Journal of Korean
Data & Information Science Society
2004, Vol. 15, No. 3, pp. 655~662

System Reliability from Common Random Stress in a Type II Bivariate Pareto Model with Bivariate Type I Censored Data

Jang Sik Cho[1] · Seung Bae Choi[2]

Abstract

In this paper, we assume that strengths of two components system follow a type II bivariate Pareto model with bivariate type I censored data. And these two components are subjected to a common stress which is independent of the strengths of the components. We obtain estimators for the system reliability based on likelihood function and relative frequency, respectively. Also we construct approximated confidence intervals for the reliability based on maximum likelihood estimator and relative frequency estimator, respectively. Finally we present a numerical study.

Keywords : Bivariate type I censored data, Common random stress, Maximum likelihood estimator, Type II bivariate Pareto distribution

1. Introduction

In many studies of two components system data, the component lifetimes were assumed to be statistically independent for the sake of simplicity of mathematical treatment. However, the assumption of independence is unrealistic as in many two components systems the component life lengths have a well-defined dependence structure.

Several bivariate distributions based on exponential model are studied by Freund(1961), Marshall and Olkin(1967) and Block and Basu(1974), etc.

---

1) First Author. Associate Professor, Department of Statistical Information Science, Kyungsung University, Busan, 608-736, Korea.
E-mail : jscho@star.ks.ac.kr

2) Full-time Lecturer, Department of Information Statistics, Dongeui University, Busan, 501-759, Korea
E-mail : csb4851@dongeui.ac.kr

On the other hand, Lindley and Singpurwalla(1986) proposed bivariate Pareto(BVP) model in the modelling of lifetimes of two components systems working in a changing environment. They considered the distribution of life lengths measured in a laboratory environment as independent exponential distributions proved that, when they work in a different environment which may be harsher, the same or gentler than the original, the resulting density of life lengths has a BVP model. The related papers are introduced by Bandyapadhyay and Basu(1990), Veenus and Nair(1994), Hanagal(1996), Jeevanand(1997), Cho, Cho, and Cha(2003), Cho, Cho, and Lee(2003) and Cho, Ko, and Kang(2003), et. al. Above all authors considered complete sample or univariate censored sample cases.

In this paper, we assume that strengths of two components system follow a type II BVP model with bivariate type I censored data, as extension of complete data or univariate random censored data. These two components are subjected to a common stress which is independent of the strengths of the components. From this common random stress, we derive maximum likelihood estimator and relative frequency estimator for the system reliability, respectively. And we construct approximated confidence intervals for the system reliability based on the proposed estimators, respectively. Also we present a numerical example by giving a data set which is generated by computer.

2. Preliminaries

Let the random variable (X, Y) be lifetimes of two components that follow a type II BVP model with parameter (η_1, η_2, η_3). Then the joint probability density function of (X, Y) is given as

$$f(x, y : \eta_1, \eta_2, \eta_3) = \begin{cases} \eta_1(\eta_2+\eta_3)x^{-\eta_1-1}y^{-(\eta_2+\eta_3)-1}, & 1<x<y<\infty, \\ \eta_2(\eta_1+\eta_3)x^{-(\eta_1+\eta_3)-1}y^{-(\eta_2+\eta_3)-1}, & 1<y<x<\infty, \\ \eta_3x^{-\eta-1}, 1<x=y<\infty, \end{cases} \tag{1}$$

where $\eta_1, \eta_2, \eta_3 > 0$ and $\eta = \eta_1+\eta_2+\eta_3$.

We note that random variables X and Y are independent if and only if $\eta_3 = 0$ and that X and Y are symmetrically distributed if and only if $\eta_1 = \eta_2$.

On the other hand, let Z be the common random stress that have a Pareto distribution with parameter $(1, \mu)$, that is, distribution function of Z is $S(z) = 1 - z^{-\mu}$. And we assume that Z is independent of (X, Y). Then the reliability of the system reliability from stress-strength relationship is computed by

$$R = P[Z < max(X, Y)].$$
$$= \mu\left[\frac{1}{\eta_1+\eta_3+\mu} + \frac{1}{\eta_2+\eta_3+\mu} - \frac{1}{\eta+\mu}\right], \quad max(X, Y) > 1. \tag{2}$$

Suppose that there are n two component systems under study and ith pair of the components have lifetimes (x_i, y_i) with bivariate type I censored data. Then ith observed lifetime (x_i, y_i) is given by

$$(x_i, y_i) = \begin{cases} (x_i, y_i), & x_i < c_x, \ y_i < c_y \\ (c_x, y_i), & x_i > c_x, \ y_i < c_x \\ (x_i, c_y), & x_i < c_x, \ y_i > c_y \\ (c_x, c_y), & x_i > c_x, \ y_i > c_y, \end{cases} \tag{3}$$

where the bivariate type I censoring times c_x and c_y are fixed values. We note that if $c_x = c_y$ then it is univariate type I censoring model.

For $j=1,2$; $k=1,2,3$; $i=1,2,\cdots,n$, we use following notations in this paper.

(a) (c_x, c_y) : bivariate type I censoring times for ith observation.

(b) $G_{1i} = I(X_i > c_x)$, $G_{2i} = I(Y_i > c_y)$, $G^*_{ji} = 1 - G_{ji}$, where $I(\cdot)$ denotes an indicator function.

(c) $R_{1i} = I(X_i < Y_i)$, $R_{2i} = I(X_i > Y_i)$, $R_{3i} = I(X_i = Y_i)$, $R^*_{ki} = 1 - R_{ki}$.

Let $\underline{\eta} = (\eta_1, \eta_2, \eta_3)$. Then the likelihood function for the sample of size n is given by

$$\begin{aligned} L(\underline{\eta}) = \prod_{i=1}^{n} \{ & [f(x_i, y_i)]^{G^*_{1i}G^*_{2i}} \cdot [\overline{F}(x_i, y_i)]^{G_{1i}G_{2i}} \\ & \cdot [\overline{F}_{X|Y=y}(x_i) f_Y(y_i)]^{G_{1i}G^*_{2i}} \\ & \cdot [\overline{F}_{Y|X=x}(y_i) f_X(x_i)]^{G^*_{1i}G_{2i}} \}^{(R_{1i}+R_{2i}+R_{3i})} \\ = {} & \eta_1^{K_1} \eta_2^{K_2} \eta_3^{K_3} (\eta_1+\eta_3)^{K_4} (\eta_2+\eta_3)^{K_5} \exp[-\eta_1 x_s - \eta_2 y_s - \eta_3 (x_s + y_s - t_s)]. \end{aligned} \tag{4}$$

where $f_X(x)$ and $f_Y(y)$ are marginal probability density functions of X and Y, respectively. And $\overline{F}_{X|Y=y}(x)$ and $\overline{F}_{Y|X=x}(y)$ are reliability functions of random variables X and Y given by $Y=y$ and $X=x$, respectively. Also $\overline{F}(x, y)$ is joint reliability function of X and Y.

$$K_1 = \sum_{i=1}^{n} (R_{1i} G^*_{1i} G^*_{2i} + R^*_{2i} G^*_{1i} G_{2i}), \quad K_2 = \sum_{i=1}^{n} (R_{2i} G^*_{1i} G^*_{2i} + R^*_{1i} G_{1i} G^*_{2i}),$$

$$K_3 = \sum_{i=1}^{n} R_{3i} G^*_{1i} G^*_{2i}, \quad K_4 = \sum_{i=1}^{n} R_{2i} G^*_{1i}, \quad K_5 = \sum_{i=1}^{n} R_{1i} G^*_{2i},$$

$$x_s = \sum_{i=1}^{n} x_i, \ \ y_s = \sum_{i=1}^{n} y_i, \ \ t_s = \sum_{i=1}^{n} \min(x_i, y_i).$$

Then $K_i, i=1,\cdots,5$ are random variables. By Cho and Choi(2004), the expected value of each K_i can be obtained as follows;

$$\begin{aligned} E(K_1) = \sum_{i=1}^{n} \{ & \eta_1/\eta - \eta_1 \exp(-\eta c_x)/\eta + \exp(-\eta c_y) - \exp(-(\eta_2+\eta_3)c_y) \\ & + (1-\exp(-\eta_1 c_x)) \cdot \exp(-(\eta_2+\eta_3)c_y) \cdot I(c_x < c_y) \\ & + \eta_3(\exp(-\eta c_y) - \exp(-\eta c_x))/\eta \cdot I(c_y < c_x)\}. \end{aligned}$$

$$\begin{aligned} E(K_2) = \sum_{i=1}^{n} \{ & \eta_2/\eta - \eta_2 \exp(-\eta c_y)/\eta + \exp(-(\eta_1+\eta_3)c_x - \eta_2 c_y) \\ & - \exp(-(\eta_1+\eta_3)c_x) \\ & + (1-\exp(-\eta_2 c_y)) \cdot \exp(-(\eta_1+\eta_3)c_x) \cdot I(c_y < c_x) \\ & + \eta_3(\exp(-\eta c_x) - \exp(-\eta c_y)))/\eta \cdot I(c_x < c_y)\}, \end{aligned}$$

$$E(K_3) = \sum_{i=1}^{n} \{(\eta_3 - \eta_3 \exp(-\eta \min(c_x, c_y)))/\eta\},$$

$$\begin{aligned} E(K_4) = \sum_{i=1}^{n} \{ & \eta_2/\eta - \eta_2 \exp(-\eta c_y)/\eta + \exp(-(\eta_1+\eta_3)c_x - \eta_2 c_y) \\ & - \exp(-(\eta_1+\eta_3)c_x) \\ & + [\exp(-\eta_2 c_y) \cdot [1-\exp(-(\eta_1+\eta_3)c_x)] \\ & + (\eta_1+\eta_3) \cdot (\exp(-\eta c_x) - 1)/\eta] \cdot I(c_x > c_y)\}, \end{aligned}$$

$$\begin{aligned} E(K_5) = \sum_{i=1}^{n} \{ & \eta_1/\eta - \eta_1 \exp(-\eta c_x)/\eta + \exp(-\eta c_y) - \exp(-(\eta_2+\eta_3)c_y) \\ & + \eta_1 \exp(-\eta c_x)/\eta - \exp(-(\eta_2+\eta_3)c_y - \eta_1 c_x)\}. \end{aligned}$$

Then the likelihood equations are given by

$$\frac{\partial}{\partial \eta_1} \log L(\underset{\sim}{\eta}) = \frac{K_1}{\eta_1} + \frac{K_4}{\eta_1+\eta_3} - \log(x_s) = 0. \tag{5}$$

$$\frac{\partial}{\partial \eta_2} \log L(\underset{\sim}{\eta}) = \frac{K_2}{\eta_2} + \frac{K_5}{\eta_2+\eta_3} - \log(y_s) = 0. \tag{6}$$

$$\frac{\partial}{\partial \eta_3} \log L(\underset{\sim}{\eta}) = \frac{K_3}{\eta_3} + \frac{K_4}{\eta_1+\eta_3} + \frac{K_5}{\eta_2+\eta_3} - \log(x_s + y_s - t_s) = 0. \tag{7}$$

The likelihood equations (5)-(7) are not easy to solve. But we can obtain MLE's ($\widehat{\eta} = (\widehat{\eta_1}, \widehat{\eta_2}, \widehat{\eta_3})$) by either Newton-Raphson procedure or Fisher's method of scoring.

Fisher information matrix is given by $I(\underset{\sim}{\eta}) = ((I_{ij}))$. And the elements of Fisher information matrix is given by $I_{ij}=E\left[-\frac{\partial^2}{\partial\eta_i\partial\eta_j}\log L(\underline{\eta})\right],\ i,j=1,2,3.$

Here $I_{11}=\frac{1}{n}\left(\frac{E(K_1)+E(K_4)}{\eta_1^2}+\frac{E(K_2)}{(\eta_1+\eta_3)^2}\right),\ I_{12}=0,\ I_{13}=\frac{E(K_2)}{n(\eta_1+\eta_3)^2},$

$$I_{22}=\frac{1}{n}\left(\frac{E(K_2)+E(K_5)}{\eta_2^2}+\frac{E(K_1)}{(\eta_2+\eta_3)^2}\right),\ I_{23}=\frac{E(K_1)}{n(\eta_2+\eta_3)^2},$$

$$I_{33}=\frac{1}{n}\left(\frac{E(K_1)}{(\eta_2+\eta_3)^2}+\frac{E(K_2)}{(\eta_1+\eta_3)^2}+\frac{E(K_3)}{\eta_3^2}\right).$$

Thus $\sqrt{n}\,(\,\hat{\underline{\eta}}'-\underline{\eta}'\,)$ has asymptotic trivariate normal distribution with mean vector zero and covariance matrix $I^{-1}(\underline{\eta})$.

3. Estimations for System Reliability

In this section, we obtain estimators for R based on the likelihood function and the relative frequency, respectively. Also we obtain approximated confidence intervals for R based on MLE(maximum likelihood estimator) and the relative frequency estimator, respectively.

Now, the estimate of system reliability R based on MLE's of $(\eta_1,\eta_2,\eta_3,\mu)$ is

$$\hat{R}_{MLE}=\hat{\mu}\left[\frac{1}{\hat{\eta}_1+\hat{\eta}_3+\hat{\mu}}+\frac{1}{\hat{\eta}_2+\hat{\eta}_3+\hat{\mu}}-\frac{1}{\hat{\eta}+\hat{\mu}}\right],\ \hat{\eta}=\hat{\eta}_1+\hat{\eta}_2+\hat{\eta}_3, \tag{8}$$

where $\hat{\mu}=n/\sum_{i=1}^{n}\log(z_i)$.

By consistency of MLE, we can see that the asymptotic distribution of $\hat{R}_{MLE}$ is normal distribution with mean R and variance $\Lambda\cdot(I^{-1}(\eta_1,\eta_2,\eta_3,\mu)/n)\cdot\Lambda'$, where $\Lambda=(\partial R/\partial\eta_1,\partial R/\partial\eta_2,\partial R/\partial\eta_3,\partial R/\partial\mu)$. Here, the elements of Fisher information matrix $I(\eta_1,\eta_2,\eta_3,\mu)$ are given by $I(\underline{\eta})=((I_{ij})),\ i,j=1,2,3$ and $I_{12}=I_{14}=I_{24}=I_{34}=0,\ \ I_{44}=1/\mu^2.$

Therefore, $100(1-\alpha)\%$ approximated confidence interval for system reliability R based on MLE is as follows;

$$\Big(\hat{R}_{MLE}-z_{\alpha/2}\sqrt{\hat{\Lambda}\cdot I^{-1}(\hat{\eta}_1,\hat{\eta}_2,\hat{\eta}_3,\hat{\mu})\cdot\hat{\Lambda}'/n}\ ,$$
$$\hat{R}_{MLE}+z_{\alpha/2}\sqrt{\hat{\Lambda}\cdot I^{-1}(\hat{\eta}_1,\hat{\eta}_2,\hat{\eta}_3,\hat{\mu})\cdot\hat{\Lambda}'/n}\Big). \tag{9}$$

We next obtain the estimate and approximated confidence interval for R based on relative frequency. Let L be the number of observations with $z_i < max(x_i, y_i)$ in the sample. Then we can see that the distribution of L is binomial distribution with parameter (n,R). The relative frequency estimate of R based on L is given by

$$\widehat{R}_{RF} = L/n, \tag{10}$$

which is asymptotic normal distribution with mean R and variance $R(1-R)/n$. Therefore, $100(1-\alpha)\%$ approximated confidence interval for R based on the relative frequency estimate is as follows;

$$\left(\widehat{R}_{RF} - z_{\alpha/2} \cdot \sqrt{\widehat{R}_{RF} \cdot (1-\widehat{R}_{RF})/n}\ ,\ \ \widehat{R}_{RF} + z_{\alpha/2} \cdot \sqrt{\widehat{R}_{RF} \cdot (1-\widehat{R}_{RF})/n} \right). \tag{11}$$

4. Numerical Example

In this section, we present a numerical example by giving a data set which is generated by computer. We generate a random samples for the strength of size 30 from a type II BVP with parameter ($\eta_1 = 3.0,\ \eta_2 = 3.0,\ \eta_3 = 0.5$) and generate common random stress sample from pareto model with $\mu = 3.0$. Also we set bivariate type I censoring times $(c_x, c_y) = (1.6,\ 1.6)$.

Then the true reliability from common random stress is $R = 0.6073$. The data are given as follows. Where, * indicates censored data.

<Table 1> Generated samples (x_1, x_2)

i	x_{1i}	x_{2i}	i	x_{1i}	x_{2i}
1	1.5230	1.5252	16	1.5978	1.5978
2	1.2099	1.1685	17	1.6487*	1.1004
3	1.2296	1.2296	18	1.6487*	1.3348
4	1.0008	1.4567	19	1.2913	1.2606
5	1.2326	1.3839	20	1.1193	1.3155
6	1.0508	1.6487*	21	1.3498	1.4909
7	1.0556	1.0556	22	1.6487*	1.3826
8	1.5484	1.0955	23	1.0133	1.0666
9	1.1145	1.5542	24	1.1988	1.0869
10	1.1059	1.3739	25	1.1883	1.0315
11	1.2720	1.1564	26	1.0058	1.0539
12	1.1970	1.6487*	27	1.0816	1.1238
13	1.3765	1.6007	28	1.3591	1.1639
14	1.0409	1.2906	29	1.2573	1.3056
15	1.6487*	1.5915	30	1.3702	1.0383

From above data set, MLE's of the parameters in BVP model are computed by $\widehat{\eta_1} = 3.0612$, $\widehat{\eta_2} = 2.9358$ and $\widehat{\eta_3} = 0.7521$, and $L = 17$, respectively.

Also two type estimates of R based on MLE and relative frequency estimator are computed by $\widehat{R}_{MLE}=0.6038$ and $\widehat{R}_{RF}=0.5666$, respectively. Also approximated confidence intervals for R based on $\widehat{R}_{MLE}$ and $\widehat{R}_{RF}$ are given by $(0.4755, 0.7322)$ and $(0.3893, 0.7439)$ from (8) and (9), respectively.

Therefore, we can see that the estimate of R based on MLE perform better than those of natural estimate based on L in viewpoint of approximated confidence interval.

In our discussions, we have concentrated on the bivariate Pareto model case. For censored samples or a more general model than above model, we can apply our results.

References

1. Bandyapadhyay, D. and Basu, A.P. (1990). On generalization of a model by Lindley and Singpurwallam, *Advanced Applied Probability*, 22, 498-500.
2. Block, H. W. and Basu, A. P. (1974). A continuous bivariate exponential extension, *Journal of the American Statistical Association*, 69, 1031-1037.
3. Cho, J. S., Cho, K. H. and Cha, Y. J. (2003). System reliability from stress -strength relationship in bivariate Pareto distribution, *Journal of the Korean Data & Information Science Society*, Vol. 14(1), 113-118.
4. Cho, J. S., Ko, J. H. and Kang, S. G. (2003). System reliabilities in bivariate Pareto model, *Far East Journal of Theoretical Statistics*, Vol. 11(1), 77-83.
5. Cho, J. S., Cho, K. H. and Lee, W. D. (2003). Reliability for series and parallel systems in bivariate Pareto model : random censorship cases, *Journal of the Korean Data & Information Science Society*, Vol. 14(1), 461-469.
6. Cho, J. S., Choi, S. B. (2004). Independence Test for Bivariate Pareto Distribution under Bivariate Type I Censorship, *Far East Journal of Theoretical Statistics*, Vol. 12(1), 53-61.
7. Freund, J. E. (1961). A bivariate extension of the exponential distribution, *Journal of the American Statistical Association*, 971-977.
8. Hanagal, D.D. (1996). A multivariate pareto distribution, *Communication in Statistics-Theory and Methods*, 25(7), 1471-1488.
9. Jeevanand, E.S. (1997). Bayes estimation of $P(X_2 < X_1)$ for a bivariate Pareto distribution, *The Statistician*, 46(1), 93-99.
10. Lindley, D.V. and Singpurwalla, N.D. (1986). Multivariate distributionsfor the life lengths of components of a system sharing a

common environment, *Journal of Applied Probability*, 23, 418-431.
11. Marshall, A.W. and Olkin, I. (1967). A multivariate exponential distribution, *Journal of the American Statistical Association*, 62, 30-44.
12. Veenus, P. and Nair, K.R.M. (1994). Characterization of a bivariate Pareto distribution, *Journal of Indian Statistical Association*, 32, 15-20.

[received date : Jun. 2004, accepted date : Aug. 2004]