

## Three Dimensional Imaging Using Wavelets

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### Abstract

The use of wavelets in three-dimensional imaging is reviewed with an example. The insufficiencies of direct two-dimensional processing is showed as a major motivating factor behind using wavelets for three-dimensional imaging. Different wavelet algorithms are used, and these are compared with the direct two-dimensional approach as well as with each other.

**Keywords** : Daubechies, Haar, Three-dimensional imaging, Wavelets

### 1. Introduction

Wavelets are functions that analyze according to scale. They are used in representing data and other functions. The main branch of mathematics leading to wavelets began with Joseph Fourier in the early nineteenth century. Fourier discovered that he could superpose sines and cosines to represent other functions. In wavelet analysis, however, the scale that is used to look at data plays a special role. This is because wavelets process data at different scales or resolutions. Analyzing using a large scale leads to gross features in a signal, while analyzing with a smaller scale leads to more precise features (Graps, 1995).

This paper discusses the motivation behind using wavelets and some specific applications where wavelets have played a vital role. By introducing the topic of seismology, the paper discusses how geophysicists took wavelets that had only been used to solve abstract mathematical problems and applied them to signal processing. Also, some one-dimensional applications, in particular signal de-noising, are discussed. The fundamental Haar and Daubechies wavelets are reviewed. Additionally, two-dimensional imaging using wavelets is reviewed.

The main emphasis of the paper is focused on the use of wavelets in three-

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dimensional imaging. The insufficiencies of direct two-dimensional processing is a major motivating factor behind using wavelets for three-dimensional imaging. Different wavelet algorithms are introduced, and these are compared with the direct two-dimensional approach as well as with each other.

## 2. Background of Wavelets

### 2.1. Motivation of Seismology

Jean Morlet, a French Geophysicist, first used wavelets, to analyze data from seismic surveys, which are comprised of lots of two-dimensional pictures placed together to produce a three-dimensional image of the rock structure below the surface of the earth. These pictures are obtained by recording the movement of the earth, a seismic trace, due to a seismic blast from start to finish (Narcowich and Boggess, 2001).

In seismology, the Fourier transform is unable to provide the oscillations that comprise the signal and only give frequency information without the oscillation occurrence time. The short-time Fourier transform contains time and frequency information. The use of this transform, however, makes it difficult to detect the times when quick, high frequency bursts occur. The inadequacies of these two analysis methods lead to the use of wavelets to solve the problem. Wavelets track time and frequency information. They allow both short bursts and slow oscillations to be examined (Narcowich and Boggess, 2001).

### 2.2. One Dimensional Introduction

The simplest basis in the wavelet analysis is the Haar basis. The Haar wavelet analysis utilizes two functions, namely the scaling function  $\phi$  and the wavelet  $\psi$ . The two functions are equally important in the analysis, as  $\phi$  is known as the father wavelet and  $\psi$  is known as the mother wavelet. The scaling function  $\phi$  is defined as:

$$\phi(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

The function can be shifted to the right or left  $k$  units by adding a shifting parameter to the original function, producing the shifted function  $\phi(x - k)$ . Let  $V_0$  be the vector space spanned by the set  $\{\phi(x - k), k \in \mathbb{Z}\}$ .  $V_0$  is a subspace of functions in  $L^2(\mathbb{R})$ . Let  $V_1$  be the vector space spanned by the set  $\{\phi(2x - k), k \in \mathbb{Z}\}$ .  $V_1$  is a subspace of  $L^2(\mathbb{R})$  and any function in  $V_0$  is included

in  $V_1$ . In general, the space of scaling functions at level  $j$ , denoted by  $V_j$ , is defined to be the space spanned by the set  $\{\phi(2^jx - k), k \in Z\}$ .  $V_j$  is included in  $V_{j+1}$  with  $\overline{\bigcup_{j=-\infty}^{\infty} V_j} = L^2(R)$  and  $\bigcap_{j \in Z} V_j = \{0\}$  (Multiresolution Property; Daubechies, 1992).

For each  $j \in Z$ , define  $W_j$  to the orthogonal complement of  $V_j$  in  $V_{j+1}$  such that  $V_{j+1} = V_j \oplus W_j$  and  $W_j \perp W_{j'}$  if  $j \neq j'$ . The Haar wavelet, defined as  $\psi(x) = \phi(2x) - \phi(2x - 1)$ , provides a way of isolating spikes that belong to  $V_j$  but not to  $V_{j-1}$ . For any  $j \in Z$ , the set of functions of  $\{\psi_{jk}(x) = 2^{j/2}\psi(2^jx - k); k \in Z\}$  is an orthonormal basis for  $W_j$  (Daubechies, 1992; Narcowich and Boggess, 2001). For any  $j \in Z, x \in R$ , the following relationships hold for the scaling function and wavelet:

$$\begin{aligned} \phi(2^jx) &= \frac{\psi(2^{j-1}x) + \phi(2^{j-1}x)}{2} \\ \phi(2^jx - 1) &= \frac{\psi(2^{j-1}x) - \phi(2^{j-1}x)}{2} \end{aligned} \tag{1}$$

Any function  $f_j \in V_j$  can be decomposed into the Haar wavelet analysis components using the relationships in (1) as:

$$f_j = w_{j-1} + w_{j-2} + \dots + w_0 + f_0$$

where  $w_{j-1} = \sum_{k \in Z} b_k^{j-1} \psi(2^{j-1}x - k) \in W_{j-1}$ ,  $f_{j-1} = \sum_{k \in Z} a_k^{j-1} \phi(2^{j-1}x - k) \in V_{j-1}$ ,  $b_k^{j-1} = (a_{2k}^j - a_{2k+1}^j) / 2$  and  $a_k^{j-1} = (a_{2k}^j + a_{2k+1}^j) / 2$ . Any function that has been decomposed  $f = f_0 + w_0 + w_1 + \dots + w_{j-1}$  can be reconstructed using the relationships:  $\phi(2^{j-1}x) = \phi(2^jx) + \phi(2^jx - 1)$  and  $\psi(2^{j-1}x) = \phi(2^jx) - \phi(2^jx - 1)$ . Depending on the application, certain components of the reconstructed signal can be eliminated.

The Haar wavelet analysis is so simple to understand the fundamentals of a multiresolution analysis; however, the components of the Haar wavelet analysis are discrete and do not approximate continuous signals very well. This leads to the need for continuous wavelets and introduces the discussion of Daubechies wavelets (Narcowich and Boggess, 2001).

Expanding the fundamentals created with the Haar wavelet analysis, Daubechies developed the hierarchical family of wavelets that better approximate continuous

signals. The simplest wavelet in the family is the Haar wavelet, and the other wavelets are compactly supported and continuous with a specified number of vanishing moments (Narcowich and Boggess, 2001; Stollnitz et al., 1996).

Daubechies constructed iteratively compactly supported scaling functions  $\phi$  and wavelets  $\psi$  satisfying orthonormality condition and the scaling equation,  $\phi(x) = \sum_{k \in Z} p_k \phi(2x - k)$ . The associated wavelet  $\psi$  in a multiresolution analysis is

given by the following relation:  $\psi(x) = \sum_{k \in Z} (-1)^k \overline{p_{1-k}} \phi(2x - k)$  (Daubechies, 1992; Narcowich and Boggess, 2001)

Just as with the Haar wavelet analysis, any function can be decomposed into the multiresolution analysis components using the decomposition formula:

$$\begin{aligned}\langle f, \phi_{j-1,l} \rangle &= 2^{-1/2} \sum_{k \in Z} \overline{p_{2k-2l}} \langle f, \phi_{jk} \rangle \\ \langle f, \psi_{j-1,l} \rangle &= 2^{-1/2} \sum_{k \in Z} (-1)^k \overline{p_{1-k+2l}} \langle f, \phi_{jk} \rangle\end{aligned}$$

Also, the function can be reconstructed using the reconstruction formula:

$$\langle f, \phi_{jk} \rangle = 2^{-1/2} \sum_{l \in Z} p_{k-2l} \langle f, \phi_{j-1,l} \rangle + 2^{-1/2} \sum_{l \in Z} (-1)^k \overline{p_{1-k+2l}} \langle f, \psi_{j-1,l} \rangle$$

These decomposition and reconstruction formulas utilizing their associated scaling function and wavelet allow for a better approximation of continuous signals than does the Haar wavelet analysis. This allows for the analysis to lend itself to certain applications associated with continuous signals, which leads to a discussion of one such application, signal de-noising (Narcowich and Boggess, 2001).

The technique associated with using wavelets to remove noise from a signal is known as the wavelet shrinkage and thresholding method. After decomposing the signal in terms of a wavelet basis, filters are employed that act as averaging filters and others are employed that produce details. Some of the resulting wavelet coefficients correspond to details in the signal. If these coefficients are small, they may be omitted without substantially affecting the main features of the signal (Graps, 1995). The technique of thresholding is to set to zero, or eliminate, all coefficients that are less than a pre-determined threshold  $\tau$ . Those above the threshold are shrunk by  $\tau$ . These new coefficients are used in an inverse wavelet transform to reconstruct the signal. Besides just eliminating unwanted noise in a signal, this technique of de-noising also allows sharp structures in a signal to remain intact throughout the process without being dulled or smoothed (Graps, 1995).

### 2.3 Two-Dimensional Introduction

So far we have only discussed approximation of objects that require only one-dimensional approximation (e.g. signals). We are now, however, prepared to use this basic framework to discuss wavelet decomposition and compression of objects that are interrelated in two dimensions. That is, objects for which any one value is related to surrounding values in both the x- and y-directions (e.g. images). Certainly the data could be linearized and analyzed in one dimension using techniques already mentioned, but to do that would be to ignore the correlations between data in the other dimension. For example, simply to attach each row of pixels in an image file to the previous row and decompose the resultant vector would totally ignore the correlations between each row. Now, this is not necessarily bad if the image (or data set) has no correlation between different rows; however, if there exists any correlation at all between rows, we should be able to use wavelets to model this correlation and hopefully achieve a greater compression than previously possible using one-dimensional techniques. So, the idea behind two-dimensional wavelets is the following: decompose each row (or column) of the data set, then decompose each column (or row, respectively) of the resultant matrix. This takes into account both correlations from row to row and from column to column to produce a better compression rate.

As just mentioned, one can make the most of correlations between data in both directions when dealing with two-dimensional images. This is probably the most widespread application of two-dimensional wavelet analysis. Simple computer bitmaps (which store a color value for each pixel) take up a lot of memory and thus are ill suited for high-speed data transmission, say, over the internet or a company local area network. It thus becomes necessary to find a way to compress such images. One widespread image format, JPEG, makes use of the discrete (Fourier) cosine transform to help compress the image. Given the shortcomings of Fourier analysis, though, this method does not work well around discontinuities in the image, like inserted text, for example. Hence a new image format, JPEG 2000, has recently been proposed. The new format scraps Fourier techniques in favor of wavelets, which offer many new advantages. A two-dimensional discrete wavelet transform is taken on the image, and the result is compressed and coded (Santa-Cruz et al., 2000). This new format promises to become the new standard, replacing conventional JPEG imaging. For a given compression ratio, image quality is generally improved using lossy JPEG 2000, and for lossless versions, JPEG 2000 achieves a higher compression ratio than lossless JPEG (Santa-Cruz et al., 2000).

## 3. Three-Dimensional Imaging

We now consider three-dimensional data sets, in specific three-dimensional images. Immediately we see the higher dimension analog of the situation that we had when we stepped from one to two dimensions. Namely, a simple two-dimensional modeling of a three-dimensional data set (e.g. via adjoining all two-dimensional arrays into one large matrix) does not take into account any correlation between data in the third dimension. Assuming they exist as they often do in images, these correlations across the third dimension can be modeled using wavelets. This, of course, can lead to better compression rates, which provide impetus for the main topic of this paper.

Three-dimensional images are represented mathematically by three-dimensional matrices. The fundamental driving force behind wavelet transformations of these three-dimensional matrices is to perform the transformation one dimension at a time across the matrices. This allows for flexibility in choosing the order in which in each dimension will be transformed. The particular image set or application most commonly determines this order.

After transformation occurs, the wavelet coefficients are set into a number of blocks, with one small block containing most of the energy and the rest of the blocks containing information in various frequency bands. Various filters can be used to actually eliminate certain wavelet coefficients and compress the image set. Some common filters used include the 9/7 filter, the Haar filter, and the Daubechies 4 filter (Wang and Huang, 1996). These filters can be applied across multiple directions of the coefficient blocks, or each direction can employ its own filter type. This, of course, is also dependent on the image set and specific application.

One obvious application of three-dimensional imaging is, of course, medical image compression. Magnetic resonance (MR) and computed tomography (CT) images are examples of three-dimensional images with widespread use in the medical field. It would be very useful if there existed some algorithms for compressing one such an image with little loss of data for the purpose of distribution throughout a hospital, consultation, or second opinion (Wang and Huang, 1996).

We decided to research this area with respect to the performance of two wavelets, Haar and Daubechies 2. It has been noted that Haar generally gives the best analysis with respect to the third (i.e. slice) dimension of medical images, given slice distances of  $>3\text{mm}$  (Wang and Huang, 1996). We were not, however, sure which wavelet would give the best general analysis across all three dimensions. Also, given that slice distances are different than pixel distances within each two-dimensional slice, we were unsure whether some hybrid of wavelet analysis would give better results or not.

We thus ran four tests on an MR image supplied by MATLAB. We compared three-dimensional Haar compression, three-dimensional Daubechies 2 compression, compression using Haar for each slice and Daubechies 2 across the pages, and

compression using Daubechies 2 for each slice and Haar across the pages. In each case we compared quality of images by the peak signal-to-noise ratio (PSNR):

$$PSNR = 20 \log \left\{ \frac{f_{\max}}{\sqrt{\sum (f(x, y, z) - f_c(x, y, z))^2 / N}} \right\}$$

where  $f$  is the original image,  $f_c$  is the compressed image, and  $f_{\max}$  is the maximum gray value in  $f$  (Wang and Huang, 1996). This was measured versus the compression ratio (CR):

$$CR = \frac{\text{number of zero coefficients}}{\text{number of overall coefficients}}$$

These values for each data set were then compared to show overall performance ability of each mode of wavelet compression.

#### 4. Results

The following figure is the data image set that was used for the image compression tests described earlier. It is an MRI file within MATLAB and represents a human brain.

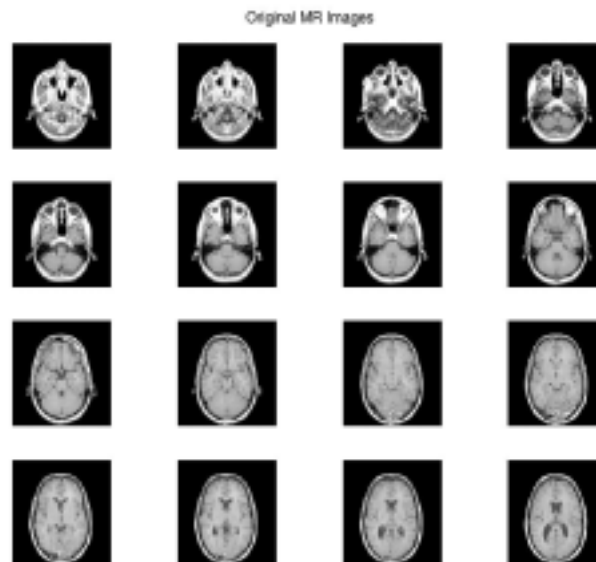


Figure 1. Original MR Image Used for Compression Tests

As discussed earlier, the main motivating factor for the use of three-dimensional image compression is the insufficiencies of a direct two-dimension compression. The following figure shows the resulting image after such compression.

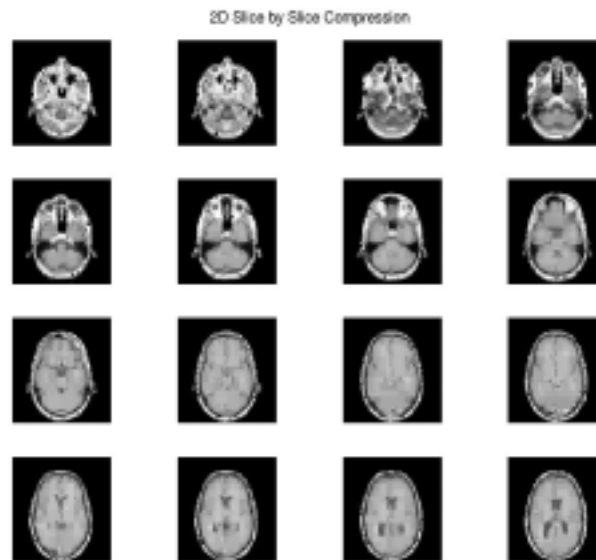


Figure 2. Two-Dimensional Slice by Slice Compression @ 95%

It is easy to see the bad image quality here. In fact, the PSNR for the above compression is about 17.7 dB. We now present the compressed images (Figures 3 – Figure 6) employing four different methods of three-dimensional wavelet analysis at compression rate of 95%. The first method uses Haar filters for all three dimensions. The second employs Daubechies 2 for all three dimensions. The third uses Haar to compress each slice, then Daubechies 2 to compress across the slices. The last uses Daubechies 2 to compress each slice before using Haar to compress across the slices. All decompose the slices 5 levels in each direction and 3 levels across the slices.



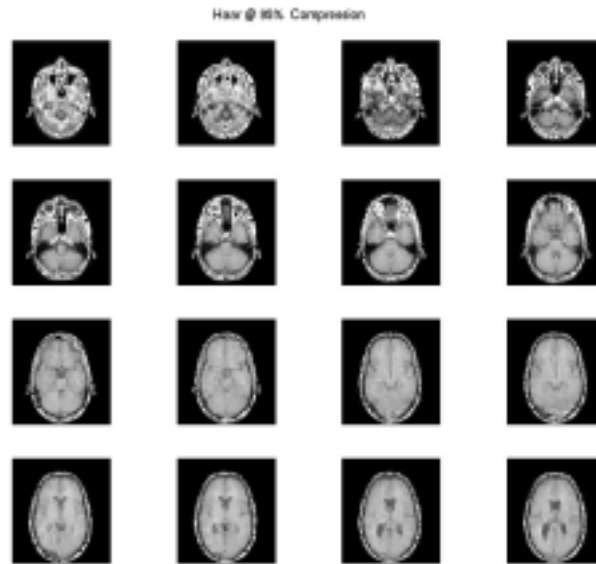


Figure 3. Haar Compression at a 95% Compression Rate

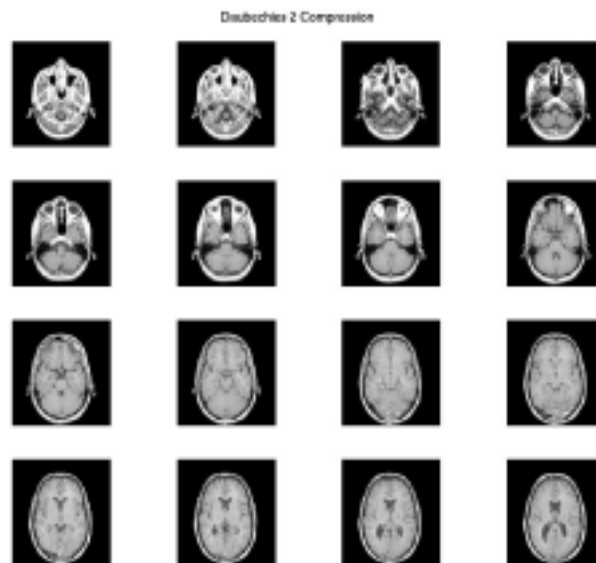


Figure 4. Daubechies 2 Compression at a 95% Compression Rate

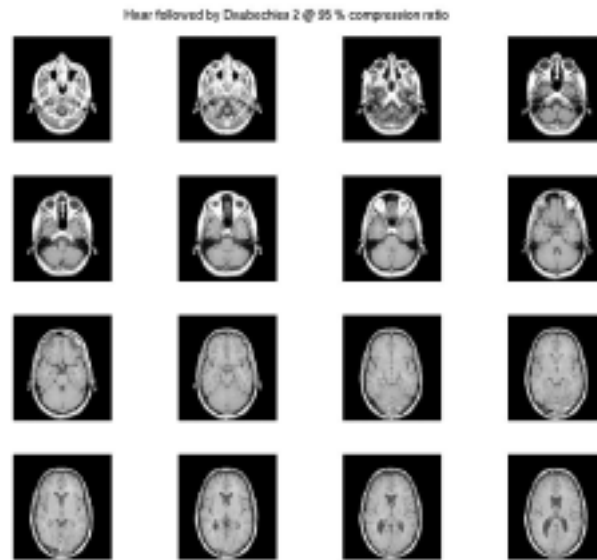


Figure 5. 2D Haar x 1D Daubechies 2 Compression at a 95% Compression Rate

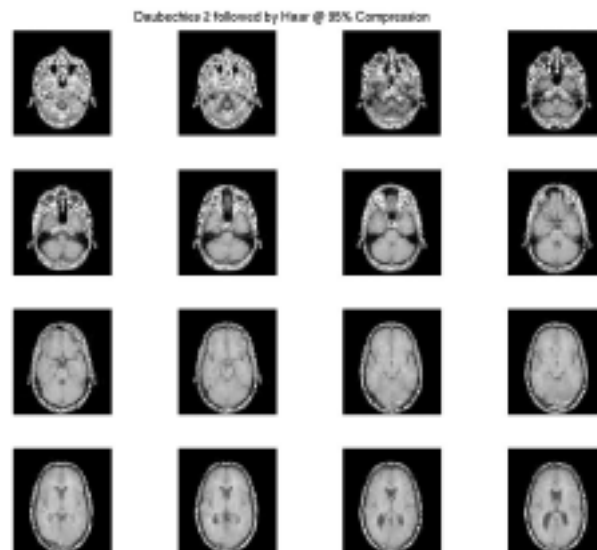


Figure 6. 2D Daubechies 2 x 1D Haar Compression at a 95% Compression Rate

As discussed before, the PSNR vs. Compression Ratio of each compression data set was used as the evaluation criterion. The following graph shows those calculations. Note that the PSNR for the two-dimensional compression technique, 177 dB, is lower than all three-dimensional methods.

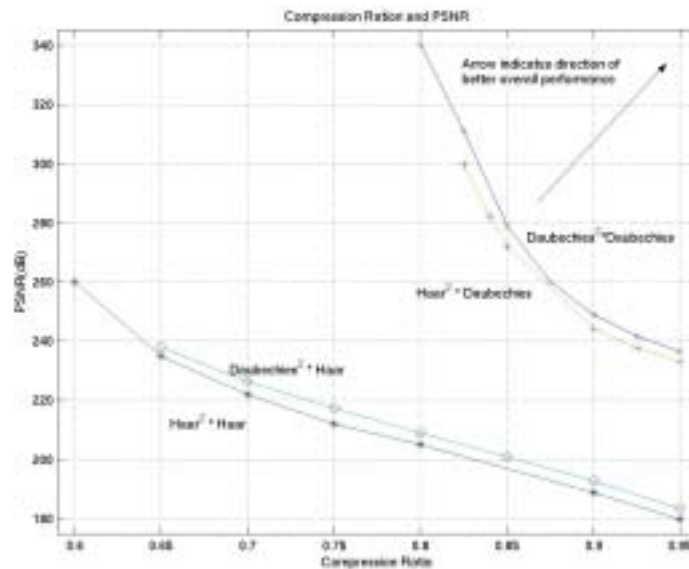


Figure 7. PSNR vs. Compression Ratio for Each Data Set

## 5. Conclusions

From the data it is easy to determine the best method from the possible candidates. Both Daubechies 2 compression (in all directions) and Haar two-dimensional compression followed by Daubechies 2 compression across the slices were front runners, giving both superior compression ratios and image quality (as measured in PSNR). However, Daubechies 2 compression in all dimensions yielded the best result for the data. It is interesting that the major distinction uncovered here is the superiority of Daubechies 2 analysis when modeling differences between slices. We believe this to be true because of the relationship between corresponding data in each slice.

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