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Estimation of Weibull Scale Parameter Based on Multiply Type-II Censored Samples

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Abstract

We consider the problem of estimating the scale parameter of the Weibull distribution based on multiply Type-II censored samples. We propose two estimators by using the approximate maximum likelihood estimation method for Weibull and extreme value distributions. The proposed estimators are compared in the sense of the mean squared error.

Keywords : Approximate maximum likelihood estimator, Extreme value distribution, Multiply Type-II censored sample, Weibull distribution

1. Introduction

The probability density function of the random variable X having a Weibull distribution with the scale parameter θ and the shape parameter δ is given by

$$f(x)=\frac{\delta}{\theta^{\delta}}x^{\delta-1}\exp\left\{-\left(\frac{x}{\theta}\right)^{\delta}\right\}, \qquad \theta>0, \quad \delta>0, \quad x>0\,. \tag{1.1}$$

Estimations for the parameters in the Weibull distribution have been studied for censored samples. In most cases of censored samples, Estimators of parameters may not be obtained explicitly by the maximum likelihood method. The approximate maximum likelihood estimating method was first developed by Balakrishnan (1989) for the purpose of providing the explicit estimators of the

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scale parameter in the Rayleigh distribution. Estimations based on censored samples have been studied by many authors. Kang et al. (2001) obtained the approximate maximum likelihood estimators (AMLEs) for the parameters in the three-parameter Weibull distribution.

For multiply Type-II censored samples, Balakrishnan et al. (1995) derived the estimators for the location and scale parameters of the extreme value distribution. Fei and Kong (1995) compared the mean squared error (MSE) of the maximum likelihood estimators (MLEs), AMLEs, best linear unbiased estimators (BLUEs) of the parameters in extreme value distribution. Kang (2003) obtained the approximate maximum likelihood estimators for the two-parameter exponential distribution based on multiply Type-II censored samples.

In this paper, we derive the AMLEs for the scale parameter of the Weibull distribution based on multiply Type-II censored samples by using the relationship of Weibull and extreme value distributions. We also compare the proposed estimators in the sense of MSE for various censored samples.

2. Estimator of the scale parameter

Consider the Weibull distribution with the pdf (1.1) and the cumulative distribution function (cdf)

$$F(x;\theta,\delta)=1-\exp\left\{-\left(\frac{x}{\theta}\right)^{\delta}\right\}. \tag{2.1}$$

We assume that the shape parameter δ is known.

Let us assume that the following multiply Type-II censored sample from a sample of size n is

$$X_{a_1:n}<X_{a_2:n}<\cdots<X_{a_s:n} \tag{2.2}$$

where $1\leq a_1<a_2<\cdots<a_s\leq n$

$$a_0=0,\quad a_{s+1}=n+1\quad F(x_{a_0})=0,\quad F(x_{a_{s+1}})=1.$$

The likelihood function based on the multiply Type-II censored sample (2.2) can be written as

$$L=n!\prod_{j=1}^{s}f(x_{a_j:n})\prod_{j=1}^{s+1}\frac{[F(x_{a_j:n})-F(x_{a_{j-1}:n})]^{a_j-a_{j-1}-1}}{(a_j-a_{j-1}-1)!}. \tag{2.3}$$

By putting $Z_{j:n}=\dfrac{X_{j:n}}{\theta}$, the likelihood function can be rewritten as

$$L=n!\prod_{j=1}^{s+1}\frac{[F(z_{a_j:n})-F(z_{a_{j-1}:n})]^{a_j-a_{j-1}-1}}{(a_j-a_{j-1}-1)!}\prod_{j=1}^{s}\frac{f(z_{a_j:n})}{\theta} \tag{2.4}$$

where $f(z)=\delta z^{\delta-1}\exp(-z^{\delta})$ and $F(z)=1-\exp(-z^{\delta})$ are the pdf and the cdf of the standard Weibull distribution, respectively.

We have the log-likelihood function as follows;

$$\ln L = \ln\left(\frac{n!}{\prod_{j=1}^{s+1}(a_j - a_{j-1} - 1)!}\right) - s\ln\theta + (a_1 - 1)\ln F(z_{a_1:n}) + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)\ln[F(z_{a_j:n}) - F(z_{a_{j-1}:n})] + (n - a_s)\ln[1 - F(z_{a_s:n})] + \sum_{j=1}^{s}\ln f(z_{a_j:n}). \quad (2.5)$$

Then the likelihood equation for θ is given by

$$\frac{\partial \ln L}{\partial\theta} = -\frac{1}{\theta}\left[s + (a_1 - 1)\frac{f(z_{a_1:n})}{F(z_{a_1:n})}z_{a_1:n} - (n - a_s)\frac{f(z_{a_s:n})}{1 - F(z_{a_s:n})}z_{a_s:n} + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)\frac{f(z_{a_j:n})z_{a_j:n} - f(z_{a_{j-1}:n})z_{a_{j-1}:n}}{F(z_{a_j:n}) - F(z_{a_{j-1}:n})} + \sum_{j=1}^{s}\frac{f'(z_{a_j:n})}{f(z_{a_j:n})}z_{a_j:n}\right] = 0\ . \quad (2.6)$$

The equation (2.6) does not admit an explicit solution for θ. But let

$$p_i = \frac{i}{n+1},\quad F^{-1}(p_i) = [-\ln(1 - p_i)]^{\frac{1}{\delta}} = \xi_i,$$

then we may expand the following functions in Taylor series around the points ξ_{a_1}, ξ_{a_s}, ξ_{a_j}, and $(\xi_{a_j}, \xi_{a_{j-1}})$, respectively.

$$\frac{f(z_{a_1:n})}{F(z_{a_1:n})}z_{a_1:n},\quad \frac{f(z_{a_s:n})}{1 - F(z_{a_s:n})}z_{a_s:n},\quad \frac{f'(z_{a_j:n})}{f(z_{a_j:n})}z_{a_j:n},\quad \frac{f(z_{a_j:n})z_{a_j:n} - f(z_{a_{j-1}:n})z_{a_{j-1}:n}}{F(z_{a_j:n}) - F(z_{a_{j-1}:n})}$$

That is, we may approximate these functions as

$$\frac{f(z_{a_1:n})}{F(z_{a_1:n})}z_{a_1:n} \simeq \alpha_1 + \beta_1 z_{a_1:n} \quad (2.7)$$

$$\frac{f(z_{a_s:n})}{1 - F(z_{a_s:n})}z_{a_s:n} \simeq \alpha_2 + \beta_2 z_{a_s:n} \quad (2.8)$$

$$\frac{f'(z_{a_j:n})}{f(z_{a_j:n})}z_{a_j:n} \simeq \alpha_{1j} + \beta_{1j} z_{a_j:n} \quad (2.9)$$

$$\frac{f(z_{a_j:n})}{F(z_{a_j:n}) - F(z_{a_{j-1}:n})}z_{a_j:n} \simeq \alpha_{2j} + \beta_{2j} z_{a_j:n} + \gamma_{2j} z_{a_{j-1}:n} \quad (2.10)$$

and

$$\frac{f(z_{a_j:n})}{F(z_{a_{j-1}:n}) - F(z_{a_j:n})}z_{a_{j-1}:n} \simeq \alpha_{3j} + \beta_{3j} z_{a_j:n} + \gamma_{3j} z_{a_{j-1}:n}, \quad (2.11)$$

where

$$\alpha_1 = -C_1\xi_{a_1}$$

$$\beta_1 = C_1 + \frac{f(\xi_{a_1})}{p_{a_1}}$$

$$C_1 = \frac{p_{a_1}(\delta - 1 - \delta\,\xi_{a_1}{}^{\delta}) - \xi_{a_1} f(\xi_{a_1})}{(p_{a_1})^2} f(\xi_{a_1})$$

$$\alpha_2 = -C_2\xi_{a_s}$$

$$\beta_2 = C_2 + \frac{f(\xi_{a_s})}{1-p_{a_s}}$$

$$C_2 = \frac{(1-p_{a_s})(\delta-1-\delta\ \xi_{a_s}{}^{\delta}) - \xi_{a_s} f(\xi_{a_s})}{(1-p_{a_s})^2} f(\xi_{a_s})$$

$$\alpha_{1j} = (\delta-1)(1+\delta\ \xi_{a_j}{}^{\delta})$$

$$\beta_{1j} = -\delta^2\ \xi_{a_j}{}^{\delta-1}$$

$$\alpha_{2j} = -\frac{(\delta-1-\delta\xi_{a_j}{}^{\delta})\xi_{a_j} f(\xi_{a_j})}{p_{a_j}-p_{a_{j-1}}} + \frac{\xi_{a_{j-1}} f(\xi_{a_{j-1}}) - \xi_{a_j} f(\xi_{a_j})}{(p_{a_j}-p_{a_{j-1}})^2}\xi_{a_j} f(\xi_{a_j})$$

$$\beta_{2j} = \frac{\delta(1-\ \xi_{a_j}{}^{\delta})}{p_{a_j}-p_{a_{j-1}}} f(\xi_{a_j}) - \frac{f^2(\xi_{a_j})\xi_{a_j}}{(p_{a_j}-p_{a_{j-1}})^2}$$

$$\gamma_{2j} = \frac{f(\xi_{a_j}) f(\xi_{a_{j-1}})\xi_{a_j}}{(p_{a_j}-p_{a_{j-1}})^2}$$

$$\alpha_{3j} = -\frac{(\delta-1-\delta\xi_{a_{j-1}}{}^{\delta})\xi_{a_{j-1}} f(\xi_{a_{j-1}})}{p_{a_j}-p_{a_{j-1}}} - \frac{\xi_{a_{j-1}} f(\xi_{a_{j-1}}) - \xi_{a_j} f(\xi_{a_j})}{(p_{a_j}-p_{a_{j-1}})^2}\xi_{a_{j-1}} f(\xi_{a_{j-1}})$$

$$\beta_{3j} = -\frac{f(\xi_{a_{j-1}}) f(\xi_{a_j})\xi_{a_{j-1}}}{(p_{a_j}-p_{a_{j-1}})^2}$$

and

$$\gamma_{3j} = \frac{\delta(1-\xi_{a_{j-1}}{}^{\delta})}{p_{a_j}-p_{a_{j-1}}} f(\xi_{a_{j-1}}) - \frac{f^2(\xi_{a_{j-1}})\xi_{a_{j-1}}}{(p_{a_j}-p_{a_{j-1}})^2}.$$

By substituting (2.7), (2.8), (2.9), (2.10), and (2.11) into (2.6), we obtain the approximate likelihood equation for θ as follows

$$\begin{aligned}\frac{\partial \ln L}{\partial \theta} \simeq & -\frac{1}{\theta}\Big[s + (a_1-1)(\alpha_1+\beta_1 z_{a_1:n}) - (n-a_s)(\alpha_2+\beta_2 z_{a_s:n}) \\ & + \sum_{j=2}^{s}(a_j-a_{j-1}-1)[(\alpha_{2j}-\alpha_{3j}) + (\beta_{2j}-\beta_{3j}) z_{a_j} + (\gamma_{2j}-\gamma_{3j}) z_{a_{j-1}:n}] \\ & + \sum_{j=1}^{s}(\alpha_{1j}+\beta_{1j} z_{a_j:n})\Big] \\ & = 0\,. \end{aligned} \tag{2.12}$$

Upon solving the equation for θ, we can derive an estimator of θ as follows;

$$\widehat{\theta}_w = \frac{B}{A} \tag{2.13}$$

where

$$A= -(a_1-1)\beta_1 x_{a_1:n}+(n-a_s)\beta_2 x_{a_s:n}$$
$$-\sum_{j=2}^{s}(a_j-a_{j-1}-1)[(\beta_{2j}-\beta_{3j})x_{a_j:n}+(\gamma_{2j}-\gamma_{3j})x_{a_{j-1}:n}]-\sum_{j=1}^{s}\beta_{1j}x_{a_j:n}$$
$$B=s+(a_1-1)\alpha_1-(n-a_s)\alpha_s+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\alpha_{1j}-\alpha_{2j})+\sum_{j=1}^{s}\alpha_{1j}\ .$$

3. Estimation based on the extreme value distribution

Let X be a random variable with pdf (1.1). Then it is easy to see that the $Y=\ln X$ has its density function to be

$$f(y;\mu,\delta)=\frac{1}{\sigma}e^{\frac{(y-\mu)}{\sigma}}\exp\left\{-e^{\frac{(y-\mu)}{\sigma}}\right\} \tag{3.1}$$

where $\sigma=\frac{1}{\delta}$, $\mu=\ln\theta$ and its cumulative distribution function to be

$$F(y;\mu,\delta)=1-\exp\left\{-e^{\frac{(y-\mu)}{\sigma}}\right\}. \tag{3.2}$$

That is, Y has the extreme value distribution with the location parameter μ and the scale parameter σ.

By putting $Z_{j:n}=\dfrac{Y_{j:n}-\mu}{\sigma}$, the likelihood function can be rewritten as

$$L\ =n!\prod_{j=1}^{s+1}\frac{[F(z_{a_j:n})-F(z_{a_{j-1}:n})]^{a_j-a_{j-1}-1}}{(a_j-a_{j-1}-1)!}\prod_{j=1}^{s}\frac{f(z_{a_j:n})}{\sigma} \tag{3.3}$$

where $f(z)=e^z\exp(-e^z)$ and $F(z)=1-\exp(-e^z)$ are the pdf and the cdf of the standard extreme value distribution.

Then, the likelihood equation for μ is given by

$$\frac{\partial\ln L}{\partial\mu}=-\frac{1}{\sigma}\Bigg[(a_1-1)\frac{f(z_{a_1:n})}{F(z_{a_1:n})}-(n-a_s)\frac{f(z_{a_s:n})}{1-F(z_{a_s:n})}$$
$$+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\frac{f(z_{a_j:n})-f(z_{a_{j-1}:n})}{F(z_{a_j:n})-F(z_{a_{j-1}:n})}+\sum_{j=1}^{s}\frac{f'(z_{a_j:n})}{f(z_{a_j:n})}\Bigg]=0. \tag{3.4}$$

We may expand the following functions in Taylor series around the points ξ_{a_1}, ξ_{a_s}, ξ_{a_j}, and $(\xi_{a_j}, \xi_{a_{j-1}})$, respectively.

$$\frac{f(z_{a_1:n})}{F(z_{a_1:n})},\quad \frac{f(z_{a_s:n})}{1-F(z_{a_s:n})},\quad \frac{f'(z_{a_j:n})}{f(z_{a_j:n})},\quad \frac{f(z_{a_j:n})-f(z_{a_{j-1}:n})}{F(z_{a_j:n})-F(z_{a_{j-1}:n})}$$

That is,

$$\frac{f(z_{a_1:n})}{F(z_{a_1:n})}\simeq\alpha_{E1}+\beta_{E1}z_{a_1:n} \tag{3.5}$$

$$\frac{f(z_{a_s:n})}{1-F(z_{a_s:n})} \simeq \alpha_{E2} + \beta_{E2} z_{a_s:n} \tag{3.6}$$

$$\frac{f'(z_{a_j:n})}{f(z_{a_j:n})} \simeq \alpha_{E1j} + \beta_{E1j} z_{a_j:n} \tag{3.7}$$

$$\frac{f(z_{a_j:n})}{F(z_{a_j:n})-F(z_{a_{j-1}:n})} \simeq \alpha_{E2j} + \beta_{E2j} z_{a_j:n} + \gamma_{E2j} z_{a_{j-1}:n} \tag{3.8}$$

and

$$\frac{f(z_{a_j:n})}{F(z_{a_{j-1}:n})-F(z_{a_j:n})} \simeq \alpha_{E3j} + \beta_{E3j} z_{a_j:n} + \gamma_{E3j} z_{a_{j-1}:n}, \tag{3.9}$$

where

$$\alpha_{E1} = \frac{f(\xi_{a_1})}{p_{a_1}} - \frac{(1-e^{\xi_{a_1}}) - f(\xi_{a_1})}{(p_{a_1})^2} f(\xi_{a_1})\xi_{a_1}$$

$$\beta_{E1} = \frac{(1-e^{\xi_{a_1}}) - f(\xi_{a_1})}{(p_{a_1})^2} f(\xi_{a_1})$$

$$\alpha_{E2} = \frac{f(\xi_{a_s})}{1-p_{a_s}} - \frac{(1-e^{\xi_{a_s}})(1-p_{a_s}) + f(\xi_{a_s})}{(1-p_{a_s})^2} f(\xi_{a_s})\xi_{a_s}$$

$$\beta_{E2} = \frac{(1-e^{\xi_{a_s}})(1-p_{a_s}) + f(\xi_{a_s})}{(1-p_{a_s})^2} f(\xi_{a_s})$$

$$\alpha_{E1j} = 1 - e^{\xi_{a_j}} + e^{\xi_{a_j}}\xi_{a_j}$$

$$\beta_{E1j} = -e^{\xi_{a_j}}$$

$$\alpha_{E2j} = \frac{f(\xi_{a_j})}{p_{a_j}-p_{a_{j-1}}} - \frac{(1-e^{\xi_{a_j}})(p_{a_j}-p_{a_{j-1}}) + f(\xi_{a_j})}{(p_{a_j}-p_{a_{j-1}})^2} f(\xi_{a_j})\xi_{a_j} - \frac{f(\xi_{a_j})f(\xi_{a_{j-1}})\xi_{a_j}}{(p_{a_j}-p_{a_{j-1}})^2}\xi_{a_{j-1}}$$

$$\beta_{E2j} = \frac{(1-e^{\xi_{a_j}})(p_{a_j}-p_{a_{j-1}}) + f(\xi_{a_j})}{(p_{a_j}-p_{a_{j-1}})^2} f(\xi_{a_j})$$

$$\gamma_{E2j} = \frac{f(\xi_{a_j})f(\xi_{a_{j-1}})\xi_{a_j}}{(p_{a_j}-p_{a_{j-1}})^2}$$

$$A_{1j} = \frac{(1-e^{\xi_{a_j}})(p_{a_j}-p_{a_{j-1}}) + f(\xi_{a_j})}{(p_{a_j}-p_{a_{j-1}})^2} f(\xi_{a_j})$$

$$B_{1j} = \frac{f(\xi_{a_j})f(\xi_{a_{j-1}})\xi_{a_j}}{(p_{a_j}-p_{a_{j-1}})^2}$$

$$\alpha_{E3j} = \frac{f(\xi_{a_{j-1}})}{p_{a_j}-p_{a_{j-1}}} + \frac{f(\xi_{a_j})f(\xi_{a_{j-1}})\xi_{a_j}}{(p_{a_j}-p_{a_{j-1}})^2}\xi_{a_j} - \frac{(1-e^{\xi_{a_{j-1}}})(p_{a_j}-p_{a_{j-1}}) + f(\xi_{a_{j-1}})}{(p_{a_j}-p_{a_{j-1}})^2} f(\xi_{a_{j-1}})\xi_{a_{j-1}}$$

$$\beta_{E3j}=-\frac{(1-e^{\xi_{a_{j-1}}})(p_{a_j}-p_{a_{j-1}})+f(\xi_{a_{j-1}})}{(p_{a_j}-p_{a_{j-1}})^2}f(\xi_{a_{j-1}})$$

and

$$\gamma_{E3j}=\frac{(1-e^{\xi_{a_{j-1}}})(p_{a_j}-p_{a_{j-1}})+f(\xi_{a_{j-1}})}{(p_{a_j}-p_{a_{j-1}})^2}f(\xi_{a_{j-1}}).$$

By substituting (3.5), (3.6), (3.7), (3.8), and (3.9) into (3.4), we can obtain the estimator of μ as follows;

$$\hat{\mu}=\frac{B_E}{A_E}$$

where

$$A_E=(a_1-1)\beta_{E1}-(n-a_s)\beta_{E2}+\sum_{j=2}^{s}(a_j-a_{j-1}-1)[(\beta_{E1j}-\beta_{E2j})+(\gamma_{E1j}-\gamma_{E2j})]+\sum_{j=1}^{s}\beta_{Ej}$$

$$\begin{aligned}B_E=&-(a_1-1)(\alpha_{E1}\sigma+\beta_{E1}Y_{a_1:n})+(n-a_s)(\alpha_{E2}\sigma+\beta_{E2}Y_{a_s:n})\\&+\sum_{j=2}^{s}(a_j-a_{j-1}-1)[(\alpha_{E2j}-\alpha_{E3j})\sigma-(\beta_{E2j}-\beta_{E3j})Y_{a_j:n}+(\gamma_{E2j}-\gamma_{E3j})Y_{a_{j-1}:n}]\\&-\sum_{j=1}^{s}(\sigma_{Ej}\sigma+\beta_{Ej}X_{a_j:n}).\end{aligned}$$

Since $\mu=\ln\theta$, we can obtain the approximate maximum likelihood estimator of β as follows.

$$\widehat{\theta_E}=e^{\hat{\mu}}. \tag{3.10}$$

We simulate the MSE for the proposed estimators of θ in the Weibull distribution and extreme value distribution for various censored samples. The simulation procedure is repeated 10,000 times in multiply censored samples with $\theta=0.5, 1, 2$ and sample size $n=5, 10$. From Table 1, the MSE of $\widehat{\theta_w}$ is smaller than the MSE of $\widehat{\theta_E}$ for $\delta=2, 3$, but the MSE of $\widehat{\theta_E}$ is smaller than the MSE of $\widehat{\theta_w}$ for $\delta=1$. In general, $\widehat{\theta_w}$ is better estimator than $\widehat{\theta_E}$ in the sense of MSE for large δ, but $\widehat{\theta_E}$ is better than $\widehat{\theta_w}$ in the sense of MSE for small δ, and the mean squared errors of the proposed estimators decrease as δ increase.

Table 1. The mean squared errors for the proposed estimators of the scale parameter θ

n	k	a_j	MSE					
			$\theta=0.5,\ \delta=1$		$\theta=1,\ \delta=1$		$\theta=2,\ \delta=1$	
			$\widehat{\theta}_w$	$\widehat{\theta}_E$	$\widehat{\theta}_w$	$\widehat{\theta}_E$	$\widehat{\theta}_w$	$\widehat{\theta}_E$
5	0	1 2 3 4 5	0.049	0.045	0.197	0.179	0.789	0.714
	1	1 2 3 4	0.061	0.056	0.244	0.223	0.977	0.892
		2 3 4 5	0.050	0.045	0.199	0.181	0.796	0.726
		1 3 4 5	0.050	0.045	0.200	0.181	0.801	0.725
	2	2 3 4	0.062	0.057	0.247	0.229	0.986	0.915
		2 3 5	0.054	0.048	0.215	0.192	0.861	0.769
		1 3 5	0.054	0.048	0.217	0.191	0.867	0.765
10	0	1 2 3 4 5 6 7 8 9 10	0.025	0.023	0.099	0.094	0.395	0.374
	1	2 3 4 5 6 7 8 9 10	0.025	0.023	0.099	0.094	0.395	0.375
		1 2 3 4 5 6 7 8 9	0.027	0.026	0.109	0.104	0.437	0.415
		1 2 3 4 6 7 8 9 10	0.025	0.023	0.099	0.094	0.396	0.375
		1 2 3 5 6 7 8 9 10	0.025	0.023	0.099	0.094	0.396	0.376
	3	4 5 6 7 8 9 10	0.025	0.024	0.099	0.095	0.398	0.379
		1 2 3 4 5 6 7	0.035	0.033	0.140	0.134	0.562	0.535
		3 4 5 6 7 8 9	0.027	0.026	0.110	0.105	0.439	0.420
		2 3 4 5 6 7 8	0.031	0.029	0.123	0.117	0.493	0.470
		2 3 4 6 8 9 10	0.025	0.024	0.100	0.094	0.399	0.378
		3 4 5 7 8 9 10	0.025	0.024	0.100	0.095	0.398	0.379
	5	3 4 5 6 7	0.035	0.034	0.141	0.136	0.565	0.544
		2 3 4 5 6	0.041	0.039	0.164	0.158	0.657	0.632
		4 5 6 7 8	0.031	0.030	0.124	0.120	0.498	0.478
		2 4 6 8 10	0.026	0.024	0.104	0.097	0.415	0.386
		1 3 5 7 9	0.028	0.026	0.112	0.106	0.446	0.423
		2 3 4 7 8	0.031	0.030	0.125	0.119	0.500	0.476
		3 4 6 8 9	0.028	0.026	0.111	0.106	0.443	0.423

Table 1. (continued)

n	k	a_j	MSE					
			$\theta=0.5,\ \delta=2$		$\theta=1,\ \delta=2$		$\theta=2,\ \delta=2$	
			$\widehat{\theta}_w$	$\widehat{\theta}_E$	$\widehat{\theta}_w$	$\widehat{\theta}_E$	$\widehat{\theta}_w$	$\widehat{\theta}_E$
5	0	1 2 3 4 5	0.012	0.013	0.049	0.050	0.194	0.201
	1	1 2 3 4	0.015	0.015	0.060	0.061	0.240	0.243
		2 3 4 5	0.012	0.012	0.049	0.050	0.195	0.200
		1 3 4 5	0.012	0.013	0.049	0.051	0.196	0.202
	2	2 3 4	0.015	0.015	0.060	0.061	0.242	0.243
		2 3 5	0.013	0.013	0.050	0.051	0.202	0.205
		1 3 5	0.013	0.013	0.051	0.052	0.202	0.207
10	0	1 2 3 4 5 6 7 8 9 10	0.006	0.006	0.025	0.025	0.099	0.101
	1	2 3 4 5 6 7 8 9 10	0.006	0.006	0.025	0.025	0.099	0.101
		1 2 3 4 5 6 7 8 9	0.007	0.007	0.027	0.028	0.109	0.110
		1 2 3 4 6 7 8 9 10	0.006	0.006	0.025	0.025	0.099	0.101
		1 2 3 5 6 7 8 9 10	0.006	0.006	0.025	0.025	0.099	0.101
	3	4 5 6 7 8 9 10	0.006	0.006	0.025	0.025	0.099	0.101
		1 2 3 4 5 6 7	0.009	0.009	0.035	0.035	0.139	0.140
		3 4 5 6 7 8 9	0.007	0.007	0.027	0.028	0.109	0.110
		2 3 4 5 6 7 8	0.008	0.008	0.031	0.031	0.122	0.123
		2 3 4 6 8 9 10	0.006	0.006	0.025	0.025	0.099	0.101
		3 4 5 7 8 9 10	0.006	0.006	0.025	0.025	0.099	0.101
	5	3 4 5 6 7	0.009	0.009	0.035	0.035	0.140	0.140
		2 3 4 5 6	0.010	0.010	0.040	0.041	0.162	0.162
		4 5 6 7 8	0.008	0.008	0.031	0.031	0.123	0.123
		2 4 6 8 10	0.006	0.006	0.025	0.026	0.101	0.102
		1 3 5 7 9	0.007	0.007	0.028	0.028	0.110	0.112
		2 3 4 7 8	0.008	0.008	0.031	0.031	0.123	0.124
		3 4 6 8 9	0.007	0.007	0.027	0.028	0.110	0.111

Table 1. (continued)

n	k	a_j	MSE					
			$\theta=0.5,\ \delta=3$		$\theta=1,\ \delta=3$		$\theta=2,\ \delta=3$	
			$\widehat{\theta}_w$	$\widehat{\theta}_E$	$\widehat{\theta}_w$	$\widehat{\theta}_E$	$\widehat{\theta}_w$	$\widehat{\theta}_E$
5	0	1 2 3 4 5	0.006	0.006	0.023	0.024	0.092	0.096
	1	1 2 3 4	0.007	0.007	0.028	0.029	0.114	0.116
		2 3 4 5	0.006	0.006	0.023	0.024	0.092	0.095
		1 3 4 5	0.006	0.006	0.023	0.024	0.093	0.097
	2	2 3 4	0.007	0.007	0.028	0.029	0.114	0.115
		2 3 5	0.006	0.006	0.024	0.024	0.094	0.097
		1 3 5	0.006	0.006	0.024	0.024	0.094	0.098
10	0	1 2 3 4 5 6 7 8 9 10	0.003	0.003	0.011	0.012	0.046	0.047
	1	2 3 4 5 6 7 8 9 10	0.003	0.003	0.011	0.012	0.046	0.047
		1 2 3 4 5 6 7 8 9	0.003	0.003	0.013	0.013	0.050	0.051
		1 2 3 4 6 7 8 9 10	0.003	0.003	0.011	0.012	0.046	0.047
		1 2 3 5 6 7 8 9 10	0.003	0.003	0.011	0.012	0.046	0.047
	3	4 5 6 7 8 9 10	0.003	0.003	0.011	0.012	0.046	0.047
		1 2 3 4 5 6 7	0.004	0.004	0.016	0.016	0.064	0.065
		3 4 5 6 7 8 9	0.003	0.003	0.013	0.013	0.050	0.051
		2 3 4 5 6 7 8	0.004	0.004	0.014	0.014	0.056	0.057
		2 3 4 6 8 9 10	0.003	0.003	0.011	0.012	0.046	0.047
		3 4 5 7 8 9 10	0.003	0.003	0.011	0.012	0.046	0.047
	5	3 4 5 6 7	0.004	0.004	0.016	0.016	0.064	0.065
		2 3 4 5 6	0.005	0.005	0.019	0.019	0.075	0.075
		4 5 6 7 8	0.004	0.004	0.014	0.014	0.056	0.057
		2 4 6 8 10	0.003	0.003	0.012	0.012	0.046	0.047
		1 3 5 7 9	0.003	0.003	0.013	0.013	0.051	0.052
		2 3 4 7 8	0.004	0.004	0.014	0.014	0.057	0.057
		3 4 6 8 9	0.003	0.003	0.013	0.013	0.050	0.051

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