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Empirical Bayes Inferences in the Burr Distribution by the Bootstrap Methods[1)]

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Abstract

We consider the empirical Bayes confidence intervals that attain a specified level of EB coverage for the scale parameter in the Burr distribution under type II censoring data. Also, we compare the coverage probabilities and the expected confidence interval lengths for these confidence intervals through simulation study.

Keywords : Bootstrap methods, Burr distribution, Empirical Bayes confidence intervals

1. Introduction

The Burr distribution has been widely used as a model for lifetime. If the parameters are appropriately chosen, the Burr distribution covers a large portion of the Pearson family. Also, the Weibull and exponential distributions are special limiting cases of the Burr distribution.

Empirical Bayes(EB) methods have become increasingly popular and have been applied to many types of problems (refer Robbins(1955), James and Stein(1961), Miller(1989), Nandram and Sedransk(1993), Pensky(1998), Ferry and Lahiri(1999)).

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Morris(1983) reviewed some parametric EB procedures, their properties, and their applications. Casella(1985) provided a readable introduction to EB idea. Parametric EB methods of point estimation was introduced by James and Stein(1961). Also, a confidence interval estimation through parametric EB methods was summarized by Laird and Louis(1987).

In many applications, EB confidence intervals are required, but computing them from the posterior based on an estimated prior (the native approach) is generally inappropriate. Since these posterior distributions fail to account for the uncertainty in estimating the prior, they may be have inappropriate shapes.

Several approaches have been proposed for incorporating this uncertainty. Morris obtained an approximate EB confidence interval for the uncertainty in the equal variance and the unequal variance cases, respectively. Laird and Louis used bootstrap methods for estimating the prior and posterior distributions and obtained EB confidence intervals based on the parametric bootstrap posterior. Carlin and Gelfand(1991) showed how bias correction can be implemented generally to a type III parametric bootstrap procedure introdeced by Laird and Louis. Nandram and Sedransk(1993) developed EB point estimation and confidence intervals for the finite population mean and made large sample comparisons with the corresponding Bayes estimators and confidence intervals.

In this paper, we consider the methods that construct the bootstrap EB confidence intervals of the scale parameter in the Burr distribution under the type II censoring data. Also, we compare the bootstrap confidence intervals with the native confidence interval in terms of the coverage probabilities and the expected confidence interval lengths through simulation study.

2. EB confidence intervals

The Burr(c, k) probability density function(pdf) is

$$f(x:\, c, k) = ckx^{c-1}(1+x^{c})^{-(k+1)},\ x>0,\ (c>0,\ k>0), \tag{1}$$

where c is the shape parameter and k is the scale parameter.

We assume that c is known throughout. Also, we consider a squared error loss function and a gamma conjugate prior with unknown parameters $(b, a+1)$ given by

$$g(k|\, a, b) = \frac{b^{a+1}}{\Gamma(a+1)} k^{a} exp(-bk),\ k>0,\ (a>-1,\ b>0). \tag{2}$$

When the parameter has the value k_{m+1}, a current sample $x_{m+1,1} < x_{m+1,2} < \cdots < x_{m+1,r}$ is obtained. At the time when the current sample is observed, there are available past observations $x_{i,1} < x_{i,2} < \cdots < x_{i,r}$, $i = 1, \cdots, m$, with past realizations $k_1, k_2, \cdots, k_m$ of the random variable k. Each sample is supposed to be a censored sample of size r obtained from a life test without replacement of n items whose life times have a Burr pdf given by equation (1).

For sample i, $i = 1, \cdots, m$, the maximum likelihood estimator of k_i is

$$\hat{k_i} = \frac{r}{T_i} \tag{3}$$

where

$$T_i = \sum_{j=1}^{r} ln(1 + x_{i,j}^c) + (n - r)\, ln(1 + x_{i,r}^c).$$

The conditional pdf of X_i for a given k_i is

$$f(x_i | k_i) = \frac{(rk_i)^r}{\Gamma(r)\, x_i^{r+1}} \exp\left(-\frac{rk_i}{x_i}\right),\ x_i > 0 \tag{4}$$

wihch is the inverted gamma pdf $IG(r, rk_i)$. By equations (2) and (4), the marginal pdf of x_i, $i = 1, \cdots, m$, is given by

$$\begin{aligned} h(x_i) &= \int_0^\infty f(x_i | k_i)\, g(k_i; a, b)\, dk_i \\ &= \frac{a^{b+1}}{B(r, b+1)} \frac{x_i^b}{(r + ax_i)^{r+b+1}},\ x_i > 0 \end{aligned} \tag{5}$$

and the posterior pdf of k_i is given by

$$f(k_i | T_i) = \frac{a + T_i^{r+b+1}}{\Gamma(r+b+1)} k_i^{r+b} exp(-a + T_i k_i),\ k_i > 0,\ (a > -1,\ b > 0).$$

Lemma 1. (Berger(1985) Let $\mu_f(k)$ and $\sigma_f^2(k)$ denote the conditional mean and variance of X (i.e. the mean and variance with respect to the density $f(x|k)$). Let μ_m and σ_m^2 denote the marginal mean and variance of X. Assuming these quantities exist, then

$$\mu_m = E^\pi[\mu_f(k)],$$
$$\sigma_m^2 = E^\pi[\sigma_f^2(k)] + E^\pi[(\mu_f(k) - \mu_m)^2].$$

Using Lemma 1, we have for all i

$$\mu_m = \frac{r}{r-1}\frac{b+1}{a} \tag{6}$$

and

$$\sigma_m^2 = \frac{r^2}{(r-1)^2}\left[\frac{1}{r-2}\frac{(b+1)(b+2)}{a^2} + \frac{(b+1)(b+2)}{a^2} - \frac{(b+1)^2}{a^2}\right]. \tag{7}$$

Since μ_m and σ_m^2 are the marginal mean and variance for X_i, $i=1,2,\cdots,m$, they can be estimated from the data.

Let

$$\widehat{\mu_m} = \frac{\sum_{i=1}^{m} X_i}{m}$$

and

$$\widehat{\sigma_m^2} = \frac{\sum_{i=1}^{m} X_i^2}{m} - \widehat{\mu_m}^2.$$

We can solve for a and b from equations (6) and (7) with $\widehat{\mu_m}$ and $\widehat{\sigma_m^2}$ for μ_m and σ_m^2, respectively. If we put

$$S_1 = \frac{r}{r-1}\widehat{\mu_m}$$

and

$$S_2 = \frac{(r-1)(r-2)}{mr^2}\sum_{i=1}^{m} X_i^2 = S_1^2 + \frac{S_1}{\hat{a}},$$

we obtain

$$\hat{a} = \frac{\hat{b}+1}{S_1} \tag{8}$$

and

$$\hat{b} = \frac{S_1^2}{S_2 - S_1^2} - 1. \tag{9}$$

Therefore, the moment estimators of a and b are obtained by

$$\widehat{b_M} = \max\left[\frac{S_1^2}{S_2 - S_1^2} - 1, -1\right] \tag{10}$$

and

$$\widehat{a_M} = \max\left[\frac{\hat{b}+1}{S_1}, 0\right] \tag{11}$$

respectively.

We will construct the native EB and some bootstrapping EB confidence intervals of the scale parameter in the Burr distribution under type II censoring data.

Let the prior parameter $\hat{\Delta} = (\widehat{a_M}, \widehat{b_M})$ be a moment estimator of $\Delta = (a, b)$ computed from the marginal distribution of X_i.

Then the estimated posterior distribution of K_i given $X_i = x_i$ is $IG(r, rx_i)$, that is,

$$f(k_i | x_i, \widehat{a_M}, \widehat{b_M}) = \frac{(rx_i)^r}{\Gamma(r) k_i^{r+1}} exp(-\frac{rx_i}{k_i}),\ k_i > 0. \quad (12)$$

Therefore, the equal tail 100(1-α)% two-sided native EB confidence interval(NEBCI) for k_i based on the estimated posterior $f(k_i | x_i, \widehat{a_M}, \widehat{b_M})$ is given by

$$(\frac{F^{-1}_{2(r+\hat{b}+1)}(\alpha)}{2(\hat{a}+T_i)}, \frac{F^{-1}_{2(r+\hat{b}+1}(1-\alpha)}{2(\hat{a}+T_i)}) \quad (13)$$

where F_k denotes the cumulative distribution function(CDF) of the chi-square distribution with k degrees of freedom.

Let us investigate the bootstrapping EB confidence intervals. To implement the bias correction we note that

$$\begin{aligned} r(\hat{\Delta}, \Delta, T_i, a) &\equiv P[k_i \leqq q_\alpha(T_i, \hat{\Delta}) | k_i \sim f(k_i | T_i, \Delta)] \\ &= F_{2(r_i+b+1)}(\frac{a+T_i}{\widehat{a_M}+T_i} F^{-1}_{2(\widehat{b_M}+r_i+1)}(\alpha)) \end{aligned} \quad (14)$$

where F is the posterior CDF and

$$R(\Delta, T_i, \alpha) \equiv E_{\hat{\Delta}| T_i, \Delta}[r(\hat{\Delta}, \Delta, T_i, \alpha)]. \quad (15)$$

Computing $R(\Delta, T_i, \alpha)$ necessitates the integration over the distribution $g(\hat{\Delta} | T_i, \Delta)$. Using the type III parametric bootstrap procedure, $R(\Delta, T_i, \alpha)$ is obtained as follows.

For the unconditional EB correction, the type III parametric bootstrap estimate of $R(\Delta, T_i, \alpha'_{(1)})$ becomes

$$\frac{1}{N}\sum_{j=1}^{N} F_{2(r_i+\hat{b}+1)}[\frac{\hat{a}+T_i}{a^*+T_i} F^{-1}_{2(b^*+r_i+1)}(\alpha'_{(1)})] = \alpha \quad (16)$$

which we equate to α and solve for $\alpha'_{(1)}$. The 100(1-α)% unconditional bias-corrected(I) NEBCI for k_i is given by

$$\left(\frac{F^{-1}_{2(r+\hat{b}+1)}(\alpha'_{(1)})}{2(\hat{a}+T_i)}, \frac{F^{-1}_{2(r+\hat{b}+1}(1-\alpha'_{(1)})}{2(\hat{a}+T_i)}\right). \tag{17}$$

We correct bias for each k_i confidence interval, but the correction in each case depends on the data through $\hat{\Delta}$. If we desire the confidence interval corrected only for the unconditional EB coverage, the bootstrap equation becomes

$$\frac{1}{N}\sum_{j=1}^{N} F_{2(r_i+\hat{b}+1)}\left[\frac{\hat{a}+T^*_{ij}}{a^*+T^*_{ij}} F^{-1}_{2(b^*+r_i+1)}(\alpha'_{(2)})\right] = \alpha \tag{18}$$

which we equate to α and solve for $\alpha'_{(2)}$.

Equation (18) differs from equation (16) only in replacing the given value T_i by the bootstrapped value T^*_{ij}.

Analogous to expression (17), the 100(1-α)% unconditional bias-corrected(II) NEBCI for k_i is given by

$$\left(\frac{F^{-1}_{2(r+\hat{b}+1)}(\alpha'_{(2)})}{2(\hat{a}+T_i)}, \frac{F^{-1}_{2(r+\hat{b}+1}(1-\alpha'_{(2)})}{2(\hat{a}+T_i)}\right). \tag{19}$$

3. Comparisons and Conclusions

We compare all the methods discussed. The EB confidence intervals are approximated by Monte Carlo method. In each iteration, we generate $k_i,\ i=1,\ \cdots, m(=10)$ from subroutin Burr as a gamma distribution $G(a, b)$ with a and b fixed. Given the k_is, we generate the lifetime $x_{ij},\ j=1,\ \cdots, n(=5, 10, 20)$ from subroutin Burr as the Burr distribution with a scale parameter k_i and a fixed shape parameter $c=2$. The random variables x_{ij} are distributed as equation (1). We order the variables $x_{i,1} < x_{i,2} < \cdots < x_{i,r}$, and compute $T_i = \sum_{j=1}^{r} ln(1+x^c_{i,j}) + (n-r)\,ln(1+x^c_{i,r})$. We assume $r_i = r$ for all i. Let us consider the censoring rate defined by $100(1-r/n)\%$ of 0% and 20%. For the given independent random variables EB confidence intervals are computed by each method with bootstrap replications $B = 1000$ times. Also, Monte Carlo sampling are repeated $R = 500$ times. The EB confidence intervals are compared in terms of the coverage probability and the expected confidence interval lengths. Let CV_k denote the coverage probability for k. If CV_k for the EB confidence interval is nearly $1-\alpha$, then the confidence interval is good. We consider the nominal coverage probability of 0.90. Let $\widehat{k_{lo}}$ and $\widehat{k_{up}}$ to be the lower limit and upper limit of the EB confidence interval for k, respectively. Define the expected confidence interval length EL_k by

$$EL_k = \frac{1}{R}\sum_{j=1}^{R}(\widehat{k_{j,up}} - \widehat{k_{j,lo}})$$

where R is the number of Monte Carlo replications. Then the smaller EL_k is the better under the same CV_k. The results of these simulations are presented in Table 1. We can observe the followings from the table;

(1) The CV_ks of the bootstrap confidence intervals obtained from the marginal estimator are better than those of the naive confidence intervals.

(2) The EL_ks of the bootstrap confidence intervals obtained from the marginal estimator are longer than those of the naive confidence intervals.

(3) The CV_ks of all the confidence intervals increase as the sample size increases.

Table 1. Comparisons of Naive and Bootstrap EB Confidence Intervals

Censoring rate= 0%						
Interval methods	n=5		n=10		n=20	
	Coverage	Length	Coverage	Length	Coverage	Length
Naive	0.716	1.096	0.812	0.792	0.876	0.552
Bias-corrected(I)	0.758	1.301	0.830	0.808	0.882	0.555
Bias-corrected(II)	0.764	1.535	0.826	1.328	0.882	0.894
Censoring rate= 20%						
Interval methods	n=5		n=10		n=20	
	Coverage	Length	Coverage	Length	Coverage	Length
Naive	0.712	1.318	0.768	0.878	0.850	0.625
Bias-corrected(I)	0.696	1.735	0.778	0.926	0.844	0.633
Bias-corrected(II)	0.730	1.611	0.784	1.081	0.856	0.803

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