([\sup_{0\le t\le (\langle f^2\rangle-\varepsilon)t}(-\frac{1}{2}t+W_t)\ge b] \\
&\quad -P([\sup_{0\le t\le (\langle f^2\rangle-\varepsilon)t}(-\frac{1}{2}t+W_t)\ge b]\cap A) \\
&\ge P[\sup_{0\le t\le a}(-\frac{1}{2}(\langle f^2\rangle t-\varepsilon t)+W_{\langle f^2\rangle t-\varepsilon t}\ge b]-\delta.
\end{aligned}
$$

This completes the proof of proposition 2.2.

Proof of theorem 2.2:
This is done by combining the above two propositions together.
An immediate advantage of this theorem is that both probabilities

$$P[\sup_{0\le t\le a}(-\frac{1}{2}(\langle f^2\rangle t+\varepsilon t)+W_{\langle f^2\rangle t+\varepsilon t}\ge b-\mu a]$$

and

$$P[\sup_{0\le t\le a}(-\frac{1}{2}(\langle f^2\rangle t-\varepsilon t)+W_{\langle f^2\rangle t-\varepsilon t}\ge b]$$

computed using formula A.1 in the appendix.

We are now ready to state the general result.

Corollary 2.1 : Suppose $X_t=X_0e^{\mu t-\frac{1}{2}\int_0^t f^2(Y_s)ds+\int_0^t f(Y_s)dB_s}$ with $X_0>0$ and Y is the same process in theorem 2.1. Then for large a,

$$P[T_b^X\le a]\approx P[\sup_{0\le t\le a}(\mu t-\frac{1}{2}\langle f^2\rangle t+W_{\langle f^2\rangle t})\ge \ln(\frac{b}{X_0})]$$
$$=\int_0^a\frac{b'}{\sqrt{2\pi x^3}}e^{-\frac{1}{2x}(b'-\mu' x)^2}dx$$

where $b'=\dfrac{\ln(\frac{b}{X_0})}{\sqrt{\langle f^2\rangle}}$ and $\mu'=\dfrac{\mu-\frac{1}{2}\langle f^2\rangle}{\sqrt{\langle f^2\rangle}}$.

Also, for any given $\varepsilon>0$, $\delta>0$, there exists an a_0 such that the error, E, for this case satisfies

$$|E|\le P[\sup_{0\le t\le a}(-\frac{1}{2}(\langle f^2\rangle t+\varepsilon t)+W_{\langle f^2\rangle t+\varepsilon t}\ge \tilde{b}-\mu a]$$
$$-P[\sup_{0\le t\le a}(-\frac{1}{2}(\langle f^2\rangle t-\varepsilon t)+W_{\langle f^2\rangle t-\varepsilon t}\ge \tilde{b}]+\delta$$

whenever $a\ge a_0$, and $\mu\ge 0$.

Proof :

$$P[T_b^X\le a]=P[\sup_{0\le t\le a}X_t\ge b]$$
$$=P[\sup_{0\le t\le a}X_0e^{\mu t-\frac{1}{2}\int_0^t f^2(Y_s)ds+\int_0^t f(Y_s)dB_s}\ge b]$$
$$=P[\sup_{0\le t\le a}e^{\mu t-\frac{1}{2}\int_0^t f^2(Y_s)ds+\int_0^t f(Y_s)dB_s}\ge\frac{b}{X_0}]$$
$$=P[\sup_{0\le t\le a}(\mu t-\frac{1}{2}\int_0^t f^2(Y_s)ds+\int_0^t f(Y_s)dB_s)\ge\ln(\frac{b}{X_0})]$$

$$= P[\sup_{0\le t\le a}(\mu t - \frac{1}{2}\int_0^t f^2(\tilde{Y}_{as})ds + \int_0^t f(\tilde{Y}_{as})dB_s) \ge \ln(\frac{b}{X_0})]$$
$$\approx P[\sup_{0\le t\le a}(\mu t - \frac{1}{2}\langle f^2\rangle t + W_{\langle f^2\rangle t}) \ge \tilde{b}]$$

for large a. The error can be handled similarly as before.

Appendix

Suppose $\{B_t, F_t\ ;\ 0\le t<\infty\}$ is a standard one dimensional Brownian Motion on (Ω, F, F_t, P). Let

$$T_b^B(\omega) = \inf\{t\ge 0 : B_t(\omega) \ge b\}$$

be the .rst passage time of B to a level $b>0$, and

$$T_L^B(\omega) = \inf\{t\ge 0 : B_t(\omega) \ge \mu t + b\}$$

be the .rst passage time to linear barrier. The distributions of T_b^B and T_L^B are given the following theorem whose proof can be found in [2].

Theorem A.1 : Let $\{B_t, F_t; 0\le t<\infty\}$ be a standard, one dimensional Brownian motion on $(\otimes, F, P)$. Let T_b^B, T_L^B be de.ned as above. Then

i. The density of T_b^B is

$$P[\,T_b^B \in dt] = \frac{|b|}{\sqrt{2\pi t^3}}\, e^{-\frac{b^2}{2t}}\, dt.$$

ii. And the density of T_L^B is given as

$$P[\,T_L^B \in dt] = \frac{|b|}{\sqrt{2\pi t^3}}\, e^{-\frac{1}{2t}(b+\mu t)^2}\, dt.$$

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