Journal of Korean
Data & Information Science Society
2004, Vol. 15, No. 2, pp. 495~498

A Note on Convergence of Expected Value of Fuzzy Variables

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Abstract

In this note, we consider several types of convergence theorems for the expected value of fuzzy variables defined by Liu and Liu [IEEE Trans. Fuzzy Systems, 10(2002), 445-450].

Keywords : Convergence, Expected value, Fuzzy variable

1. Introduction

There are many convergence concepts in probability theory. They play a very important role in both theory and applications. For detailed expositions, we may consult Ash(1972) on Chow and Teicher(1980). Recently Liu(2003) gave some new convergence concepts of fuzzy sequence and discussed the relationship among them. Liu and Liu(2002) also presented a novel concept of expected values of fuzzy variable, which is essentially a type of Choquet integral and coincides with that of random variables. In this note, we concentrate on the convergence of expected value of fuzzy variables defined by Liu and Liu(2002). One question is whether the analogous convergence theorems (Denominated Convergence Theorem, Monotone Convergence Theorem, Fatou's Lemma) hold in fuzzy set theory. The answer is in the negative. We are going to give simple counter-example for this problem.

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2. Definitions

Possibility theory was proposed by Zadeh(1978), and developed by many researchers such as Dubois and Prade(1988). Let ξ be a fuzzy variable defined on the possibility space $(\Theta, P(\Theta), \mathrm{Pos})$. The necessity of a fuzzy event is defined as the impossibility of the opposite event, i.e.,

$$\mathrm{Nec}\{\xi \le r\} = 1 - \mathrm{Pos}\{\xi > r\}. \tag{1}$$

Thus a necessity measure is the dual of possibility measure. The credibility of a fuzzy event is defined as the average of its possibility and necessity. That is,

$$\mathrm{Cr}\{\xi \le r\} = \frac{1}{2}(\mathrm{Pos}\{\xi \le r\} + \mathrm{Nec}\{\xi \le r\}). \tag{2}$$

Note that the credibility measure Cr is self-dual,
i.e., $\mathrm{Cr}\{\xi \le r\} + \mathrm{Cr}\{\xi > r\} = 1$ for any r.

Definition 1. (Liu and Liu(2002)) Let ξ be a fuzzy variable. Then the expected value of ξ is defined by

$$\mathrm{E}\xi = \int_0^{+\infty} \mathrm{Cr}\{\xi \ge r\}\,\mathrm{dr} - \int_{-\infty}^0 \mathrm{Cr}\{\xi \le r\}\,\mathrm{dr} \tag{3}$$

provided that at least one of the two integrals is finite.

This definition is not only applicable to continuous case but also discrete case. An important property proved by Liu and Liu(2002) is the lineality of expected value operator. That is, if ξ and η are fuzzy variables with finite expected values, then $\mathrm{E}(a\xi + b\eta) = a\,\mathrm{E}\xi + b\,\mathrm{E}\eta$ for any real numbers a and b. For the detailed expositions, the interested reader may consult the book by Liu(2002).

Definition 2. Suppose that $\{\xi_i\}$ is a sequence of fuzzy variables. The sequence $\{\xi_i\}$ is said to be convergent a.s. to the fuzzy variable ξ if and only if there exists a set $\theta \in A$ with $\mathrm{Cr}\{A\} = 1$ such that

$$\lim_{i\to\infty} |\xi_i(\theta) - \xi(\theta)| = 0$$

for every $\theta \in A$. In that case we write $\xi_i \to \xi$, a.s.

Definition 3. Suppose that $\{\xi_i\}$ is a sequence of fuzzy variables. We say that the sequence $\{\xi_i\}$ convergences in credibility to the fuzzy variable ξ if

$$\lim_{i\to\infty} \mathrm{Cr}\{|\xi_i - \xi| \ge \varepsilon\} = 0$$

for every $\varepsilon > 0$.

Definition 4. (Liu(2003)) Suppose that $\{\xi_i\}$ is a sequence of fuzzy variables with finite expected values. We say that the sequence $\{\xi_i\}$ convergence in mean to the fuzzy variable ξ if

$$\lim_{i\to\infty} \mathrm{E}|\xi_i - \xi| = 0.$$

3. Counter example

Let $\{\xi_n\}$, ξ be random variables, then the following convergence theorems hold.

- If $0 \le \xi_n \uparrow \xi$ a.s., then $\mathrm{E}\,\xi_n \uparrow \mathrm{E}\,\xi$ (monotone convergence theorem).
- If $0 \le \xi_n$, then $\mathrm{E}\,\liminf_{n\to\infty} \xi_n \le \liminf_{n\to\infty} \mathrm{E}\,\xi_n$ (Fatou's lemma).
- If $0 \le \xi_n \to \xi$ a.s. and $\sup \xi_n$ is integrable, then $\lim \mathrm{E}\,\xi_n = \mathrm{E}\,\xi$ (dominated convergence theorem).

Can we prove that the analogous convergence theorems hold in fuzzy case?
The answer is negative. We give counter-examples.

Example. Let $\Theta = (0,1]$, $\mathrm{Pos}\{\theta\} = 1$ for $0 < \theta \le 1$ and the fuzzy variables are defined by

$$\xi_n(\theta) = \theta^{\frac{1}{n}} \text{ for } 0 < \theta \le 1$$

for $n = 1, 2, \cdots$ and $\xi = 1$. Then the sequence $\{\xi_n\}$ converges monotonically to ξ. However, for any $n = 1, 2, \cdots$, we have

$$\mathrm{C_r}\{\xi_n \ge r\} = \begin{cases} \frac{1}{2} & 0 < r \le 1, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\mathrm{C_r}\{\xi \ge r\} = \begin{cases} 1 & 0 < r \le 1, \\ 0 & \text{otherwise}, \end{cases}$$

and hence $\mathrm{E}\,\xi_n = \frac{1}{2}$ for $n = 1, 2, \cdots$ and $\mathrm{E}\,\xi = 1$. Hence this simple example does imply above three basic convergences, that is, $\mathrm{E}\,\xi_n$ does not converge monotonically increasing to $\mathrm{E}\,\xi$, $\mathrm{E}\,\xi_n \not\to \mathrm{E}\,\xi$ and $1 = \mathrm{E}\,\xi = \mathrm{E}\,\liminf \xi_n \not\le \liminf \mathrm{E}\,\xi_n = \frac{1}{2}$.

In the following, we introduce a simple convergence theorem.

Theorem. Let $\{\xi_n\}$ be a uniformly bounded sequence of fuzzy variables. If the sequence $\{\xi_n\}$ converges in credibility to a fuzzy variable ξ, then $\{\xi_n\}$ converges in mean to ξ.

Proof. By the definition of expected value, we have

$$E|\xi_n - \xi| = \int_0^\infty C_r\{|\xi_n - \xi| \geq r\} dr.$$

Since ξ_n are uniformly bounded, there exists $M > 0$ such that

$$E|\xi_n - \xi| = \int_0^M C_r\{|\xi_n - \xi| \geq r\} dr.$$

Now, the result follows from dominated convergence theorem since for any $r > 0$, $C_r\{|\xi_n - \xi| \geq \varepsilon\} \to 0$ and $C_r\{|\xi_n - \xi| \geq r\} \leq 1$.

By Theorem 6 of Liu(2003), convergence in mean imply convergence in credibility. The following result is immediate.

Corollary. Let $\{\xi_n\}$ be a uniformly bounded sequence of fuzzy variables. Then the sequence $\{\xi_n\}$ converges in mean to ξ if and only if $\{\xi_n\}$ converges in credibility to ξ.

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[received date : Dec. 2003 , accepted date : Mar. 2004]